

Twin Polynomials and Kernels Matrix

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Abstract Polynomials and matrices have played a very important role in the development of different branches of mathematics. Indeed, several mathematicians have introduced classical polynomials very useful for the scientific community such as the Lagrange’s interpolation polynomials, the Chebyshev’s polynomials and the Bernstein’s polynomials [1,2]. Also, there is a strong link between polynomials and matrices through the notions of the determinant, the characteristic polynomial and the minimal polynomial. Similarly, we will introduce in this article two polynomials which we will call twin polynomials as well as a matrix called kernels matrix. Finally, we will present some applications such as the resolution of recurrent sequences with second member and the establishment of several sum formulas.

Keywords: factorial mean, twin polynomials, factorial means staircase, kernels matrix, kernels determinant

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1. Introduction

Polynomials and matrix calculus continue to attract the intension of researchers thanks to the simplifications they allow for the various scientific problems encountered. In addition, the strong links between these two branches make their use frequent. The twin polynomials introduced in this article show links with other classical mathematical tools such as binomial coefficients, trigonometric kernels, the Fibonacci sequence as well as recurrent sequences of order 3. The kernels matrix is a unitary matrix which will reverse several Hermitian matrices. We will then deduce several formulas and present some applications of these new tools.

2. Definitions

2.1. Absolute Value in Complex Sense

If z is a complex number then the notation $\langle z \rangle$ represents the absolute value of z in the sense of complex numbers which is equal to:

$$\text{If } z = a + ib \text{ then } \langle z \rangle = \begin{cases} |z| = |a| & \text{if } b = 0 \text{ (} z \in \mathbb{R} \text{)} \\ |a + i|b|| & \text{if } b \neq 0 \end{cases}$$

We write thus: $\forall z \in \mathbb{C}, \sqrt{z^2} = \langle z \rangle$.

2.2. Delta and Gamma Functions

To simplify, throughout the rest of this research, the following four complex-valued functions are defined while respecting the convention mentioned above:

$$\delta(x) = x + \sqrt{x^2 - 1}, \text{ and } \bar{\delta}(x) = x - \sqrt{x^2 - 1}, x \in \mathbb{C}$$

$$\gamma(x) = \sqrt{x^2 + 1} + x, \text{ and } \bar{\gamma}(x) = \sqrt{x^2 + 1} - x, x \in \mathbb{C}.$$

$$\text{we also note: } \begin{cases} \Re_{\delta}(x) = x \text{ and } I_{\delta}(x) = \sqrt{x^2 - 1} \\ \Re_{\gamma}(x) = \sqrt{x^2 + 1} \text{ and } I_{\gamma}(x) = x \end{cases}$$

2.3. Factorial Mean

2.3.1. Definition

We define Factorial Mean by:

$$\forall n \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad k \leq n \quad \text{and} \quad n - k \equiv 0[2]:$$

$$M_n^k = C_{n+k}^{\frac{n-k}{2}} = \binom{n+k}{\frac{n-k}{2}}$$

2.3.2. Property1. Factorial Mean Equation (FME)

The factorial Mean verify the following relations:

$$\forall n \in \mathbb{N}^* \forall k \in \mathbb{N} \quad 1 \leq k \leq n \quad \begin{cases} M_{2n+2}^{2k} = M_{2n+1}^{2k-1} + M_{2n}^{2k} \\ M_{2n+3}^{2k+1} = M_{2n+2}^{2k} + M_{2n+1}^{2k+1} \end{cases}$$

Proof. For $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$ such that $k \leq n$:

$$\begin{aligned} M_{2n+2}^{2k} &= \frac{(n+1+k)!}{(2k)!(n+1-k)!} = \frac{(n+1-k+2k)(n+k)!}{(2k)!(n+1-k)!} \\ &= \frac{(n+k)!}{(2k-1)!(n+1-k)!} + \frac{(n+k)!}{(2k)!(n-k)!} = M_{2n+1}^{2k-1} + M_{2n}^{2k} \end{aligned}$$

$$\begin{aligned}
 M_{2n+3}^{2k+1} &= \frac{(n+2+k)!}{(2k+1)!(n+1-k)!} \\
 &= \frac{(n+1-k+2k+1)(n+1+k)!}{(2k+1)!(n+1-k)!} \\
 &= \frac{(n+1+k)!}{(2k)!(n+1-k)!} + \frac{(n+1+k)!}{(2k+1)!(n-k)!} = M_{2n+2}^{2k} + M_{2n+1}^{2k+1}.
 \end{aligned}$$

