

Certain Topological Indices of Some Derived Graphs

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Abstract Derived graphs are the graphs obtained by some kind of operation on a given and usually smaller graph. By studying the relations between a graph and its derived graph, one can obtain information on one depending on the information on the other. In this paper, we consider certain popular topological indices of two of the derived graphs, namely the central graph and the image graph.

Keywords: topological indices, derived graphs, central graph, image graph, Zagreb indices

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1. Introduction

Several topological graph indices have been defined and studied by many mathematicians and chemists. They are defined as topological graph invariants measuring several physical, chemical, pharmacological, pharmaceutical, biological, etc. properties of graphs which are modelling real life situations. They can be grouped into three classes according to the way they are defined: by vertex degrees, by matrices or by distances. In this paper, we consider degree based topological indices of two derived graphs.

Let $G = (V, E)$ be a simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges where $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_i v_j \mid v_i, v_j \in V(G)\}$. That is, we do not allow loops or multiple edges within the set of edges. For any vertex $v \in V(G)$, we denote the degree of v by $d_G(v)$ or d_v .

If v_i and v_j are adjacent vertices of G , and if the edge e connects them, this situation will be denoted by $e = v_i v_j$. In such a case, the vertices v_i and v_j are called adjacent vertices and the edge e is said to be incident with v_i and v_j . Adjacency and incidence play a very important role in the spectral graph theory, the sub-area of graph theory dealing with linear algebraic study of graphs. The smallest and biggest vertex degrees in a graph will be denoted by δ and Δ , respectively.

Written with multiplicities, a degree sequence in general is written as

$$DS = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \dots, \Delta^{(a_\Delta)}\}. \quad (1)$$

Let D be a set of some non-decreasing non-negative integers. We say that a graph G is a realization of the set D if the degree sequence of G is equal to D . Also D is said to be realizable if there is at least one graph having D as its degree sequence.

Definition 1 [1] Let $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \dots, \Delta^{(a_\Delta)}\}$ be a realizable degree sequence and its realization be the graph G . The $\Omega(G)$ of G is defined in terms of the degree sequence as

$$\begin{aligned} \Omega(G) &= a_3 + 2a_4 + 3a_5 + \dots + (\Delta - 2)a_\Delta - a_1 \\ &= \sum_{i=1}^{\Delta} a_i(i - 2) \end{aligned} \quad (2)$$

The omega value of a graph G is shown to have many nice and practical applications. In [1,2], some fundamental properties of omega invariant are obtained. It can be used to find the number of fundamental cycles, loops, multiple edges, the size of the largest cycle, etc. amongst all realizations of a given degree sequence.

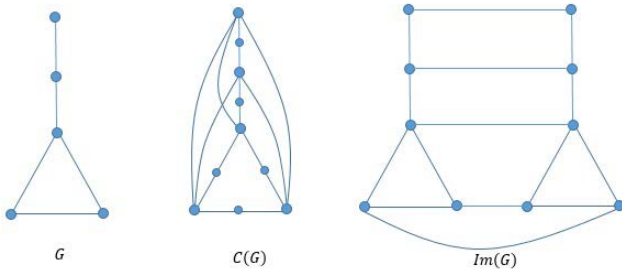
The subdivision graph $S(G)$ of a graph G is a new graph of order $n + m$ and size $2m$ which is obtained by inserting an additional vertex to each edge of G . The vertex set of the subdivision graph $S(G)$ is $V(S(G)) = V(G) \cup C$, where $C = \{c_{ij} \mid c_{ij} \in v_i v_j\}$. Note that the degree of any new vertex c_{ij} is 2, and the edge set of $S(G)$ is $E(S(G)) = \{v_i c_{ij}, v_j c_{ij} \mid v_i v_j \in E(G)\}$.

The central graph $C(G)$ of a graph G is a graph of order $n + m$ and size $\binom{n}{2} + m$ which is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$. The vertex set of

the central graph $C(G)$ is $V(C(G))=V(S(G))$ and the edge set of $C(G)$ is

$$E(C(G)) = E(S(G)) \cup \{v_i v_j \mid v_i v_j \notin E(G)\}.$$

The image graph $Im(G)$ of a graph G is a graph of order $2n$ and size $2m+n$ which is obtained by taking a copy of G exactly once and joining each vertex of G with itself in the copy of G to form an edge.



