Turkish Journal of Analysis and Number Theory, 2020, Vol. 8, No. 6, 97-106
Available online at http://pubs.sciepub.com/tjant/8/6/1
Published by Science and Education Publishing
DOI:10.12691/tjant-8-6-1

Certain Generating Functions Involving Some Hypergeometric Series of Four Variables by Means of Operational Representations

Jihad A. Younis[1], Maged G. Bin-Saad[1], Kottakkaran S. Nisar[2,*]

[1]Department of Mathematics, Aden University, Aden, Yemen
[2]Department of Mathematics, College of Arts and Sciences, Wadi Aldawaser, Prince Sattam bin Abdulaziz University, Saudi Arabia
*Corresponding author: n.sooppy@psau.edu.sa

Received October 06, 2020; Revised November 07, 2020; Accepted November 15, 2020

Abstract The main aim of this present paper is to present certain generating functions of some hypergeometric functions in four variables by using the integral and symbolic representations for these quadruple functions. A few interesting special cases have also been considered.

Keywords: *Laplace integrals, symbolic representations, quadruple hypergeometric functions, generating functions*

Cite This Article: Jihad A. Younis, Maged G. Bin-Saad, and Kottakkaran S. Nisar, "Certain Generating Functions Involving Some Hypergeometric Series of Four Variables by Means of Operational Representations." *Turkish Journal of Analysis and Number Theory*, vol. 8, no. 6 (2020): 97-106. doi: 10.12691/tjant-8-6-1.

1. Introduction

Several families of generating functions have been established in diverse ways. These are playing important roles in the theory of special functions of applied mathematics and mathematical physics. One can refer the extensive work of Srivastava and Manocha [1] for a systematic introduction to, and several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of hypergeometric functions and polynomials in one, two or more variables, among much abundant literature. In fact, a remarkable large number of generating functions involving a variety of hypergeometric functions have been developed by many authors such as [2,3,4,5,6].

Bin-Saad and Younis introduced in [7] thirty new quadruple hypergeometric series, ten of them defined below

$$X_1^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p+q}(a_2)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1}$$

$$X_2^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2}$$

$$X_4^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{3}$$

$$X_{13}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{4}$$

$$X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{5}$$

$$X_{18}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{6}$$

$$X_{19}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p+q}(a_2)_{q+n}(a_3)_p}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{7}$$

$$X_{20}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{8}$$

$$X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{n+p}(a_3)_{p+q}}{(c_1)_{m+p}(c_2)_n(a_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{9}$$

$$X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_n(a_3)_p(a_4)_{p+q}}{(c_1)_{m+n+p}(c_2)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{10}$$

Here $(a)_m$ is the Pochhammer symbol which defined as

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}, a, m \in C$$

with the assumed $(a)_0 = 1$.

It may be recalled the Laplace integral representations of the above functions, see e.g. [8,9,10,11] respectively, as below

$$\begin{aligned} &X_{1}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_2,c_1,c_1,c_3;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2y+stz\right) \\ &\times {}_0F_1\left(-;c_2;s^2x\right) {}_0F_1\left(-;c_3;stu\right) ds\,dt, \\ &\left(R(a_1)>0, R(a_2)>0\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &X_{2}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x\right) {}_0F_1\left(-;c_2;sty\right) \\ &\times {}_0F_1\left(-;c_3;t^2z\right) {}_0F_1\left(-;c_4;stu\right) ds\,dt, \\ &\left(R(a_1)>0, R(a_2)>0\right), \end{aligned} \tag{12}$$

$$\begin{aligned} &X_{4}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_1,c_3;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &{}_0F_1\left(-;c_1;s^2x+t^2z\right) \\ &\times {}_0F_1\left(-;c_2;sty\right) {}_0F_1\left(-;c_3;stu\right) ds\,dt, \\ &\left(R(a_1)>0, R(a_2)>0\right), \end{aligned} \tag{13}$$

$$\begin{aligned} &X_{13}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x+sty\right) {}_1F_1\left(a_3;c_2;sz+tu\right) ds\,dt, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0\right), \end{aligned} \tag{14}$$

$$\begin{aligned} &X_{14}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u) \\ &= \frac{1}{\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_2-1} t^{a_3-1} \\ &\times \mathrm{H}_6\left(a_1;c_1;x,sy+tz\right) {}_0F_1\left(-;c_2;stu\right) dsdt, \\ &\left(\mathrm{Re}(a_2)>0, \mathrm{Re}(a_3)>0\right), \end{aligned} \tag{15}$$