2.4. Parity of Polynomials of Two Variables

Let P a polynomial of two variables. We say that P is Even if:

$$\forall (x, y) \in \mathbb{C}^2 \quad P(-x, -y) = P(x, y)$$

We say that P is Odd if:

$$\forall (x, y) \in \mathbb{C}^2 \quad P(-x, -y) = -P(x, y).$$

3. Twin Polynomials

3.1. Definition

For every recurrence relation, there is a unique polynomial of degree n such that:

$$\begin{cases}
 A_0(X, Y) = 1 & \text{and} & A_1(X, Y) = X \\
 A_{n+2}(X, Y) = XA_{n+1}(X, Y) - Y^2A_n(X, Y)
 \end{cases}$$

$$\begin{cases}
 B_0(X, Y) = 1 & \text{and} & B_1(X, Y) = X \\
 B_{n+2}(X, Y) = XB_{n+1}(X, Y) + Y^2B_n(X, Y)
 \end{cases}$$

Proof. This is due to the setting of the initial conditions of the recurrence relations. Indeed, suppose there is another polynomial P_n verify the same recurrence relation such as: $P_0(X, Y) = 1$ and $P_1(X, Y) = X$. We consider the polynomial $Q_n = A_n - P_n$. We have $Q_0 = Q_1 = 0$. The polynomial Q_n verify the same recurrence relation. By recurrence, we show that: $\forall n \in \mathbb{N}, Q_n = 0$ hence $A_n = P_n$. We follow the same reasoning to demonstrate the unicity of B_n .

3.2. First Expressions of Twin Polynomials

The first expressions of A_n and B_n for $n = 1 \dots 5$:

$$\begin{aligned}
 A_0(X, Y) &= 1 \\
 A_1(X, Y) &= X \\
 A_2(X, Y) &= X^2 - Y^2 \\
 A_3(X, Y) &= X^3 - 2Y^2X \\
 A_4(X, Y) &= X^4 - 3Y^2X^2 + Y^4 \\
 A_5(X, Y) &= X^5 - 4Y^2X^3 + 3Y^4X
 \end{aligned}$$

$$\begin{aligned}
 B_0(X, Y) &= 1 \\
 B_1(X, Y) &= X \\
 B_2(X, Y) &= X^2 + Y^2 \\
 B_3(X, Y) &= X^3 + 2Y^2X \\
 B_4(X, Y) &= X^4 + 3Y^2X^2 + Y^4 \\
 B_5(X, Y) &= X^5 + 4Y^2X^3 + 3Y^4X
 \end{aligned}$$

3.3. Property 2

$$\forall n \in \mathbb{N} \quad \begin{cases} A_n(X, iY) = B_n(X, Y) \\ A_n(X, Y) = B_n(X, iY) \end{cases}$$

Proof. By recurrence, for $n = 0$ and $n = 1$:

$$\begin{cases}
 A_0(X, iY) = 1 = B_0(X, Y) \\
 A_1(X, iY) = iY = B_1(X, Y)
 \end{cases}$$

For $n+1$:

$$\begin{aligned}
 A_{n+2}(X, iY) &= XA_{n+1}(X, iY) - (iY)^2A_n(X, iY) \\
 &= XB_{n+1}(X, Y) + Y^2B_n(X, Y) \\
 &= B_{n+2}(X, Y).
 \end{aligned}$$

3.4. Lemma 1

A_n and B_n have same parity of n : A_{2n} and B_{2n} are even and A_{2n+1} and B_{2n+1} are odd.

Proof. It's easy to prove it by recurrence.

3.5. Lemma 2

All the monomials of A_n and B_n are of degree n .

Proof. It's easy to prove it by recurrence.

3.6. Coefficients of A_n and B_n

The polynomials are given by:

$$\begin{cases}
 A_{2n}(X, Y) = \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} X^{2k} Y^{2(n-k)} \\
 A_{2n+1}(X, Y) = \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} X^{2k+1} Y^{2(n-k)} \\
 B_{2n}(X, Y) = \sum_{k=0}^n M_{2n}^{2k} X^{2k} Y^{2(n-k)} \\
 B_{2n+1}(X, Y) = \sum_{k=0}^n M_{2n+1}^{2k+1} X^{2k+1} Y^{2(n-k)}
 \end{cases}$$

Proof. We will present the demonstration for A_n and then using the property 2, we can easily establish the demonstration for B_n . Using preceding lemmas, distinction is made according to the parity of n . we note:

$$\begin{cases} A_{2n}(X, Y) = \sum_{k=0}^n a_{2n,2k} X^{2k} Y^{2(n-k)} \\ A_{2n+1}(X, Y) = \sum_{k=0}^n a_{2n+1,2k+1} X^{2k+1} Y^{2(n-k)} \end{cases} = \sum_{k=1}^{n+1} (-1)^{n+1-k} M_{2n+1}^{2k-1} X^{2k} Y^{2(n+1-k)} + \sum_{k=0}^n (-1)^{n+1-k} M_{2n}^{2k} X^{2k} Y^{2(n+1-k)}$$

