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Fixed Point Theorems in Partial Metric Spaces for Four Weakly Compatible Maps

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Abstract In 2011, E. Karapinar *et al.* [12] proved some common fixed point theorems for four weakly compatible self-maps in complete partial metric spaces. In this paper, we extend these theorems using E.A. property and (CLR)-property in complete partial metric spaces.

Keywords: *partial metric space, fixed point, E.A. property, (CLR) property*

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1. Introduction

Partial metric spaces, introduced by Matthews [2,3], are a generalization of the notion of the metric space in which in definition of metric the condition $d(x,x)=0$ is replaced by the condition $d(x,x) \leq d(x,y)$. Matthews discussed some properties of convergence of sequence and proved the fixed point theorems for contractive mapping on partial metric spaces: any mapping T of a complete partial metric spaces X into itself that satisfies, where $0 \leq k < 1$, the inequality $d(Tx,Ty) \leq kd(x,y)$, for all $x,y \in X$, has a unique common fixed point. Recently, many authors [4-12] have focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces.

The definition of partial metric space was given by Matthews [1] as follows:

Definition 1.1. [1] Let X be a nonempty set and let $p: X \times X \to \mathbb{R}_0^+$ satisfy

(PM1) $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$,

(PM2) $p(x,x) \leq p(x,y)$,

(PM3) $p(x,y) = p(y,x)$,

(PM4)

$$p(x,y) \leq p(x,z) + p(z,y) - p(z,z). \tag{1.1}$$

for all $x,y \in X$, where $\mathbb{R}_0^+ = [0,\infty]$. Then the pair (X,p) is called a partial metric space (in short PMS) and p is called a partial metric on X.

Let (X,p) be a PMS. Then, the functions $d_p, d_m: X \times X \to \mathbb{R}_0^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$$

are usual metrics on X. It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the open $p-$ balls $\{B_p(x,\varepsilon): x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X: p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

E. Karapinar *et al.* [13] proved the following theorem for four weakly compatible mappings in partial metric spaces.

Theorem 1.2. [13] Let (X,p) be a complete PMS. Suppose that T,S,F,G are self-mappings on X, and F and G are continuous. Suppose also that T,F and S,G are continuing pairs and that

$$T(X) \subset F(X), S(X) \subset G(X). \tag{1.2}$$

If there exists an $r \in [0,1)$, and $m,n \in \mathbb{N}$ such that

$$p(Tx,Sy) \leq rM(x,y) \tag{1.3}$$

for any x,y in X, where

$$M(x,y) = \max\left\{ \begin{array}{l} p(Tx,Gx), p(Sy,Fy), p(Gx,Fy), \\ \frac{1}{2}\left[p(Tx,Fy) + p(Sy,Gx)\right] \end{array} \right\} \tag{1.4}$$

Then T,S,F and G have a unique common fixed point z in X.

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.3. [1] Let (X,d) be a metric space and $f, g: X \to X$. Then f and g are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some t in X.

In 2012, Sintunavarat and Kumam [14] introduced the notion of (CLR) property as follows:

Definition 1.4. [14] Two self-mappings f and g of a metric spaces (X,d) are said to satisfy (CLR_f) property if there exists a sequences $\{x_n\}$ in X such that,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx,$$

for some x in X.

2. Main Results

Theorem 2.1. Let T, S, F and G be four self-mappings on a complete partial metric space (X,p) satisfying (1.3), (1.4) and the followings:

(1.5) pairs$(T,G)$and (S,F) are weakly compatible,

(1.6) pair (T,G) or (S,F) satisfy E.A. property.

If any one of TX, SX, FX and GX is a complete subspace of X, then T, S, F and G have a unique common fixed point.

Proof: Let us suppose (T,G)satisfies the E.A. property. Then, ∃ a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Gx_n = z$, for some z in X.

Since $TX \subset FX$, ∃ a sequence $\{y_n\}$ in X such that $Tx_n = Fy_n$.

Hence, $\lim_{n\to\infty} Fy_n = z$.

We shall show that $\lim_{n\to\infty} Sy_n = z$.