Two of the most important topological graph indices are called the first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$\begin{aligned} M_1(G) &= \sum_{u \in V(G)} d_G^2(u) \\ M_2(G) &= \sum_{uv \in E(G)} d_G(u)d_G(v) \end{aligned} \tag{3}$$

They were first defined in 1972 by Gutman and Trinajstić, [3], and are often referred to due to their uses in Chemistry for QSAR and QSPR studies. In [4], some results on the first Zagreb index together with some other indices are given. For some graph operations, these indices are calculated in [5].

The F -index or forgotten index of a graph G denoted by $F(G)$ or $M_3(G)$ is defined as

$$F(G) = \sum_{u \in V(G)} d_G^3(u). \tag{4}$$

They were first appeared in the study of structure-dependency of total π -electron energy in 1972, [3]. Recently, this sum was named as the forgotten index or the F -index by Furtula and Gutman, [7].

The hyper-Zagreb index was defined as a variety of the classical Zagreb indices as

$$HM(G) = \sum_{uv \in E} (d_u + d_v)^2, \tag{5}$$

see e.g. [7].

Inspired by the study of heat formation for heptanes and octanes in [11], Furtula et. al. proposed an index, is called augmented Zagreb index which gives a better prediction power. It is defined by

$$AZI(G) = \sum_{uv \in E} \frac{d_u d_v}{d_u + d_v - 2}. \tag{6}$$

The harmonic index was introduced by Zhang [8] who found that it correlates well with π -electron energy of benzenoid hydrocarbons and was defined as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}. \tag{7}$$

[6] introduced the redefined versions of the some Zagreb indices, i.e. the redefined first, second and third Zagreb indices for a graph G as

$$ReZG_1(G) = \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v}, \tag{8}$$

$$ReZG_2(G) = \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v}, \tag{9}$$

$$ReZG_3(G) = \sum_{uv \in E(G)} d_u d_v (d_u + d_v). \tag{10}$$

[9] reformulated the Zagreb indices in terms of the edge degrees instead of the vertex degrees as

$$EM_1(G) = \sum_{e=uv \in E(G)} (d(e))^2. \tag{11}$$

Note that the edge degree of an edge $e = uv$ is equal to $d(e) = d_u + d_v - 2$.

Aram and Dehgard, [10], introduced the concept of reformulated F -index as

$$RF(G) = \sum_{e=uv \in E(G)} (d(e))^3. \tag{12}$$

Topological indices of some derived graphs such as subdivision, total, semitotal, line and paraline graphs have already been studied in [12,13]. In this paper, we examine some degree based topological indices of image graph and central graph which are two of the derived graphs, and find relations between these topological indices.

2. Main Results

Now we will determine some well-known topological indices of central and image graphs of G in terms of the Zagreb indices, the order and size, and also the omega invariant of the graph G :

Theorem 2 Let G be a graph of order n and $C(G)$ and $Im(G)$ be the central graph and image graph of G , respectively. Then

- (i) $M_1(C(G)) = n(n-1)^2 + 4m$,
- (ii) $M_2(C(G)) = \frac{n-1}{2} [n(n-1)^2 - 2m(n-2)]$,
- (iii) $M_1(Im(G)) = 2[M_1(G) + 2\Omega(G) + 5n]$,
- (iv) $M_2(Im(G)) = 3M_1(G) + 2M_2(G) + \Omega(G) + m + 5n$

Proof. We can divide the set of vertices of $C(G)$ into two parts according to their degrees:

1. The vertices $v_i \in V(G)$ with $d_{C(G)}(v_i) = |V(G)| - d_G(v_i) - 1 + d_G(v_i) = n - 1$. The total number of these vertices in $C(G)$ is n .
2. The remaining vertices c_{ij} of $C(G)$ so that $c_{ij} \in S(G) - G$. These vertices have degree $Im(G)$

$d_{C(G)}(c_{ij}) = 2$ and the total number of the vertices lying in this category is m .