Now, using recurrence relation:

$$A_{n+2}(X, Y) = X A_{n+1}(X, Y) - Y^2 A_n(X, Y).$$

We get following system equations:

$$\begin{cases} a_{2n+2,2n+2} = a_{2n+1,2n+1} = 1 \\ a_{2n+2,0} = -a_{2n,0} = \dots = (-1)^{n+1} a_{0,0} = (-1)^{n+1} \quad (E). \\ a_{2n+2,2k} = a_{2n+1,2k-1} - a_{2n,2k} \quad 1 \leq k \leq n \end{cases}$$

Using recurrence, we prove that: $a_{2n+1,1} = (-1)^n (n+1)$.

For $n = 0$, we have $a_{1,1} = 1 = (-1)^0 (0+1)$. We suppose that it's true for $n \in \mathbb{N}$. For $n + 1$:

$$\begin{aligned} a_{2n+3,1} &= a_{2n+2,0} - a_{2n+1,1} = (-1)^{n+1} - (-1)^n (n+1) \\ &= (-1)^{n+1} (n+2) \end{aligned}$$

Using variable change

$$a_{n,p} = (-1)^{\frac{n-p}{2}} v_{n,p} \quad (n-p \equiv 0[2]):$$

$$(E) \Rightarrow \begin{cases} v_{2n+2,2n+2} = v_{2n+1,2n+1} = 1 \\ v_{2n+2,0} = \dots = v_{0,0} = 1 \text{ and } v_{2n+1,1} = n+1 \\ v_{2n+2,2k} = v_{2n+1,2k-1} + v_{2n,2k} \quad 1 \leq k \leq n \end{cases}$$

$(v_{n,k})_{n,k}$ verify Factorial Mean Equation (FME). We

conclude that: $v_{n,k} = M_n^k$ and we get:

$$a_{n,p} = (-1)^{\frac{n-p}{2}} M_n^k \quad (n-p \equiv 0[2])$$

Which completes the demonstration.

We can also check these formulas by using recurrence relation. For $n = 0$ and $n = 1$: they are verified.

$$\begin{aligned} A_{2n+2}(X) &= X A_{2n+1}(X, Y) - Y^2 A_{2n}(X) \\ &= X \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} X^{2k+1} Y^{2(n-k)} \\ &\quad - Y^2 \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} X^{2k} Y^{2(n-k)} \\ &= \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} X^{2(k+1)} Y^{2(n-k)} \\ &\quad + Y^2 \sum_{k=0}^n (-1)^{n+1-k} M_{2n}^{2k} X^{2k} Y^{2(n-k)} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^n (-1)^{n+1-k} \left(M_{2n+1}^{2k-1} + M_{2n}^{2k} \right) X^{2k} Y^{2(n+1-k)} \\ &\quad + (-1)^{n+1} Y^{2(n+1)} + X^{2(n+1)} \\ &= \sum_{k=0}^{n+1} (-1)^{n+1-k} M_{2n+2}^{2k} X^{2k} Y^{2(n+1-k)} \end{aligned}$$

$$\begin{aligned} A_{2n+3}(X) &= X A_{2n+2}(X) - Y^2 A_{2n+1}(X) \\ &= X \sum_{k=0}^{n+1} (-1)^{n+1-k} M_{2n+2}^{2k} X^{2k} Y^{2(n+1-k)} \\ &\quad - Y^2 \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} X^{2k+1} Y^{2(n-k)} \\ &= \sum_{k=0}^{n+1} (-1)^{n+1-k} M_{2n+2}^{2k} X^{2k+1} Y^{2(n+1-k)} \\ &\quad + \sum_{k=0}^n (-1)^{n+1-k} M_{2n+1}^{2k+1} X^{2k+1} Y^{2(n+1-k)} \\ &= \sum_{k=0}^n (-1)^{n+1-k} \left(M_{2n+2}^{2k} + M_{2n+1}^{2k+1} \right) X^{2k+1} Y^{2(n+1-k)} \\ &\quad + X^{2(n+1)+1} \\ &= \sum_{k=0}^{n+1} (-1)^{n+1-k} M_{2n+3}^{2k+1} X^{2k+1} Y^{2(n+1-k)}. \end{aligned}$$

3.7. Differential Equations

Twin polynomials A_n and B_n verify the following differential equations:

$$(4Y^2 - X^2) \frac{\partial^2 A_n}{\partial X^2} - 3X \frac{\partial A_n}{\partial X} + n(n+2) A_n = 0 \quad (*)$$

$$(4Y^2 + X^2) \frac{\partial^2 B_n}{\partial X^2} + 3X \frac{\partial B_n}{\partial X} - n(n+2) B_n = 0 \quad (**)$$