$$\begin{aligned} &X_{18}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \\ &\times {}_0F_1\left(-;c_1;s^2x+sty\right) \Phi_3\left(a_3;c_2;sz,stu\right) dsdt, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0\right), \end{aligned} \tag{16}$$

$$\begin{aligned} &X_{19}^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} \\ &\times t^{a_2-1} v^{a_3-1} {}_0F_1\left(-;c_1;s^2x+sty+svz\right) \\ &\times {}_0F_1\left(-;c_2;stu\right) ds\,dt\,dv, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0, \mathrm{Re}(a_3)>0\right), \end{aligned} \tag{17}$$

$$\begin{aligned} &X_{20}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_3)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_3-1} \\ &\times {}_0F_1\left(-;c_1;s^2x\right) \\ &\times \Psi_2^{(3)}\left(a_2;c_2,c_3,c_4;sy,tz,su\right) dsdt, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_3)>0\right), \end{aligned} \tag{18}$$

$$\begin{aligned} &X_{24}^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} \\ &\times t^{a_2-1} v^{a_3-1} {}_0F_1\left(-;c_1;s^2x+tvz\right) \\ &\times {}_0F_1\left(-;c_2;sty\right) {}_0F_1\left(-;c_3;svu\right) ds\,dt\,dv, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_2)>0, \mathrm{Re}(a_3)>0\right), \end{aligned} \tag{19}$$

$$\begin{aligned} &X_{26}^{(4)}(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_4)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_4-1} \\ &\times \Phi_3^{(3)}\left(a_2,a_3;c_1;sy,tz,s^2x\right) \\ &\times {}_0F_1\left(-;c_2;stu\right) dsdt, \\ &\left(\mathrm{Re}(a_1)>0, \mathrm{Re}(a_4)>0\right), \end{aligned} \tag{20}$$

where ${}_0F_1$, ${}_1F_1$, Φ_3, H_6, $\Psi_2^{(3)}$ and $\Phi_3^{(3)}$ denote the confluent hypergeometric functions [12].

The aim of this paper is to investigate the various generating functions for the quadruple hypergeometric functions $X_j^{(4)}(j=1,2,4,10,13,18,19,20,24,26)$. In Section 2, our main generating functions are obtained with the help of integral representations of Laplace-type. In Section 3, we introduce symbolic representations for the functions. The aim of Section 4 is to use symbolic representations to derive a number of generating functions.

2. Generating Functions via Laplace Integral Representations

Here, in this Section, we used the Laplace integral representations for the hypergeometric series of four variables defined in pervious Section to determine generating functions which are listed in the form of following relations (21) to (34):