Let, if possible, $\lim_{n\to\infty} Sy_n = t = z$.

From (1.3), we have

$$p(Tx_n, Sy_n) \le rM(x_n, y_n)$$

Letting limit as $n \to \infty$, we get

$$\lim_{n\to\infty} p(Tx_n, Sy_n) \le r \lim_{n\to\infty} M(x_n, y_n), \qquad (1.7)$$

where

$$\lim_{n\to\infty} M(x_n, y_n)$$
$$= \lim_{n\to\infty}\left(\max\left\{\begin{array}{l} p(Tx_n,Gx_n),\, p(Sy_n,Fy_n), \\ p(Gx_n,Fy_n), \\ \frac{1}{2}\left[p(Tx_n,Fy_n)+p(Sy_n,Gx_n)\right]\end{array}\right\}\right)$$
$$= \max\left\{p(z,z),\, p(t,z),\, p(z,z), \frac{1}{2}\left[p(z,z)+p(t,z)\right]\right\}$$
$$= p(t,z).$$

Thus, from (1.7), we get

$$p(z,t) \le r\left(p(z,t)\right) < r\left(p(z,t)\right),$$

which is a contradiction.

Therefore, $t = z$, that is, $\lim_{n\to\infty} Sy_n = z$.

Suppose that SX is a complete subspace of X. Then, $z = Su$ for some u in X.

Subsequently, we have

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Fy_n = z = Fu.$$

Now, we shall show that $Su = Fu$.

Let, if possible, $Su \ne Fu$.

From (1.3), we have

Letting limit as $n \to \infty$, we get

$$\lim_{n\to\infty} p(Tx_n, Su) \le r \lim_{n\to\infty} M(x_n, u), \qquad (1.8)$$

where

$$\lim_{n\to\infty} M(x_n, u)$$
$$= r\lim_{n\to\infty}\left(\max\left\{\begin{array}{l} p(Tx_n,Gx_n),\, p(Su,Fu),\, p(Gx_n,Fu), \\ \frac{1}{2}\left[p(Tx_n,Fu)+p(Su,Gx_n)\right]\end{array}\right\}\right)$$
$$= \max\left\{\begin{array}{l} p(z,z),\, p(Su,z),\, p(z,z), \\ \frac{1}{2}\left[p(z,z)+p(Su,z)\right]\end{array}\right\}$$
$$= p(Su,z).$$

Thus, from (1.8), we get

$$p(z,Su) \le r\left(p(z,Su)\right) < r\left(p(z,Su)\right),$$

a contradiction.

Therefore, $Su = Fu = z$.

Since, S and F are weakly compatible, therefore, $SFu = FSu$, implies that $SSu = SFu = FSu = FFu$.

Since, $SX \subset TX$, there exists $v \epsilon X$ such that

$$Su = Tv.$$

Now, we claim that $Tv = Fv$.

Let, if possible, $Tv \ne Fv$.

From (1.3), we have

Letting limit as $n \to \infty$, we get

$$\lim_{n\to\infty} p(Tx_n, Sy_n) \le r \lim_{n\to\infty} M(x_n, y_n),$$

$$p(Tv, Su) \le r \lim_{n\to\infty} M(v,u), \qquad (1.9)$$

where

$$\lim_{n\to\infty} M(v,u)$$
$$= \lim_{n\to\infty}\left(\max\left\{\begin{array}{l} p(Tx_n,Gx_n),\, p(Sy_n,Fy_n),\, p(Gx_n,Fy_n), \\ \frac{1}{2}\left[p(Tx_n,Fy_n),+p(Sy_n,Gx_n)\right]\end{array}\right\}\right)$$
$$= \max\left\{\begin{array}{l} p(Tv,Gv),\, p(Su,Fu),\, p(Gv,Fu), \\ \frac{1}{2}[p(Tv,Fu+p(Su,Gv)]\end{array}\right\}$$
$$= p(Tv,Gu) = p(Tv,Su).$$

Then, from (1.9), we get $P(Tv,Su) \le rp(Tv,Su)$, a contradiction.