Proof. Let P polynomial solution of (*) such that

$$P_n(X, Y) = \sum_{k=0}^n p_{n,k}(Y) X^k. \text{ We have:}$$

$$\frac{\partial P_n}{\partial X} = \sum_{k=1}^n k p_{n,k}(Y) X^{k-1}$$

$$\frac{\partial^2 P_n}{\partial X^2} = \sum_{k=0}^{n-2} (k+1)(k+2) p_{n,k+2}(Y) X^k$$

Replacing in the equation (*), we find:

$$\begin{aligned} & (4Y^2 - X^2) \sum_{k=0}^{n-2} (k+1)(k+2) p_{n,k+2}(Y) X^k \\ & - 3X \sum_{k=1}^n k p_{n,k}(Y) X^{k-1} + n(n+2) \sum_{k=0}^n p_{n,k}(Y) X^k = 0 \end{aligned}$$

We get so the following system:

$$\begin{cases} 8Y^2 p_{n,2} + n(n+2) p_{n,0} = 0 \\ 24Y^2 p_{n,3} + (n-1)(n+3) p_{n,1} = 0 & 2 \leq k \leq n-2 \\ 4Y^2 (k+1)(k+2) p_{n,k+2} + (n-k)(n+k+2) p_{n,k} = 0 \end{cases}$$

We conclude that:

$$\begin{aligned} \forall k \in \mathbb{N} \quad 0 \leq k \leq n-2 \\ 4Y^2 (k+1)(k+2) p_{n,k+2} + (n-k)(n+k+2) p_{n,k} = 0. \end{aligned}$$

Or even better:

$$\frac{p_{2n,2k+2}}{p_{2n,2k}} = -\frac{1}{2Y^2} \frac{(n-k)(n+k+1)}{(k+1)(2k+1)} \quad (0 \leq k \leq n-2).$$

For even numbers:

$$\begin{aligned} \prod_{k=0}^p \frac{p_{2n,2k+2}}{p_{2n,2k}} &= \prod_{k=0}^p \left(-\frac{1}{2Y^2} \right) \prod_{k=0}^p \frac{(n-k)(n+k+1)}{(k+1)(2k+1)} \\ &= \frac{(-1)^{p+1} [(n-p)..(n-1)n][(n+1)..(n+1+p)]}{Y^{2(p+1)} 2^{p+1} [1 \times 2..(p+1)][1 \times 3..(2p+1)]} \\ \frac{p_{2n,2p+2}}{p_{2n,0}} &= \frac{(-1)^{p+1} [(n-p)..(n-1)n][(n+1)..(n+1+p)]}{Y^{2(p+1)} 2^{p+1} [1 \times 2..(p+1)][1 \times 3..(2p+1)]} \\ &= \frac{(-1)^p}{Y^{2p}} \frac{(n+p)!}{(n-p)!(2p)!} (-1)^n Y^{2n} \\ p_{2n,2p+2} &= \frac{(-1)^p}{Y^{2p}} \frac{(n+p)!}{(n-p)!(2p)!} (-1)^n Y^{2n} p_{2n,0} \\ &= (-1)^{n-p} M_{2n}^{2p} Y^{2(n-p)} \end{aligned}$$

For odd numbers:

$$\begin{aligned} \prod_{k=0}^p \frac{p_{2n+1,2k+3}}{p_{2n+1,2k+1}} &= \prod_{k=0}^p \left(-\frac{1}{2Y^2} \right) \prod_{k=0}^p \frac{(n-k)(n+k+2)}{(k+1)(2k+3)} \\ \frac{p_{2n+1,2p+3}}{p_{2n+1,1}} &= \frac{(-1)^{p+1} [(n-p)..(n-1)n][(n+2)..(n+2+p)]}{Y^{2(p+1)} 2^{p+1} [1 \times 2..(p+1)][3..(2p+3)]} \\ &= \frac{(-1)^{p+1}}{Y^{2(p+1)}} \frac{(n+2+p)!}{(n+1)(2p+3)!(n-p-1)!}. \end{aligned}$$

$$\begin{aligned} p_{2n+1,2p+1} &= \frac{(-1)^p}{Y^{2p}} \frac{(n+1+p)!}{(n+1)(2p+1)!(n-p)!} p_{2n+1,1} \\ &= (-1)^{n-p} Y^{2(n-p)} M_{2n+1}^{2p+1} \end{aligned}$$

Now, to demonstrate the equation (**), we can easily use property 2 and equation (*):

$$\begin{aligned} & (4(iY)^2 - X^2) \frac{\partial^2 A_n(X, iY)}{\partial X^2} - 3X \frac{\partial A_n(X, iY)}{\partial X} \\ & + n(n+2) A_n(X, iY) = 0 \end{aligned}$$

We get:

