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Gaps between Operator Norm and Spectral and Numerical Radii of the Tensor Product of Operators

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Abstract One of the fundamental problem of the Spectral Theory of Linear Operators is to determine of the geometric place of the spectrum of the given operator and calculate the spectral and numerical radii of this operator. Other important problem in this theory is to explained the situation the spectral (numerical) radius is equal or not to operator norm. The only known way to calculate the spectral radius to date is the classical Gelfand formula, which often presents great technical challenges. Also, there is not yet a method of calculating the numerical radius for an operator. It should be noted that the finding the numerical range and numerical radius means maximally localizing the spectrum of an operator. The main purpose of this paper is to determine the relations gaps between operator norm and spectral and numerical radii of the tensor product operators associated with the compatible gaps of coordinate operators.

Keywords: *operator norm, spectral radius, numerical radius*

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1. Introduction

As is known from the mathematical literature, in a Banach space one of the most fundamental questions of the Spectral Theory of Linear Operators is to clearly determine the geometric location of the spectrum or resolvent sets of the operator and calculate the spectral radius of the operator. Although in many cases, serious theoretical and technical difficulties are encountered in finding the spectrum set of nonselfadjoint linear operators, as it is known from the literature, the only formula used in the spectral radius of linear bounded operators calculation is Gelfand formula. Let $\mathcal{X}$ be a Banach space and A be a linear bounded operator in $\mathcal{X}$, i.e. $A \in L(\mathcal{X})$, Gelfand formula is as follows:

$$r(A) = \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}}$$

[1]. As it is known $r(A) \le \| A\|$. Let A be a linear bounded normal operator in a Hilbert space H. It is true the relation $gap(A) = 0$ (see [2]). But generally, these equations may not be provided for non-normal linear bounded operators. For example, the gap between operator norm and spectral radius of the operator $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ is $gap(A) = \sqrt{2} - 1$. Moreover, let A be a quasinilpotent operator. It is clear that $gap(A) \ge 0$, $A \in L(\mathcal{X})$, $A \ne 0$. The determination and definition the specific sub-classes of the operators family defined by $A \in L(\mathcal{X})$, $gap(A) = \| A\| - r(A) > 0$ is theoretically important as well as providing great benefits in applied mathematics.

Let H be a Hilbert space and $A \in L(H)$. It is known that

$$\frac{1}{2}\| A\| \le w(A) \le \| A\|$$

[3]. If $A \in L(H)$ is a normal operator, then $w(A) = \| A\|$ (see [4]). But this inequality may not be true for non-normal operators. Generally, for $A \in L(H)$ the following inequality is true $r(A) \le w(A)$ (see [5,6]). One of the most fundamental questions is to identify specific subclasses of the family of operators that satisfy the $A \in L(H)$, $wgap(A) = \| A\| - w(A) > 0$ condition.

In this study, the spectral and numerical gap characteristics of the tensor product of linear bounded operators defined on the tensor product of a finite number of Hilbert spaces are expressed with the appropriate characteristic numbers of the coordinate operators that form the tensor product.

Demuth's open problem in 2015 had a great impact on the emergence and shaping of the subject examined in this article (see [7]).

2. Gap between Operator Norm and Spectral Radius of the Tensor Product of Operators

In this section, gap between operator norm and spectral radius of the tensor product of operator (for more information of tensor product see [8]) will be investigated. It is known that for an operator $A \in L(H)$ the spectral radius $r(A)$ is defined as

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

(see [5]). Namely, it is true the following result.

Theorem 1 *Let H_1, H_2 be two separable Hilbert spaces and $A_1 \in L(H_1), A_2 \in L(H_2)$. For the tensor product*

$$A = A_1 \otimes A_2 : H \to H,\, H = H_1 \otimes H_2$$

is valid

$$gap(A) = gap(A_1)gap(A_2) + r(A_1)gap(A_2) + r(A_2)gap(A_1).$$

Proof

By the definition

$$gap(A) = \| A\| - r(A)$$

it is known that

$$\| A\| = \| A_1\| \; \| A_2\|.$$

From [9], for the spectrum of A it is known that

$$\sigma(A) = \sigma(A_1)\sigma(A_2).$$

Hence,

$$\begin{aligned} gap(A) &= \| A_1\| \; \| A_2\| - r(A_1)r(A_2) \\ &= (\| A_1\| - r(A_1))(\| A_2\| - r(A_2)) + r(A_1)\| A_2\| \\ &\quad + r(A_2)\| A_1\| - 2r(A_1)r(A_2) \\ &= (\| A_1\| - r(A_1))(\| A_2\| - r(A_2)) \\ &\quad + r(A_1)(\| A_2\| - r(A_2)) + r(A_2)(\| A_1\| - r(A_1)) \\ &= gap(A_1)gap(A_2) + r(A_1)gap(A_2) + r(A_2)gap(A_1). \end{aligned}$$