Then, using Eq. (3), we have

$$\begin{aligned} M_1(C(G)) &= \sum_{v_i \in V(C(G))} d_{C(G)}^2(v_i) \\ &= \sum_{v_i \in V(C(G))} (n-1)^2 + \sum_{c_{ij} \in V(C(G))} 2^2 = n(n-1)^2 + 4m. \end{aligned}$$

Also we can divide the incident edges of $C(G)$ into two categories:

1. Corresponding to every pair of adjacent vertices v_i and v_j in G , there is a pair of incident edges in $C(G)$ such that $v_i c_{ij} \sim v_j c_{ij}$ and the total number of them is $2m$.
2. The remaining incident edges of $C(G)$ are $v_i v_j$ such that $v_i v_j \notin E(G)$ and the total number of incident edges lying in this category is $\frac{1}{2} \sum_{v_i \in V(G)} (n-1-d_G(v_i))$.

Thus, using Eq. (3), we get

$$\begin{aligned} M_2(C(G)) &= \sum_{v_i v_j \in E(G)} d_{C(G)}(v_i) d_{C(G)}(v_j) \\ &\quad + \sum_{v_i c_{ij} \in E(S(G))} d_{C(G)}(v_i) d_{C(G)}(c_{ij}) \\ &= \frac{1}{2} \sum_{v_i \in V(G)} (n-1-d_G(v_i))(n-1)(n-1) \\ &\quad + \sum_{v_i c_{ij} \in E(S(G))} 2(n-1) \\ &= \frac{1}{2} \sum_{v_i \in V(G)} (n-1-d_G(v_i))(n-1)^2 + 4m(n-1) \\ &= \frac{n-1}{2} [n(n-1)^2 - 2m(n-1) + 2m]. \end{aligned}$$

According to the definition of the image graph of G , for all $v_i \in V(\text{Im}(G))$, $d_{\text{Im}(G)}(v_i) = d_G(v_i) + 1$. If we consider the degree sequence D of G and using Eq. (3), we have

$$\begin{aligned} M_1(\text{Im}(G)) &= 2 \sum_{i=1}^{\Delta} a_i (i+1)^2 = 2 \left[M_1(G) + \sum_{i=1}^{\Delta} a_i (2i+1) \right] \\ &= 2 [M_1(G) + 2\Omega(G) + 5n]. \end{aligned}$$

We can divide the incident edges of $\text{Im}(G)$ into two cases:

Case 1. The incident edges of $\text{Im}(G)$ are $v_i v_j$ such that $v_i v_j \in E(G)$ and they also form edges in the copy of G , say G' , so $v_i v_{j'} \in E(G')$ and the total number of incident edges in this case is $2m$.

Case 2. Each vertex in G is joined to itself in G' to form an edge, so the remaining incident edges of are $v_i v_{i'}$ such that $v_i v_{i'} \in E(\text{Im}(G))$ and the total number of these edges is n .

$$\begin{aligned} M_2(\text{Im}(G)) &= 2 \sum_{v_i v_j \in E(G)} (d_G(v_i) + 1)(d_G(v_j) + 1) \\ &\quad + 2 \sum_{v_i v_{i'} \in E(\text{Im}(G))} (d_G(v_i) + 1)(d_{G'}(v_{i'}) + 1) \\ &= 2 \left[\sum_{v_i v_j \in E(G)} d_G(v_i) d_G(v_j) \right. \\ &\quad \left. + \sum_{v_i v_j \in E(G)} (d_G(v_i) + d_G(v_j)) \right] + \sum_{i=1}^{\Delta} a_i (i+1)^2 \\ &\quad + \sum_{v_i v_j \in E(G)} 1 \\ &= 3M_1(G) + 2(M_2(G) + \Omega(G) + m) + 5n. \end{aligned}$$

Theorem 3. Let G be a graph as in Theorem 2. Then the third Zagreb index, also called the forgotten index, of $C(G)$ and $\text{Im}(G)$ are

- i) $F(C(G)) = n(n-1)^3 + 8m$,
- ii) $F(\text{Im}(G)) = 2[F(G) + 3M_1(G) + 3\Omega(G) + 7n]$.

Proof. Using Eq. (4), we have

$$\begin{aligned} F(C(G)) &= \sum_{v_i \in V(C(G))} d_{C(G)}(v_i)^3 \\ &= \sum_{v_i \in V(C(G))} (n-1)^3 + \sum_{c_{ij} \in V(C(G))} 2^3 \\ &= n(n-1)^3 + 8m. \end{aligned}$$

Since

$$F(\text{Im}(G)) = 2 \sum_{i=1}^{\Delta} a_i (i+1)^3,$$

the result immediately follows by using the process in the proof of Theorem 2.