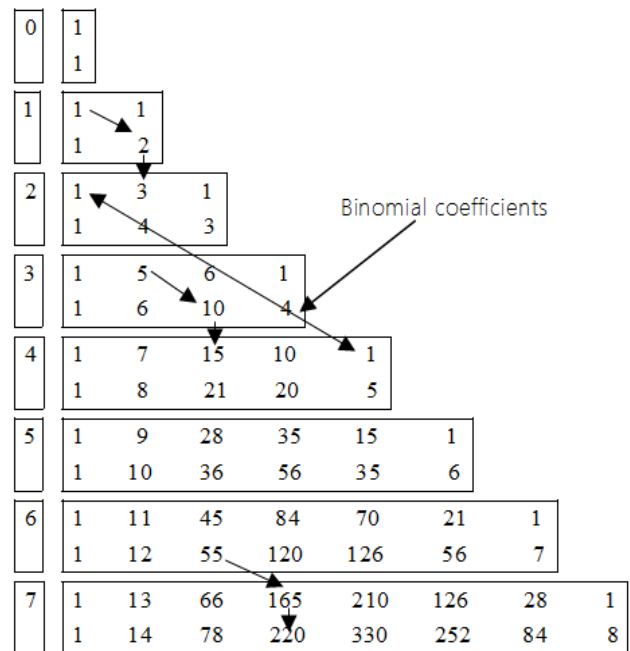
$$(4Y^2 + X^2) \frac{\partial^2 B_n}{\partial X^2} + 3X \frac{\partial B_n}{\partial X} - n(n+2) B_n = 0.$$

3.8. Factorial Means Staircase

The staircase of factorial means gathers in its steps the factorial means. The first column of the staircase represents the integer natural number n . Each step of the staircase contains two lines representing the factorial means for the even integers and for the odd integers in decreasing order: for the index step n (from the first column), the first line of the step represents the factorial means for $2n$ (k even varying from $2n$ to 0) and the second line represents the factorial means for $2n+1$ (k odd varying from $2n+1$ to 1).

To build the staircase, we use the Factorial Mean Equation (MFE). To simplify the building procedure, we have put arrows on the stairs below. The oblique arrow means "+" (addition) and the downward vertical arrow means "=".

The following staircase is by way of example constructed for $n = 7$:



Note. It should be noted that the factorial means staircase also contains the binomial coefficients of Newton's identity (the diagonals) [3,4] as well as the elements of the Fibonacci sequence [5].

3.8. Link with the Fibonacci Sequence

3.8.1. Definition

The Fibonacci's sequence is defined by [5]:

$$\forall n \in \mathbb{N} \quad \begin{cases} F_0 = 1 \text{ et } F_1 = 1 \\ F_{n+2} = F_{n+1} + F_n \end{cases}$$

3.8.2. Theorem

If we note $A_n = A_n(1,1)$ and $B_n = B_n(1,1)$ [6]:

$$\begin{cases} A_{2n} = \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} \\ A_{2n+1} = \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} \end{cases} \text{ and } \begin{cases} B_{2n} = \sum_{k=0}^n M_{2n}^{2k} \\ B_{2n+1} = \sum_{k=0}^n M_{2n+1}^{2k+1} \end{cases}$$

Then: $\forall n \in \mathbb{N}, B_n = F_n$ and $A_n = \frac{2}{\sqrt{3}} \cos\left(\frac{2n-1}{6} \pi\right)$.

Proof. We have $B_0 = M_0^0 = 1 = F_0$ and $B_1 = M_1^1 = 1 = F_1$.

By proceeding by recurrence, we assume that: $B_{2n} = F_{2n}$ and $B_{2n+1} = F_{2n+1}$.

$$\begin{aligned} B_{2n} + B_{2n+1} &= \sum_{k=0}^n M_{2n}^{2k} + \sum_{k=0}^n M_{2n+1}^{2k+1} \\ &= 2 + \sum_{k=1}^n (M_{2n}^{2k} + M_{2n+1}^{2k-1}) \\ &= 2 + \sum_{k=1}^n M_{2n+2}^{2k} = \sum_{k=0}^{n+1} M_{2n+2}^{2k} = B_{2n+2} \end{aligned}$$

$$\begin{aligned} B_{2n+1} + B_{2n+2} &= \sum_{k=0}^n M_{2n+1}^{2k+1} + \sum_{k=0}^{n+1} M_{2n+2}^{2k} \\ &= 1 + \sum_{k=0}^n (M_{2n+1}^{2k+1} + M_{2n+2}^{2k}) \\ &= 1 + \sum_{k=0}^n M_{2n+3}^{2k+1} = \sum_{k=0}^{n+1} M_{2n+3}^{2k+1} = B_{2n+3} \end{aligned}$$