Therefore, $Tv = Su = Gv$.

Thus, we have $Fu = Su = Tv = Gv$.

The weak compatibility of T and G implies that $TTv = TGv = GTv = GGv$.

Now, we claim that, Su is the common fixed point of $T, S, F$and G.

Suppose that, $SSu \ne Su$.

From (1.3), we have

$$p(Su, SSu) = p(Tv, SSu) \le rM(v, Su), \qquad (1.10)$$

where

$$M(v,Su)=\max\left\{\begin{array}{l}p(Tv,Gv),\,p(FSu,SSu),\,p(Gv.SSu),\\ \frac{1}{2}\left[p(Tv,SSu)+p(SFu,Gv)\right]\end{array}\right\}$$

$$=\max\{p(Su,SSu),0,0,p(Su,SSu)\},$$

$$=p(Su,SSu).$$

Thus, from (1.10), we have $p(Su,SSu)\le rp(Su,SSu)$, a contradiction.

Therefore, $Su=SSu=FSu$.

Hence, Su is the common fixed point of $S$and F.

Similarly, we prove that Gv is the common fixed point of T and G. Since, $Fu=Gv, Fu$ is the common fixed point of $T,S,F$and G. The proof is similar when TX is assumed to be a complete subspace of X are similar to the cases in which FXor SX, respectively is complete subspace of X, since $T(X)\subset F(X)$ and $S(X)\ \subset G(X)$.

Now, we shall prove that the common fixed point is unique.

If possible, let c and d be two common fixed points of T,S,F and G such that $c\neq d$.

From (1.3), we have

$$p(c,d)=p(Tc,Sd)\le rM(c,d),\qquad(1.11)$$

where

$$M(c,d)=\max\left\{\begin{array}{l}p(Tc,Gc),\,p(Sd,Fd),\,p(Gd,Fd),\\ \frac{1}{2}\left[p(Tc,Fd)+p(Sd,Gd)\right]\end{array}\right\}$$

$$=\max\{p(c,d),0,0,p(c,d)\}$$

$$=p(c,d)$$

Thus, from (1.11), we get $p(c,d)\le rp(c,d)$, a contradiction.

Therefore, $c=d$ and the uniqueness follows.

Theorem 2.2. Let $T,\ S,\ F\ and\ G$ be four self-mappings on a complete partial metric space (X,p) satisfying (1.3), (1.5) and the followings:

$$\begin{array}{l}T(X)\subset F(X),\\ \textit{and the pair }(T,G)\textit{ satisfies }(CLR_G)\textit{ property, or}\\ S(X)\subset G(X),\\ \textit{and the pair }(S,F)\textit{ satisfies }(CLR_F)\textit{ property.}\end{array}\qquad(1.12)$$

Then, $T,S,F\ and\ G$ have a unique common fixed point.

Proof: Without loss of generality, assume that $T(X)\subset F(X)$ and the pair (T,G) satisfies (CLR_G) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}G\,x_n=Gx$, for some x in X.

Since $T(X)\subset F(X)$, there exists a sequence $\{y_n\}$ in X such that $Tx_n=Fy_n$.

Hence, $\lim_{n\to\infty}F\,y_n=Gx$.

We shall show that $\lim_{n\to\infty}S\,y_n=Gx$.

Let, if possible $\lim_{n\to\infty}S\,y_n=z\neq Gx$.

From (1.3), we have

$$p(Tx_n,Sy_n)\le rM(x_n,y_n).$$

Letting limit as $n\to\infty$, we get

$$\lim_{n\to\infty}p(Tx_n,Sy_n)\le r\lim_{n\to\infty}M(x_n,y_n),\qquad(1.13)$$

where

$$\lim_{n\to\infty}M(x_n,y_n)$$

$$=\lim_{n\to\infty}\left(\max\left\{\begin{array}{l}p(Tx_n,Gx_n),\,p(Sy_n,Fy_n),\,P(Gx_n,Fy_n),\\ \frac{1}{2}\left[p(Tx_n,Fy_n)+p(Sy_n,Gx_n)\right]\end{array}\right\}\right)$$

$$=\max\left\{\begin{array}{l}p(Gx,Gx),\,p(z,Gx),\,p(Gx,Gx),\\ \frac{1}{2}\left[p(Gx,Gx)+p(z,Gx)\right]\end{array}\right\}$$

$$=p(z,Tx).$$

Thus, from (1.13), we get

$$p(Gx,z)\le rp(z,Gx)<rp(Gx,z),$$

a contradiction.