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_1^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_2+k;c_2,c_1,c_1,c_3;x^2,y,z,u)$$
$$=(1+2x)^{-a_1}\sum_{k,m=0}^{\infty}\frac{(a_1+k)_m\left(c_2-\frac{1}{2}\right)_m}{(2c_2-1)_m\,k!\,m!}\left(\frac{w}{1+2x}\right)^k\left(\frac{4x}{1+2x}\right)^m \tag{21}$$
$$\times X_3\left(a_1+k+m,a_2+k;c_1,c_3;\frac{y}{(1+2x)^2},\frac{z}{1+2x},\frac{u}{1+2x}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_1^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_2+k;c_2,c_1,c_1,c_3;x,y,z,u\tau)$$
$$=(1+u)^{-a_1}(1+\tau)^{-a_2}\sum_{k,q,r=0}^{\infty}\frac{(a_1+k)_q(a_2+k)_r(c_3)_{q+r}}{(c_3)_q(c_3)_r\,k!\,q!\,r!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k\left(\frac{u}{1+u}\right)^q\left(\frac{\tau}{1+\tau}\right)^r \tag{22}$$
$$\times X_1\left(a_1+k+q,\,a_2+k+r;c_2,\,c_1;\frac{x}{(1+u)^2},\frac{y}{(1+u)^2},\frac{z}{(1+u)(1+\tau)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_2^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_2+k,a_2+k;c_1,c_2,c_3,c_4;x^2,y,z,u)$$
$$=(1+2x)^{-a_1}\sum_{k,m=0}^{\infty}\frac{(a_1+k)_m\left(c_1-\frac{1}{2}\right)_m}{(2c_1-1)_m\,k!\,m!}\left(\frac{w}{1+2x}\right)^k\left(\frac{4x}{1+2x}\right)^m \tag{23}$$
$$\times X_4\left(a_2+k,a_1+k+m;c_3,c_2,c_4;z,\frac{y}{1+2x},\frac{u}{1+2x}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_2^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_2+k,a_2+k;c_1,c_2,c_3,c_4;x^2,y,z^2,u)$$
$$=(1+2x)^{-a_1}(1+2z)^{-a_2}\sum_{k,p=0}^{\infty}\frac{(a_2+k)_p\left(c_3-\frac{1}{2}\right)_p}{(2c_3-1)_p\,k!\,p!}\left(\frac{w}{(1+2x)(1+2z)}\right)^k\left(\frac{4z}{1+2z}\right)^p \tag{24}$$
$$\times F_E\left(a_1+k,a_1+k,a_1+k,c_1-\tfrac{1}{2},a_2+k+p,a_2+k+p;2c_1-1,c_2,c_4;\lambda_1x,\lambda_2y,\lambda_2u\right),$$
$$\left(\lambda_1=\frac{4}{1+2x},\lambda_2=\frac{1}{(1+2x)(1+2z)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_2^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_2+k,a_2+k;c_1,c_2,c_3,c_4;x^2,y,z^2,u)$$
$$=(1+2x)^{-a_1}(1+2z)^{-a_2}\sum_{k,q=0}^{\infty}\frac{(a_1+k)_q(a_2+k)_q}{(c_4)_q\,k!\,q!}\left(\frac{w}{(1+2x)(1+2z)}\right)^k\left(\frac{u}{(1+2x)(1+2z)}\right)^q \tag{25}$$
$$\times F_K\left(c_1-\tfrac{1}{2},a_2+k+q,a_2+k+q,a_1+k+q,c_3-\tfrac{1}{2},a_1+k+q;2c_1-1,2c_3-1,c_2;\lambda_1x,\lambda_2z,\lambda_3y\right),$$
$$\left(\lambda_1=\frac{4}{1+2x},\lambda_2=\frac{4}{1+2z},\lambda_3=\frac{1}{(1+2x)(1+2z)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_4^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_2+k,a_2+k\,;c_1,c_2,c_1,c_3;x,y\tau,z,u)$$

$$=(1+y)^{-a_1}(1+\tau)^{-a_2}\sum_{k,n,r=0}^{\infty}\frac{(a_1+k)_n\,(a_2+k)_r\,(c_2)_{n+r}}{(c_2)_n\,(c_2)_r\,k!\,n!\,r!}\left(\frac{w}{(1+y)(1+\tau)}\right)^k\left(\frac{y}{1+y}\right)^n\left(\frac{\tau}{1+\tau}\right)^r \quad (26)$$

$$\times X_{11}\left(a_1+k+n,\,a_2+k+r\,;c_1,\,c_3;\frac{x}{(1+y)^2},\frac{u}{(1+y)(1+\tau)},\frac{z}{(1+\tau)^2}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_4^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_2+k,a_2+k\,;c_1,c_2,c_1,c_3;x,y\tau,z,u)$$

$$=(1+y)^{-a_1}(1+\tau)^{-a_2}\sum_{k,m,p=0}^{\infty}\frac{(a_1+k)_{2m}\,(a_2+k)_{2p}}{(c_1)_{m+p}\,k!\,m!\,p!}\left(\frac{w}{(1+y)(1+\tau)}\right)^k\left(\frac{x}{(1+y)^2}\right)^m\left(\frac{z}{(1+\tau)^2}\right)^p \quad (27)$$

$$\times H_B\left(c_2,\,a_1+k+2m,\,a_2+k+2p\,;c_2,\,c_3,c_2;\frac{y}{1+y},\frac{u}{(1+y)(1+\tau)},\frac{\tau}{1+\tau}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{13}^{(4)}(a_1+k,a_1+k,a_1+k,a_2+k,a_1+k,a_2+k,c_2,c_2;\,c_1,c_1,c_1,c_2,c_2;x,y,z,u)$$

$$=(1-z)^{-a_1}(1-u)^{-a_2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1-z)(1-u)}\right)^k \quad (28)$$

$$\times H_3\left(a_1+k,\,a_2+k\,;c_1;\frac{x}{(1-z)^2},\frac{y}{(1-z)(1-u)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{14}^{(4)}(a_1,a_1,a_1,a_2+k,a_1,a_2+k,a_3+k,a_3+k\,;c_1,c_1,c_1,c_2;x,y,z,\tau\,u)$$

$$=(1+\tau)^{-a_2}(1+u)^{-a_3}\sum_{k,q,r=0}^{\infty}\frac{(a_3+k)_q\,(a_2+k)_r\,(c_2)_{q+r}}{(c_2)_q\,(c_2)_r\,k!\,q!\,r!}\left(\frac{w}{(1+\tau)(1+u)}\right)^k\left(\frac{u}{1+u}\right)^q\left(\frac{\tau}{1+\tau}\right)^r \quad (29)$$