Corollary 1 *Under the assumptions of the last theorem it is true*

1. $gap(A) \ge gap(A_1)gap(A_2)$,
2.

$$\begin{aligned} &\min(r(A_1), r(A_2))\min(gap(A_1), gap(A_2)) \\ &\le \frac{1}{2}(gap(A) - gap(A_1)gap(A_2)) \\ &\le \max(r(A_1), r(A_2))\max(gap(A_1), gap(A_2)). \end{aligned}$$

Example 1 *Let* $H_1 = H_2 = \mathbb{C}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $A_2 = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. *In this case,* $\| A_1\| = \sqrt{2}$, $\sigma(A_1) = \{0,1\}$, $r(A_1) = 1$ *and* $\| A_2\| = 2\sqrt{2}$, $\sigma(A_2) = \{0,2\}$, $r(A_2) = 2$. *Then,* $gap(A_1) = \sqrt{2} - 1$ *and* $gap(A_2) = 2(\sqrt{2} - 1)$. *Consequently, by Theorem 1 it is obtain that*

$$\begin{aligned} &gap(A_1 \otimes A_2) \\ &= (\sqrt{2} - 1)(2\sqrt{2} - 2) + (2\sqrt{2} - 2) + 2(\sqrt{2} - 1) \\ &= 2. \end{aligned}$$

Corollary 2 *By using the last theorem for the tensor product*

$$A = A_1 \otimes A_2 \otimes A_3 : H \to H,\, H = H_1 \otimes H_2 \otimes H_3$$

of the operators $A_1 \in L(H_1), A_2 \in L(H_2), A_3 \in L(H_3)$ *is true*

$$\begin{aligned} &gap(A) = gap(A_1)gap(A_2)gap(A_3) \\ &+ r(A_1)gap(A_2)gap(A_3) + r(A_2)gap(A_1)gap(A_3) \\ &+ r(A_3)gap(A_1)gap(A_2) + r(A_1)r(A_2)gap(A_3) \\ &+ r(A_1)r(A_3)gap(A_2) + r(A_2)r(A_3)gap(A_1). \end{aligned}$$

Generally, using the Theorem 1 it can be proved the following proposition.

Corollary 3 *For the tensor product*

$$A = A_1 \otimes A_2 \otimes \ldots A_n : H \to H,\, H = \bigoplus_{m=1}^{n} H_m$$

of the operators $A_m \in L(H_m), 1 \le m \le n$ *it is true*

$$\begin{aligned} gap(A) &= \prod_{m=1}^{n} gap(A_m) + \sum_{m=1}^{n} r(A_m) \prod_{\substack{k=1 \\ k \ne m}}^{n} gap(A_k) \\ &+ \sum_{\substack{m,k=1 \\ m \ne k}}^{n} r(A_m)r(A_k) \prod_{\substack{i=1 \\ i \ne m, i \ne k}}^{n} gap(A_i) + \ldots \\ &+ \sum \Big\{ r(A_{m_1})r(A_{m_2})\ldots r(A_{m_{n-1}})gap(A_{m_n}) : \\ &\quad 1 \le m_1, m_2, \ldots, m_n \le n; \\ &\quad m_j \ne \{m_1, m_2, \ldots, m_{j-1}, m_{j+1}, \ldots m_n\}, 1 \le j \le n \Big\}. \end{aligned}$$