Theorem 4 Let G be a graph as in Theorem 2. Then the hyper-Zagreb index of $C(G)$ and $\text{Im}(G)$ are

- (i) $HM(C(G)) = 2(n-1)^2 [n(n-1) - 2m] + 2m(n+1)^2$,
- (ii) $HM(\text{Im}(G)) = 12M_1(G) + 2HM(G) + 8\Omega(G) + 8m + 20n$.

Proof. From Eq. (5), we have

$$\begin{aligned} HM(C(G)) &= \sum_{v_i v_j \in E(C(G))} (d_{C(G)}(v_i) + d_{C(G)}(v_j)) \\ &= \frac{1}{2} \sum_{v_i \in V(G)} (n-1-d_G(v_i))(2n-2)^2 + 2m(2+n-1)^2 \end{aligned}$$

and

$$\begin{aligned} HM(\text{Im}(G)) &= 2 \sum_{v_i v_j \in E(G)} [(d_G(v_i) + 1) + (d_G(v_j) + 1)]^2 \\ &\quad + 4 \sum_{i=1}^{\Delta} a_i (i+1)^2 \end{aligned}$$

and the result follows.

Theorem 5 Let G be a graph as in Theorem 2. The augmented Zagreb index of $C(G)$ is

$$AZI(C(G)) = \frac{(n-1)^6}{16(n-2)^3} [n(n-1) - 2m] + 16m.$$

Proof. Using Eq. (6), we obtain

$$\begin{aligned} AZI(C(G)) &= \sum_{v_i v_j \in E(C(G))} \left(\frac{d_{C(G)}(v_i) d_{C(G)}(v_j)}{d_{C(G)}(v_i) + d_{C(G)}(v_j) - 2} \right)^3 \\ &= \sum_{v_i \in V(G)} \left(\frac{(n-1)(n-1)}{2n-4} \right)^3 \frac{n-1-d_G(v_i)}{2} + \left[\frac{2(n-1)}{n-1} \right]^3 2m \\ &= \frac{(n-1)^6}{16(n-2)^3} [n(n-1) - 2m] + 16m. \end{aligned}$$

Theorem 6 Let G be a graph as in Theorem 2. The harmonic index of $C(G)$ is

$$H(C(G)) = \frac{n}{2} + \frac{m(3n-5)}{n^2-1}.$$

Proof. We know that $H(G) = \sum_{v_i v_j \in E(G)} \frac{2}{d_{v_i} + d_{v_j}}$, i.e.,

$$\begin{aligned} H(C(G)) &= \sum_{v_i v_j \in E(C(G))} \frac{2}{d_{C(G)}(v_i) + d_{C(G)}(v_j)} \\ &= \sum_{v_i \in V(G)} \left(\frac{2}{(n-1) + (n-1)} \right) \frac{n-1-d_G(v_i)}{2} + \frac{2}{(n-1)+2} 2m \\ &= \frac{n}{2} + \frac{m(3n-5)}{n^2-1}. \end{aligned}$$

Theorem 7 Let G be a graph as in Theorem 2. The redefined versions of Zagreb indices of $C(G)$ and $\text{Im}(G)$ are

- (i) $\text{Re} ZG_1(C(G)) = \frac{n(n-1) + m(n+1) - 2m}{n-1},$
- (ii) $\text{Re} ZG_1(\text{Im}(G)) = \sum_{uv \in E(G)} \frac{d_G(u) + d_G(v) + 2}{(d_G(u)+1)(d_G(v)+1)} + \sum_{i=1}^{\Delta} \frac{2a_i}{i+1},$
- (iii) $\text{Re} ZG_2(C(G)) = \frac{n-1}{4} [n(n-1) - 2m] + \frac{4m(n-1)}{n+1},$
- (iv) $\text{Re} ZG_2(\text{Im}(G)) = \sum_{uv \in E(G)} \frac{(d_G(u)+1)(d_G(v)+1)}{d_G(u) + d_G(v) + 2} + \frac{1}{2} (\Omega(G) + 3n),$
- (v) $\text{Re} ZG_3(C(G)) = (n-1)^3 [n(n-1) - 2m] + 4m(n^2 - 1),$
- (vi) $\text{Re} ZG_3(\text{Im}(G)) = 2 [3\Omega(G) + 6M_1(G) + 2M_2(G) + F(G) + HM(G) + \text{Re} ZG_3(G) + 7n + 2m].$