$$\times X_5\left(a_1,a_2+k+r,\,a_3+k+q;\,c_1;x,\frac{y}{1+\tau},\frac{z}{1+u}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{18}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3,a_2+k\,;c_1,c_1,c_2,c_2;x,y,z,u)$$

$$=(1-z)^{-a_1}\left(\frac{z-u}{z}\right)^{-a_2}\sum_{k,p,q=0}^{\infty}\frac{(a_1+k)_p\,(1-a_3-q)_p\,(-q)_p\,(a_2+k)_{q-p}}{(c_2)_p\,k!\,p!\,q!} \quad (30)$$

$$\times\left(\frac{w\,z}{(1-z)(z-u)}\right)^k\left(\frac{z(z-u)}{u(1-z)}\right)^p\left(\frac{u}{u-z}\right)^q H_3\left(a_1+k+p,\,a_2+k+q-p\,;c_1\,;\frac{x}{(1-z)^2},\frac{y\,z}{(1-z)(z-u)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{19}^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2+k,a_3+k,a_2+k\,;\,c_1,c_1,c_1,c_2;x,y,z,\tau\,u)$$

$$=(1+\tau)^{-a_1}(1+u)^{-a_2}\sum_{k,m,q=0}^{\infty}\frac{(a_1+k)_{2m}\,(a_2+k)_q\,(c_2)_q}{(c_1)_m\,(c_2)_q\,k!\,m!\,q!}\left(\frac{w}{(1+\tau)(1+u)}\right)^k\left(\frac{x}{(1+\tau)^2}\right)^m\left(\frac{u}{1+u}\right)^q \quad (31)$$

$$\times F_G\left(a_1+k+2m,a_1+k+2m,a_1+k+2m,c_2+q,a_2+k+q,a_3+k\,;c_2,c_1+m,c_1+m;\lambda_1\tau,\lambda_2 y,\lambda_1 z\right),$$

$$\left(\lambda_1=\frac{1}{1+\tau},\,\lambda_2=\frac{1}{(1+\tau)(1+u)}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{20}^{(4)}(a_1+k,a_1+k,a_2,a_1+k,a_1+k,a_2,a_3+k,a_2;c_1,c_2,c_3,c_4;x^2,y,z,u)$$
$$=(1+2x)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2x}\right)^k \tag{32}$$
$$\times F_4^{(4)}\left(a_1+k,\ a_1+k,\ a_1+k,\ a_3+k,a_2,a_2,c_1-\tfrac{1}{2},a_2;\ c_2,c_4,\ 2c_1-1,\ c_3;\frac{y}{1+2x},\frac{u}{1+2x},\frac{4x}{1+2x},z\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{24}^{(4)}(a_1+k,a_1+k,a_2+k,a_1+k,a_1+k,a_2+k,a_3+k,a_3+k;c_1,c_2,c_1,c_3;x,y,z,u\tau)$$
$$=(1+u)^{-a_1}(1+\tau)^{-a_3}\sum_{k,q,r=0}^{\infty}\frac{(a_1+k)_q\,(a_3+k)_r\,(c_3)_{q+r}}{(c_3)_q\,(c_3)_r\,k!\,q!\,r!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k\left(\frac{u}{1+u}\right)^q\left(\frac{\tau}{1+\tau}\right)^r \tag{33}$$
$$\times X_{16}\left(a_1+k+q,\ a_2+k,a_3+k+r;\ c_1,c_2;\frac{x}{(1+u)^2},\frac{y}{1+u},\frac{z}{1+\tau}\right),$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_{26}^{(4)}(a_1+k,a_1+k,a_3,a_1+k,a_1+k,a_2,a_4+k,a_4+k;c_1,c_1,c_1,c_2;x,y,z,u\tau)$$
$$=(1+u)^{-a_1}(1+\tau)^{-a_4}\sum_{k,q,r=0}^{\infty}\frac{(a_1+k)_q\,(a_4+k)_r\,(c_2)_{q+r}}{(c_2)_q\,(c_2)_r\,k!\,q!\,r!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k\left(\frac{u}{1+u}\right)^q\left(\frac{\tau}{1+\tau}\right)^r \tag{34}$$
$$\times X_{18}\left(a_1+k+q,\ a_2,a_3,a_4+k+r;\ c_1;\frac{x}{(1+u)^2},\frac{y}{1+u},\frac{z}{1+\tau}\right),$$

where H_3 is the Horn function defined in [12], F_E, F_G and F_K are the Lauricella functions defined in [13], H_B is the Srivastava function defined in [12]. $X_1, X_3, X_4, X_5, X_{11}, X_{16}$ and X_{18} are the Exton functions defined in [14] and $F_4^{(4)}$ is the Sharma and Parihar function defined in [15].