Therefore, $Gx=z$, that is, $\lim_{n\to\infty}Sy_n=Gx$.

Subsequently, we have

$$\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}Gx_n=\lim_{n\to\infty}Fy_n=\lim_{n\to\infty}Sy_n=Gx=z.$$

Now, we will show that $Tx=z$.

Let, if possible, $Tx\neq z$.

From (1.3), we have

$$p(Tx,Sy_n)\neq rM(x,y_n).$$

Letting limit as $n\to\infty$, we get

$$\lim_{n\to\infty}\left(p(Tx,Sy_n)\right)\le r\lim_{n\to\infty}M(x,y_n),\qquad(1.14)$$

where,

$$\lim_{n\to\infty}M(x,y_n)$$

$$=\lim_{n\to\infty}\left(\max\left\{\begin{array}{l}p(Tx,Gx),\,p(Sy_n,Fy_n),\,P(Gx,Fy_n),\\ \frac{1}{2}\left[p(Tx,Fy_n)+p(Sy_n,Gx)\right]\end{array}\right\}\right)$$

$$=\max\left\{\begin{array}{l}p(Tx,z),\,p(z,z),\,p(z,z),\\ \frac{1}{2}\left[p(Tx,z)+p(z,z)\right]\end{array}\right\}$$

$$=p(Tx,z).$$

Thus, from (1.14), we get $p(Tx,z)\le p(Tx,z)$, a contradiction.

Therefore, $Tx=z=Gx$.

Since, the pair (T,G) is weakly compatible, it follows that $Tz=Gz$.

Also, since $T(X)\subset F(X)$, there exists some y in X such that $Tx=Fy$, that is, $Fy=z$.

Now, we show that $Sy=z$.

Let, if possible, $Sy\neq z$.

From (1.3), we have $p(Tx_n,Sy)\le rM(x_n,y)$.

Letting limit as $n\to\infty$, we get

$$\lim_{n\to\infty}p(Tx_n,Sy)\le r\lim_{n\to\infty}M(x_n,y),\qquad(1.15)$$

where,

$$\lim_{n\to\infty} M(x_n, y)$$
$$= \lim_{n\to\infty}\left(\max\left\{\begin{array}{l} p(Tx_n, Gx_n), p(Sy, Fy), p(Gx_n, Fy), \\ \frac{1}{2}\left[p(Tx_n, Fy) + p(Sy, Gx_n)\right] \end{array}\right\}\right)$$
$$= \max\left\{p(z,z), p(Sy,z), p(z,z), \frac{1}{2}\left[p(z,z) + p(Sy,z)\right]\right\}$$
$$= p(Sy, z).$$

Thus, from (1.15), we get

$$p(z, Sy) \le rp(Sy, z) < rp(z, Sy),$$

a contradiction.

Hence, $z = Sy = Fy$.

Since the pair (S, F) is weakly compatible, it follows that $Sz = Fz$.

Let, if possible, $Tz \neq Sz$.

From (1.3), we have

$$p(Tz, Sz) \le rM(z, z), \quad (1.16)$$

where,

$$M(z,z) = \max\left\{\begin{array}{l} p(Tz, Sz), p(Sz, Fz), p(Gz, Fz), \\ \frac{1}{2}\left[p(Tz, Fz) + p(Sz, Gz)\right] \end{array}\right\}$$
$$= p(Tz, Sz).$$

Thus, from (1.16), we get $p(Tz, Sz) \le rp(Tz, Sz)$, a contradiction.

Therefore, $Gz = Sz$, that is, $Tz = Gz = Fz = Sz$.