Proof. From Eq. (8), we have,

$$\begin{aligned} \text{Re} ZG_1(C(G)) &= \sum_{v_i v_j \in E(C(G))} \frac{d_{C(G)}(v_i) + d_{C(G)}(v_j)}{d_{C(G)}(v_i) d_{C(G)}(v_j)} \\ &= \sum_{v_i \in V(G)} \frac{2(n-1)}{(n-1)^2} \frac{n-1-d_G(v_i)}{2} + \frac{n+1}{2(n-1)} 2m \end{aligned}$$

$$= \frac{n(n-1) + m(n+1) - 2m}{n-1}.$$

$\text{Re} ZG_3(\text{Im}(G))$

$$\begin{aligned} &= \sum_{v_i v_j \in E(\text{Im}(G))} [(d_G(v_i) + 1) + (d_G(v_j) + 1)] \\ &\quad \cdot [(d_G(v_i) + 1)(d_G(v_j) + 1)] + 2 \sum_{i=1}^{\Delta} a_i (i+1)^3 \\ &= \sum_{v_i v_j \in E(\text{Im}(G))} [d_G(v_i) d_G(v_j) (d_G(v_i) + d_G(v_j)) \\ &\quad + 2d_G(v_i) d_G(v_j) + (d_G(v_i) + d_G(v_j))^2 2(d_G(v_i) \\ &\quad + d_G(v_j)) + (d_G(v_i) + d_G(v_j)) + 2] \\ &\quad + 2 \sum_{i=1}^{\Delta} a_i (i+1)^3. \end{aligned}$$

The others also follow combinatorically.

Theorem 8 Let G be a graph as in Theorem 2. The reformulated Zagreb indices of G are

- (i) $EM_1(C(G)) = 2(n-2)^2 [n(n-1) - 2m] + 2m(n-1)^2,$
- (ii) $RF(C(G)) = 8(n-2)^3 [n(n-1) - 2m] + 2m(n-1)^3,$
- (iii) $EM_1(\text{Im}(G)) = 2 [\text{Re} M_1(G) + 6M_1(G) - 4m],$
- (iv) $RF(\text{Im}(G)) = 2 [\text{Re} M_2(G) + 6\text{Re} M_1(G) + 12M_1(G) + 4F(G) - 16m].$

Proof. Using Eq. (11), we have,

$$\begin{aligned} EM_1(C(G)) &= \sum_{e=v_i v_j \in E(C(G))} (d(e))^2 \\ &= (2n-4)^2 \sum_{v_i \in V(G)} \frac{n-1-d_G(v_i)}{2} + 2m(n-1)^2 \\ &= (2n-4)^2 [n(n-1) - 2m] + 2m(n-1)^2. \end{aligned}$$

For the proof of (ii), similar methods can be applied by using Eq. (12). Similar ways can be employed to prove (iii) and (iv) by means of Eqs. (11) and (12).

$$\begin{aligned} EM_1(\text{Im}(G)) &= 2 \sum_{e=v_i v_j \in E(G)} (d_G(e) + 2)^2 + 4 \sum_{i=1}^{\Delta} a_i i^2 \\ &= 2 \sum_{e=v_i v_j \in E(G)} ((d_G(e))^2 + 2d_G(e) + 4) + 4 \sum_{i=1}^{\Delta} a_i i^2 \end{aligned}$$

and the result immediately follows.

$$RF(\text{Im}(G)) = 2 \sum_{e=v_i v_j \in E(G)} (d_G(e) + 2)^3 + 8 \sum_{i=1}^{\Delta} a_i i^3$$

and we have the desired result.

3. Conclusions

In this paper, we obtained the expressions for topological indices of two derived graphs, central and image graphs and found these expressions in terms of the

topological indices of main graph, the size, order and also a recently defined graph invariant called Ω .

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