Proof. To prove (21), for convenience and simplicity, by denoting the left-hand side of (21) with δ and using (11), we have

$$\delta=\sum_{k=0}^{\infty}\frac{w^k}{k!\,\Gamma(a_1+k)\,\Gamma(a_2+k)}\int_0^{\infty}\int_0^{\infty}e^{-(s+t)}\,s^{a_1+k-1}\,t^{a_2+k-1}$$
$$\times{}_0F_1\left(-;\,c_1;\,s^2y+stz\right)\,{}_0F_1\left(-;\,c_2;\,s^2x^2\right)\,{}_0F_1\left(-;\,c_3;\,stu\right)ds\,dt.$$

Using the Kummer's transformation (see [16])

$${}_0F_1\left(-;a;x^2\right)=e^{-2x}\,{}_1F_1\left(a-\tfrac{1}{2};\,2a-1;\,4x\right)$$

after a little simplification, we get

$$\delta=\sum_{k,m,n,p,q=0}^{\infty}\frac{\left(c_2-\frac{1}{2}\right)_m w^k\,(4x)^m\,y^n\,z^p u^q}{\left[\dfrac{(c_1)_{n+p}\,(2c_2-1)_m\,(c_3)_q\,k!\,m!\,n!\,q!}{\Gamma(a_1+k)\,\Gamma(a_2+k)}\right]}$$
$$\int_0^{\infty}\int_0^{\infty}e^{-s(1+2x)}\,e^{-t}\,s^{a_1+k+2m+2n+p+q-1}\,t^{a_2+k+p+q-1}\,ds\,dt\,.$$

Using the following well-known formula (see [16,17]) $\Gamma(z)=s^z\int_0^{\infty}e^{-st}\,t^{z-1}\,dt$, $\operatorname{Re}(z)>0$, after a little simplification, we easily arrive at the right-hand side of (21). This completes the proof of (21). The proof of the relations (22) to (34) runs in same way.

If we take $k=0$ in (32) , we shall obtain the following relation:

$$X_{20}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x^2,y,z,u\right)=(1+2x)^{-a_1}$$
$$\times F_4^{(4)}\left(a_1,\,a_1,\,a_1,\,a_3,a_2,a_2,c_1-\tfrac{1}{2},a_2;\,c_2,c_4,\,2c_1-1,\,c_3;\,\lambda\,y,\lambda\,u,4\lambda\,x,z\right),$$
$$\left(\lambda=\frac{1}{1+2x}\right),$$

which, for $u=0$, yields the well-known result (see [12]).

By setting $x=0$ in (22) and (33), we have the following generating relations:

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_3(a_1+k,a_2+k;\,c_1,c_3;\,y,z,u\tau)$$
$$=(1+u)^{-a_1}(1+\tau)^{-a_2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k$$
$$\times X_{11}^{(4)}\left(\begin{array}{l}a_1+k,a_1+k,a_1+k,\ a_2+k,a_1+k,\\ a_2+k,c_3,c_3;c_1,\ c_1,c_3,c_3;\lambda_1^2y,\lambda_1\lambda_2z,\lambda_1u,\lambda_2\tau\end{array}\right),$$
$$\left(\lambda_1=\frac{1}{1+u},\ \lambda_2=\frac{1}{1+\tau}\right),$$

where $X_{11}^{(4)}$ is quadruple hypergeometric series defined by Bin-Saad and Younis (see [9] and [7]).

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}H_B\left(a_1+k,a_2+k,a_3+k;c_2,c_1,c_3;y,z,u\tau\right)=\left(1+u\right)^{-a_1}\left(1+\tau\right)^{-a_3}\times\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{(1+u)(1+\tau)}\right)^k$$
$$\times F_7^{(4)}\left(a_1+k,a_1+k,a_3+k,a_3+k,a_2+k,c_3,a_2+k,c_3;c_2,c_3,c_1,c_3;\lambda_1 y,\lambda_1 u,\lambda_2 z,\lambda_2\tau,\right),$$
$$\left(\lambda_1=\frac{1}{1+u},\lambda_2=\frac{1}{1+\tau}\right),$$

which, for $k=0$, we obtain new connection between Srivastava function H_B and Sharma and Parihar function $F_7^{(4)}$.