Which proves that: $\forall n \in \mathbb{N} \quad B_n = F_n$. For $(A_n)_{n \in \mathbb{N}}$:

$$\begin{aligned} A_{2n+2} &= 1 + (-1)^{n+1} - \sum_{k=1}^n (-1)^{n-k} (M_{2n}^{2k} + M_{2n+1}^{2k-1}) \\ &= - \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} + \sum_{k=1}^n (-1)^{n-k} M_{2n+1}^{2k+1} = A_{2n+1} - A_{2n}. \end{aligned}$$

So $A_{2n+2} = A_{2n+1} - A_{2n}$. Using the characteristic equation $r^2 - r + 1 = 0$, we obtain:

$$\begin{aligned} A_n &= \frac{1}{\sqrt{3}} \left[\bar{\delta} \left(\frac{\sqrt{3}}{2} \right) \delta \left(\frac{1}{2} \right)^n + \delta \left(\frac{\sqrt{3}}{2} \right) \bar{\delta} \left(\frac{1}{2} \right)^n \right] \\ &= \frac{1}{\sqrt{3}} \left[e^{i \frac{2n-1}{6} \pi} + e^{-i \frac{2n-1}{6} \pi} \right] = \frac{2}{\sqrt{3}} \cos \left(\frac{2n-1}{6} \pi \right). \end{aligned}$$

3.8.3. Consequences

a) In the factorial means staircase, at step number n , the sum of the elements of the first line is equal to the number F_{2n} and the sum of the elements of the second line equal to the number F_{2n+1} .

b) In the factorial means staircase, the sum of all the elements of step number n is equal to F_{2n+2} .

c) We note $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio. We have the following limits [5]:

$$\begin{cases} \lim_{n \rightarrow +\infty} \frac{B_{2n+1}}{B_{2n}} = \lim_{n \rightarrow +\infty} \frac{\sum_{k=0}^n M_{2n+1}^{2k+1}}{\sum_{k=0}^n M_{2n}^{2k}} = \varphi \\ \lim_{n \rightarrow +\infty} \frac{B_{2n+2} + B_{2n+1}}{B_{2n} + B_{2n+1}} \\ = \lim_{n \rightarrow +\infty} \frac{\sum_{k=0}^{n+1} (M_{2n+2}^{2k} + M_{2n+3}^{2k+1})}{\sum_{k=0}^n (M_{2n}^{2k} + M_{2n+1}^{2k+1})} = \varphi^2 \end{cases}$$

3.9. Expressions of Polynomial Functions

Let $(x, y) \in \mathbb{C}^2$. The characteristic equations of sequences $(A_n(x, y))_{n \in \mathbb{N}}$ and $(B_n(x, y))_{n \in \mathbb{N}}$ are:

$$(E_1): r^2 - xr + y^2 = 0 \text{ and } (E_2): r^2 + xr + y^2 = 0.$$

- Case 1: $y = 0$. $A_n(x, y) = B_n(x, y) = x^n$
- Case 2: $y \neq 0$. We prove that:

$$r_1 = \langle y \rangle \delta \left(\frac{x}{2 \langle y \rangle} \right) \text{ and } r_2 = \bar{r}_1 = \langle y \rangle \bar{\delta} \left(\frac{x}{2 \langle y \rangle} \right).$$

Thus, we can write the expressions as follows:

$$\begin{cases} A_0(x; y) = 1 \text{ et } A_1(x; y) = x \\ A_n(x; y) = \langle y \rangle^n \left[a \delta \left(\frac{x}{2 \langle y \rangle} \right)^n + b \bar{\delta} \left(\frac{x}{2 \langle y \rangle} \right)^n \right] \\ B_0(x; y) = 1 \text{ et } B_1(x; y) = x \\ B_n(x; y) = \langle y \rangle^n \left[c \gamma \left(\frac{x}{2 \langle y \rangle} \right)^n + (-1)^n d \bar{\gamma} \left(\frac{x}{2 \langle y \rangle} \right)^n \right] \end{cases}$$

Using the initial conditions, one can easily find the constants a, b, c and d . After solving the systems of equations, we obtain the following analytical expressions:

$$\left\{ \begin{aligned} A_n(x, y) &= \frac{\langle y \rangle^n}{2I_\delta \left(\frac{x}{2\langle y \rangle} \right)} \left[\delta \left(\frac{x}{2\langle y \rangle} \right)^{n+1} - \bar{\delta} \left(\frac{x}{2\langle y \rangle} \right)^{n+1} \right] \\ B_n(x, y) &= \frac{y^n}{2\Re_\gamma \left(\frac{x}{2\langle y \rangle} \right)} \left[\gamma \left(\frac{x}{2\langle y \rangle} \right)^{n+1} + (-1)^n \bar{\gamma} \left(\frac{x}{2\langle y \rangle} \right)^{n+1} \right] \end{aligned} \right. \quad (1)$$