Corollary 4 *From the Corollary 3 we have*

(1) $gap(A) \ge \prod_{m=1}^{n} gap(A_m)$,

(2) $gap(A) = 0$ *if and only if*

$$\prod_{m=1}^{n} gap(A_m) = 0,$$

$$r(A_m) \prod_{\substack{k=1 \\ k \ne m}}^{n} gap(A_k) = 0, 1 \le m \le n,$$

$$r(A_m)r(A_k) \prod_{\substack{i=1 \\ i \ne m, i \ne k}}^{n} gap(A_i) = 0, 1 \le m, k \le n,$$

and etc.,

(3) *in special case when* $A_1 = A_1^*$ *and* $gap(A) = 0$, *then*

$$r(A_1) \prod_{k=2}^{n} gap(A_k) = 0,$$

(4) *in special case when*

$$gap(A_m) = 0, 1 \le m \le n - 1,$$

then in order to $gap(A)=0$ *the necessary and sufficient condition is*

$$r(A_1)\dots r(A_{n-1})gap(A_n)=0.$$

3. Gap between Operator Norm and Numerical Radius of the Tensor Product of Operators

In this section, the gap between operator norm and numerical radius of the tensor product of operators associated with the gap of coordinate operators will be investigated.

Definition 1 [3] *The numerical range and the numerical radius of the linear bounded operator* T *in any Hilbert space* $\mathcal{H}$ *define by*

$$W(T)=\{(Tx,x)_{\mathcal{H}}:\|x\|_{\mathcal{H}}=1\}$$

and

$$w(T)=\sup\{|\lambda|:\lambda\in W(T)\},$$

respectively.

It is easy to prove the following proposition.

Theorem 2 *For the numerical range and numerical radius of tensor product operator* $A=\bigotimes_{m=1}^{n}A_m$ *of the operators* $A_m\in L(H_m), 1\le m\le n$ *in tensor product* H *of Hilbert spaces* $H_m, 1\le m\le n$ *are true.*

$$W(A)=co\left(W(A_1)W(A_2)\dots W(A_n)\right)$$

and

$$w(A)=w(A_1)w(A_2)\dots w(A_n),$$

where $co(\Omega), \Omega\subset\mathbb{C}$ *denotes the convex hull of the set* Ω.

Note that the analogous types result of Theorem 1 are true in this case too. For example, it is true the following result.

Theorem 3 *For the tensor product* $A=\bigotimes_{m=1}^{n}A_m$ *of the operators* $A_m\in L(H_m), 1\le m\le n$ *in the tensor product* $H=\bigotimes_{m=1}^{n}H_m$ *of Hilbert spaces* $H_m, 1\le m\le n$ *the numerical gap is of the form*

$$\begin{aligned}
& wgap(A)=\|A\|-w(A)\\
&=\|A_1\|\ \|A_2\|\dots\|A_n\|-w(A_1)w(A_2)\dots w(A_n)\\
&=\prod_{m=1}^{n}wgap(A_m)+\sum_{m=1}^{n}w(A_m)\prod_{\substack{k=1\\k\ne m}}^{n}wgap(A_k)\\
&+\sum_{\substack{m,k=1\\m\ne k}}^{n}w(A_m)w(A_k)\prod_{\substack{i=1\\i\ne m,i\ne k}}^{n}wgap(A_i)+\dots\\
&+\sum\Big\{w(A_{m_1})w(A_{m_2})\dots w(A_{m_{n-1}})wgap(A_{m_n}):\\
&\quad 1\le m_1,m_2,\dots,m_n\le n;\\
&\quad m_j\ne\{m_1,m_2,\dots,m_{j-1},m_{j+1},\dots m_n\}\\
&\quad 1\le j\le n\Big\}.
\end{aligned}$$

Example 2 *Let* $H_1=\mathbb{C}^2, H_2=L^2((0,1),\mathbb{R})$ *and* $A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A_2=V$ *Volterra integration operator in* $L^2((0,1),\mathbb{R})$. *In this case, we get*

$$\|A_1\|=1, w(A_1)=1, wgap(A_1)=0$$

and from [5] *and* [10]

$$\|A_2\|=\frac{2}{\pi}, w(A_2)=\frac{1}{2}, wgap(A_2)=\frac{2}{\pi}-\frac{1}{2}=\frac{4-\pi}{2\pi}.$$

Consequently, by Theorem 3 we have

$$wgap(A_1\otimes A_2)=0.\frac{4-\pi}{2\pi}+1.\frac{4-\pi}{2\pi}+\frac{1}{2}.0=\frac{4-\pi}{2\pi}.$$

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