3. Symbolic Representations

In this section, we consider the symbolic images for the quadruple functions (1) - (10). Our results presented here are obtained with the help of the following operator formulas:

$$D_x^n x^\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}x^{\alpha-n}, \tag{35}$$

$$D_x^{-n} x^\alpha=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}x^{\alpha+n}, \tag{36}$$

$$n\in N\cup\{0\},\ \alpha\in C-\{-1,-2,...\},$$

where the operators D_x and D_x^{-1} are the derivative operator and the inverse of the derivative, respectively, (see [18]). Now, we introduce certain operational representations involving quadruple functions as below:

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)\\&\times\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)^{-a_2}\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\ X_1^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right),\end{aligned}\tag{37}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1\alpha_2\right]\right)\\&\times\left(1-z\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\left[D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\alpha_2\right]\right)^{-b}\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\}\\&=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\ X_2^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right),\end{aligned}\tag{38}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]\right)^{-b}\\&\times\exp\left(y\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]+z\left[D_{\alpha_2}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2^2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_2\right]\right)\\&\times\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{b-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\ X_4^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_1,c_3;x,y,z,u\right),\end{aligned}\tag{39}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+z\left[D_{\alpha_1}D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_3\right]\right)\\&\times\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)^{-a_2}\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\ X_{13}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right),\end{aligned}\tag{40}$$

$$\begin{aligned}&\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\alpha_2\right]\right)\\&\times\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\right)^{-a_3}\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}\\&=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_1,c_1,c_2;x,y,z,u\right),\end{aligned}\tag{41}$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)$$
$$\times\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_3}\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \quad (42)$$
$$=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\ X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_2,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)$$
$$\times\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_3}\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \quad (43)$$
$$=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\ X_{19}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_2;c_1,c_1,c_1,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-z\left[D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_3\right]-u\left[D_{\alpha_1}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1\right]\right)^{-a_2}$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\} \quad (44)$$
$$=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\ X_{20}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+u\left[D_{\alpha_1}D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_3\right]\right)$$
$$\times\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-z\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]\right)^{-a_2}\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\} \quad (45)$$
$$=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\ X_{24}^{(4)}\left(a_1,a_1,a_2,a_1,a_1,a_2,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)\times\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)^{-a_2}\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]\right)^{-a_3}$$
$$\times\exp\left(u\left[D_{\alpha_1}D_{\alpha_4}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_4\right]\right)\times\left\{\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \quad (46)$$
$$=\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}X_{26}^{(4)}\left(a_1,a_1,a_3,a_1,a_1,a_2,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u\right),$$

where e^x is the exponential expansion defined by (see [16])

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}. \quad (47)$$

Proof. To prove formula (37), denote, for convenience, the left-hand side of assertion (37) by δ. Then by the exponential expansion (47) and binomial theorem, we have

$$\delta=\sum_{m,n,p,q=0}^{\infty}\frac{(a_2)_{p+q}\,x^m y^n z^p u^q \beta_1^{-(n+p)}\beta_2^{-m}\beta_3^{-q}}{m!n!p!q!}$$
$$\times D_{\alpha_1}^{2m+2n+p+q}D_{\beta_1}^{-(n+p)}D_{\beta_2}^{-m}D_{\beta_3}^{-q}$$
$$\times\left\{\alpha_1^{a_1+2m+2n+p+q}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}.$$

Using (35) and (36) and in view of the definition (1), we obtain the desired result (37). The proofs of formulas (38) to (46) run parallel to that of formula (37), so are skipped details.

If we take $x=0$ in (40) and (41), we get operational representations involving Srivastava's function H_A (see [12]). Relation (44) with $x=y=0$, yield the Bin-Saad's and Maisoon's results [19].

4. Generating Functions via Symbolic Representations

In this section, we establish some generating relations by using the symbolic images obtained in previous section. Following are these generating relations:

$$\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)\right)$$
$$\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k=0}^{\infty}\frac{t^k}{k}X_1^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,-k,-k;c_2,c_1,c_1,c_3;x,y,z,u\right), \quad (48)$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1\alpha_2\right]\right)$$
$$\times\exp\left(t\left(1-z\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\left[D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\alpha_2\right]\right)\frac{\alpha_1}{\beta}\right)\times\left\{\alpha_1^{-1}\alpha_2^{a_2-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\} \tag{49}$$
$$=\alpha_1^{-1}\alpha_2^{a_2-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\sum_{k=0}^{\infty}\frac{\alpha_1^k t^k}{\beta^k k!}X_2^{(4)}\left(k,k,a_2,k,k,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right),$$