3.10. Generating Function of Twin Polynomials

Generating function of polynomial A_n is obtained as follow:

$$\begin{aligned} xt \sum_{n=0}^{+\infty} A_n(x, y) t^n &= xt + t \sum_{n=1}^{+\infty} [A_{n+1}(x, y) + y^2 A_{n-1}(x, y)] t^n \\ &= y^2 t^2 \sum_{n=0}^{+\infty} A_n(x, y) t^n + \sum_{n=0}^{+\infty} A_n(x, y) t^n - 1 \\ \sum_{n=0}^{+\infty} A_n(x, y) t^n &= \frac{1}{y^2 t^2 - xt + 1}. \end{aligned}$$

Generating function of polynomial B_n is obtained as follow:

$$\begin{aligned} xt \sum_{n=0}^{+\infty} B_n(x, y) t^n &= xt + t \sum_{n=1}^{+\infty} [B_{n+1}(x, y) - y^2 B_{n-1}(x, y)] t^n \\ &= -y^2 t^2 \sum_{n=0}^{+\infty} B_n(x, y) t^n + \sum_{n=0}^{+\infty} B_n(x, y) t^n - 1 \\ \sum_{n=0}^{+\infty} B_n(x, y) t^n &= -\frac{1}{y^2 t^2 + xt - 1}. \end{aligned}$$

3.11. Factorization of Twin Polynomials

3.11.1. Theorem

For $n \geq 1$, we can factorize twin polynomials:

$$\left\{ \begin{aligned} A_n(X, Y) &= \prod_{k=1}^n \left[X - 2\langle Y \rangle \cos\left(\frac{k\pi}{n+1}\right) \right] \\ B_n(X, Y) &= \prod_{k=1}^n \left[X - 2\langle iY \rangle \cos\left(\frac{k\pi}{n+1}\right) \right] \end{aligned} \right.$$

In particular (2):

$$\left\{ \begin{aligned} A_{2n}(X, Y) &= \prod_{k=1}^n \left[X^2 - \left(2\cos\left(\frac{k\pi}{2n+1}\right) Y \right)^2 \right] \\ A_{2n+1}(X, Y) &= X \prod_{k=1}^n \left[X^2 - \left(2\cos\left(\frac{k\pi}{2(n+1)}\right) Y \right)^2 \right] \end{aligned} \right.$$

$$\left\{ \begin{aligned} B_{2n}(X, Y) &= \prod_{k=1}^n \left[X^2 + \left(2\cos\left(\frac{k\pi}{2n+1}\right) Y \right)^2 \right] \\ B_{2n+1}(X, Y) &= X \prod_{k=1}^n \left[X^2 + \left(2\cos\left(\frac{k\pi}{2(n+1)}\right) Y \right)^2 \right]. \end{aligned} \right.$$

Proof. Using (1) and by setting $x = 2\langle y \rangle \cos(\theta)$, we get:

$$\begin{aligned} A_n(\theta, y) &= \frac{\langle y \rangle^n}{2I_\delta(\cos(\theta))} \left[\delta(\cos(\theta))^{n+1} - \bar{\delta}(\cos(\theta))^{n+1} \right] \\ &= \frac{\langle y \rangle^n}{2i \sin(\theta)} \left[e^{i(n+1)\theta} - e^{-i(n+1)\theta} \right] \\ &= \frac{\langle y \rangle^n}{2i \sin(\theta)} 2i \sin((n+1)\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)} \langle y \rangle^n \end{aligned}$$

Thus, the roots $X_{k,n}$ of the polynomial A_n are given by:

$$\begin{aligned} \sin((n+1)\theta) &= 0 \text{ and } \sin(\theta) \neq 0 \\ \Rightarrow (n+1)\theta &= k\pi \quad k \in \llbracket 1, n \rrbracket \Rightarrow \theta = \frac{k\pi}{n+1} \quad k \in \llbracket 1, n \rrbracket \end{aligned}$$

$$X_{k,n} = 2\langle y \rangle \cos\left(\frac{k\pi}{n+1}\right) \quad k \in \llbracket 1, n \rrbracket.$$

The factorization of polynomial A_n is:

$$\begin{aligned} A_n(X, Y) &= \prod_{k=1}^n [X - X_{k,n}] \\ &= \prod_{k=1}^n \left[X - 2\langle Y \rangle \cos\left(\frac{k\pi}{n+1}\right) \right]. \end{aligned}$$

Using property 2, we deduce easily:

$$B_n(X, Y) = \prod_{k=1}^n \left[X - 2\langle iY \rangle \cos\left(\frac{k\pi}{n+1}\right) \right]$$

Noticing that $X_{n+1-k,n} = X_{k,n}$ $k \leq n$ and $\langle iY \rangle^2 = -\langle Y \rangle^2$, we deduce easily (2).