Now, we shall show that $z = Sz$.

Let, if possible, $z \neq Sz$.

From (1.3), we have

$$p(Tx, Sz) \le rM(x, z), \quad (1.17)$$

where,

$$M(x,z) = \max\left\{\begin{array}{l} p(Tx, Gx), p(Sz, Fz), p(Gx, Fz), \\ \frac{1}{2}\left[p(Tx, Fz) + p(Sz, Gx)\right] \end{array}\right\}$$
$$= p(Tx, Sz) = p(z, Sz).$$

Thus, from (1.17), we get $p(z, Sz) \le r\, p(z, Sz)$, a contradiction.

Therefore, $z = Sz = Tz = Fz = Gz$.

Hence, z is the common fixed point T, S, F and G.

Now, we shall prove that the common fixed point is unique.

Let u be the another common fixed point of T, S, F and G.

Let, if possible, $z \neq u$.

From (1.3), we have

$$p(u, z) = p(Tu, Sz) \le rM(u, z),$$

where,

$$M(u,z) = \max\left\{\begin{array}{l} p(Tu, Gu), p(Sz, Fz), p(Gu, Fz), \\ \frac{1}{2}\left[p(Tu, Fz) + p(Sz, Gu)\right] \end{array}\right\}$$
$$= p(u, z).$$

Therefore, we get $p(u, z) \le p(u, z)$, a contradiction.

Thus, $u = z$, and hence the uniqueness follows.

3. Conclusion

In this paper, some results in complete partial metric spaces are proved using two important properties in fixed point theory, viz., E.A. property and (CLR) property. The results proved are the extended version of the result proved by E. Karapinar *et al.* for weakly compatible maps.

References

[1] Aamri M., Moutawakil D. El., Some new common fixed point theorems under strict Contractive conditions, J. Math. Anal. Appl., 27(1) (2002), 181-188.

[2] S.G. Matthews, Partial metric topology, Research report 212, Department of Computer Science,University of Warwick, (1992).

[3] S.G. Matthews, Partial metric topology, in proceedings of the 8th Summer Conference, Queen's College, General Topology and its Appl. Proc. 728(1992), 183-197.

[4] S. Oltra, and O. Valero, Banach's fixed point theorem for partial metric spaces, Rendiconti dell' Istitutodi Mathematica dell' Universiia di Trieste, 36(1-2) (2004), 17-26.

[5] O. Valero, On Banach fixed point theorems for partial metric spaces, Applied General Topology, 6(2) (2005), 229-240.

[6] S. Oltra, S. Romaguera and E.A. Sanchez--Perez, The canonical partial metric and the uniform convexity on normed spaces, Applied General Topology, 6(2)(2005), 185-194.

[7] I.A. Rus, Fixed point theory in partial metric spaces, Analele Universitattii de Vest, Timitsoara, 46(2) (2008), 149-160.

[8] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, Topology and its Applications, 157(18) (2010), 2778-2785.

[9] I. Altun and A. Erduran, Fixed Point Theorems for monotone mappings on partial metric spaces, Fixed Point Theory and applications, 2011(2011), Article ID 508730, 10 pages.

[10] E. Karapinar, Weak ϕ-contraction on partial metric spaces, Journal of Computational Analysis and Applications (In press).

[11] E. Karapinar, Generalizations of Caristi Kirk's theorem on partial metric spaces, Fixed Point Theory and applications, 2011 (2011).

[12] E. Karapinar and I.M. Erhan, Fixed point theorems for operators on partial metric spaces, Applied Mathematics Letters, 24(11) (2011), 1900-1904.

[13] E. Karapinar and U. Yuksel, Some common fixed point theorems in partial metric spaces, Journal of Applied Mathematics, 2011 (2011), Article ID 263621.

[14] Sintunavarat W. and Kumam P., Common Fixed Point Theorems for a Pair of Weakly compatible mappings in Fuzzy Metric Spaces, Hindawi Publishing Corporation, Journal of Applied mathematics, 2011(2011), Article ID 637958, 14 pages.