$$\exp\left(y\left[D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_2\right]+z\left[D_{\alpha_2}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2^2\right]+u\left[D_{\alpha_1}D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_2\right]\right)$$
$$\times\exp\left(t\left(1-x\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\left[D_{\alpha_1}\beta^{-1}D_{\beta}^{-1}\alpha_1\right]\right)\frac{\alpha_1}{\beta}\right)\times\left\{\alpha_1^{-1}\alpha_2^{a_2-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\} \tag{50}$$
$$=\alpha_1^{-1}\alpha_2^{a_2-1}\beta^{-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k=0}^{\infty}\frac{\alpha_1^k t^k}{\beta^k k!}X_4^{(4)}\left(k,k,a_2,k,k,a_2,a_2,a_2;c_1,c_2,c_1,c_3;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+z\left[D_{\alpha_1}D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_3\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \tag{51}$$
$$=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{13}^{(4)}\left(a_1,a_1,a_1,-k,a_1,-k,a_3,a_3;c_1,c_1,c_2,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\alpha_2\right]+t\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_2\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \tag{52}$$
$$=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{14}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_2,-k,-k;c_1,c_1,c_1,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)\left(1-z\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \tag{53}$$
$$=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{18}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,-k,-k,-k;c_1,c_1,c_2,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \tag{54}$$
$$=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{19}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,-k,-k,-k;c_1,c_1,c_1,c_2;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+t\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-z\left[D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_3\right]-u\left[D_{\alpha_1}\beta_4^{-1}D_{\beta_4}^{-1}\alpha_1\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\right\} \tag{55}$$
$$=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\beta_4^{c_4-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{20}^{(4)}\left(a_1,a_1,-k,a_1,a_1,-k,a_3,-k;c_1,c_2,c_3,c_4;x,y,z,u\right).$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+u\left[D_{\alpha_1}D_{\alpha_3}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\alpha_3\right]+t\left(1-y\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]-z\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\} \tag{56}$$
$$=\alpha_1^{a_1-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{24}^{(4)}\left(a_1,a_1,-k,a_1,a_1,-k,a_3,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right),$$

$$\exp\left(x\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]+u\left[D_{\alpha_1}D_{\alpha_4}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_4\right]+t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\} \quad (57)$$
$$=\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}X_{26}^{(4)}\left(a_1,a_1,-k,a_1,a_1,-k,a_4,a_4;c_1,c_1,c_1,c_2;x,y,z,u\right).$$

Proof. To prove relation (48), taking $a_2=-k$ in (37) and multiplying both the sides with $\frac{t^k}{k!}$, we obtain

$$\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)$$
$$\times\frac{\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)^{-k}}{k!}$$
$$\times\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}$$
$$\times\frac{t^k}{k!}X_1^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,-k,-k;c_2,c_1,c_1,c_3;x,y,z,u\right),$$

and then taking the double sum of both sides we get

$$\exp\left(x\left[D_{\alpha_1}^2\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1^2\right]+y\left[D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1^2\right]\right)$$
$$\times\left(\sum_{k=o}^{\infty}\frac{\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]-u\left[D_{\alpha_1}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_1\right]\right)^{-k}}{k!}\right)$$
$$\times\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}$$
$$\times\sum_{k=0}^{\infty}\frac{t^k}{k!}X_1^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,-k,-k;c_2,c_1,c_1,c_3;x,y,z,u\right).$$

Now, by using the definition (47), then, after some simplification gives the result. In a similar manner, one can prove the relations (49) to (57).

If we choose $x=0$ in (54) and (57), and simplifying, we have the following new generating functions:

$$\exp\left(t\begin{pmatrix}1-u\left[D_{\alpha_1}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\\-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\end{pmatrix}\left(1-z\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}=\alpha_1^{a_1-1}\beta_1^{c_1-1}\beta_2^{c_2-1}$$
$$\times\sum_{k=0}^{\infty}\frac{t^k}{k!}F_F\left(a_1,a_1,a_1,-k,-k,-k;c_2,c_1,c_1;u,z,y\right),$$

$$\exp\begin{pmatrix}u\left[D_{\alpha_1}D_{\alpha_4}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\alpha_4\right]\\+t\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]\right)\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)\end{pmatrix}$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\right\}=\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\beta_2^{c_2-1}$$
$$\times\sum_{k=0}^{\infty}\frac{t^k}{k!}F_P\left(a_1,-k,a_1,a_4,a_4,-k;c_2,c_1,c_1;u,z,y\right),$$

which, for $u=0$, we have generating functions involving Appell's series [12] F_3

$$\exp\left(t\left(1-y\left[D_{\alpha_1}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_1\right]\right)\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]\right)\right)$$
$$\times\left\{\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\right\}$$
$$=\alpha_1^{a_1-1}\alpha_4^{a_4-1}\beta_1^{c_1-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}F_3\left(a_1,-k,-k,a_4;c_1;y,z\right),$$

which, for $z=0$, after a little simplification, we have interesting result

$$\exp\left(t\left(1-y\left[D_\alpha\beta^{-1}D_\beta^{-1}\alpha\right]\right)\right)\times\left\{\alpha^{a-1}\beta^{c-1}\right\}$$
$$=\alpha^{a-1}\beta^{c-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}{}_2F_1\left(a,-k;c;y\right),$$

where ${}_2F_1$ is the Gaussian hypergeometric function (see [12]).

5. Conclusion

Based on the integral and operational representations for the hypergeometric functions of four variables defined in (1) to (10), we established several generating functions for these quadruple functions. Some particular cases and the consequences of our main results are also considered. We concluded this investigation by remarking that the scheme suggested in the derivation of the results can be applied to find other new generating functions for other quadruple hypergeometric functions and study their special cases.

Conflict of Interest

The authors have no competing interests.

References

[1] Srivastava, M. H and Manocha, H. A., Treatise on Generating Functions. Halsted Press, Bristone, London, New York and Toronto, 1985.

[2] Agarwal, P., Chand, M and Purohit, S. D., A note on generating functions involving the generalized Gauss hypergeometric function. National Acad. Sci. Lett., 3, 457-459, 2014.

[3] Bin-Saad, M. G and Younis, J. A., Certain generating functions of some quadruple hypergeometric series. Eurasian Bulletin Math., 2, 56-62, 2019.

[4] Desale, B. S and Qashash, G. A., Generating functions of special triple hypergeometric functions. International Mathematical Forum, 9, 1677-1693, 2014.

[5] Liu, H and Wang, W., Some generating relations for extended Appell's and Lauricella's hypergeometric functions. Rocky Mountain J. Math., 44, 1987-2007, 2014.

[6] Singh, M., Pundhir, S and Singh, M. P., Generating function of certain hypergeometric functions by means of fractional calculus. International J. Comput. Eng. Res. (IJCER), 11, 40-47, 2017.

[7] Bin-Saad, M. G and Younis, J. A., On connections between certain class of new quadruple and known triple hypergeometric series. Tamap Journal of Mathematics and Statistics, Volume 2019.

[8] Bin-Saad, M. G and Younis, J. A., Certain quadruple hypergeometric series and their integral representations. Appl. Appl. Math., 14, 1085-1098, 2019.

[9] Bin-Saad, M. G and Younis, J. A., Certain integrals associated with hypergeometric functions of four variables. Earthline J. Math. Sci., 2, 325-341, 2019.

[10] Bin-Saad, M. G and Younis, J. A., Certain integral representations of some quadruple hypergeometric series. Palestine J. Math., 9, 132-141, 2020.

[11] Bin-Saad, M. G and Younis, J. A., Some integrals connected with a new quadruple hypergeometric series, Universal J. Math. Appl., 3, 19-27, 2020.

[12] Srivastava, M. H and Karlsson, P. W., Multiple Gaussian Hypergeometric Series. Ellis Horwood Lt1., Chichester, 1984.

[13] Lauricella, G., Sull funzioni ipergeometric a pi variabili. Rend. Cric. Mat. Palermo, 7, 111-158, 1893.

[14] Exton, H., Hypergeometric functions of three variables. J. Indian Acad. Math., 4, 113-119, 1982.

[15] Sharma, C and Parihar, C. L., Hypergeometric functions of four variables (I). J. Indian Acad. Math., 11, 121-133, 1989.

[16] Erdélyi, A., Magnus, W., Oberhettinger, F and Tricomi, F. G., Higher Transcendental Functions. Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[17] Rainville, E. D., Special Functions. The Macmillan Company: New York, NY, USA, 1960; Reprinted by Chelsea Publishing Company, Bronx, NY, USA, 1971.

[18] Miller, K. S and Ross, B., An Introduction to Fractional Calculus and Fractional Differential Equations. Wiley, New York, 1993.

[19] Bin-Saad, M. G and Hussein, M. A., Operational images and relations of two and three variable hypergeometric series. J. Progr. Res. Math., 2, 39-46, 2015.