3.11.2. Consequences

Using the different expressions of these two polynomials, we can derive several formulas and several sums. we cite as an example some:

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{C} \quad x^2 \neq 1$$

$$\sum_{k=0}^n (-1)^{n-k} (2x)^{2k} M_{2n}^{2k} = \frac{1}{2\sqrt{x^2-1}} \left(\delta(x)^{2n+1} - \bar{\delta}(x)^{2n+1} \right)$$

$$\forall n \in \mathbb{N} \quad \forall x \in \mathbb{C} \quad x^2 \neq 1$$

$$\begin{aligned} \sum_{k=0}^n (-1)^{n-k} (2x)^{2k+1} M_{2n+1}^{2k+1} \\ = \frac{1}{2\sqrt{x^2-1}} \left(\delta(x)^{2n+2} - \bar{\delta}(x)^{2n+2} \right) \end{aligned}$$

$$\forall n \in \mathbb{N}^* \quad \sum_{k=0}^n (-1)^{n-k} 2^{2k} M_{2n}^{2k} = 2n+1$$

$$\forall n \in \mathbb{N}^* \quad \sum_{k=0}^n (-1)^{n-k} 2^{2k} M_{2n+1}^{2k+1} = n+1$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} \frac{2^{2k}}{2k+1} M_{2n}^{2k} = \frac{1}{2n+1}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n \frac{2^{2k+1}}{k+1} M_{2n+1}^{2k+1} = \frac{1+(-1)^n}{n+1}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n C_{p+k}^k = \sum_{k=0}^n C_{n+p+k}^p = C_{n+p+1}^{p+1}$$

$$\forall n \in \mathbb{N} \quad \prod_{k=0}^n \cos\left(\frac{k\pi}{2n+1}\right) = \frac{1}{2^n}$$

$$\sum_{1 \leq k_1 < \dots < k_{2p} \leq n} \prod_{j=1}^{2p} \cos\left(\frac{k_j \pi}{n+1}\right)$$

$$= (-1)^p \frac{M_n^{n-2p}}{2^{2p}} = (-1)^p \frac{C_{n-p}^p}{2^{2p}}$$

$$\sum_{1 \leq k_1 < k_2 \leq 2n} \cos\left(\frac{k_1 \pi}{2n+1}\right) \cos\left(\frac{k_2 \pi}{2n+1}\right) = -\frac{2n-1}{4}$$

$$\sum_{k=1}^n \cos^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n-1}{4};$$

$$\sum_{k=1}^n \sin^2\left(\frac{k\pi}{2n+1}\right) = \frac{2n+1}{4} \quad (a)$$

$$\sum_{k=1}^n \cos^2\left(\frac{k\pi}{2(n+1)}\right) = \frac{n}{2};$$

$$\sum_{k=1}^n \sin^2\left(\frac{k\pi}{2(n+1)}\right) = \frac{n}{2} \quad (b)$$

$$\sum_{1 \leq k_1 < k_2 \leq 2n+1} \cos\left(\frac{k_1 \pi}{2(n+1)}\right) \cos\left(\frac{k_2 \pi}{2(n+1)}\right) = -\frac{n}{2}$$

$$\begin{aligned} \forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} (2X)^{2k} \\ = \sum_{k=0}^n (-1)^k M_{2(n+k+1)}^{2(n-k)} X^{2(n-k)} (1-X^2)^k \end{aligned}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} 2^{2k} \frac{C_{n+k}^{n-k}}{n-k+1} = \frac{2^{2n+1}-1}{n+1}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^k M_{2n}^{2k} (2\sin(x))^{2k} = \frac{\cos((2n+1)x)}{\cos(x)}$$

$$\forall n \in \mathbb{N}$$

$$\sum_{k=0}^n (-1)^k M_{2n+1}^{2k+1} \left(2\sin\left(\frac{x}{2}\right)\right)^{2k} = \frac{\sin((n+1)x)}{\sin(x)}$$

$$\forall n \in \mathbb{N}^*$$

$$\sum_{k=1}^n (-1)^{k-1} M_{2n-1}^{2k-1} \frac{(2\sin(x))^{2k}}{k} = \frac{2}{n} (1 - \cos(2nx))$$

$$\forall n \in \mathbb{N}$$

$$\sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} (2\cos(x))^{2k+1} = \frac{\sin(2(n+1)x)}{\sin(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} (2\cos(x))^{2k} = \frac{\sin((2n+1)x)}{\sin(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} (2ch(x))^{2k} = \frac{sh((2n+1)x)}{sh(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} M_{2n+1}^{2k+1} \left(2ch\left(\frac{x}{2}\right)\right)^{2k} = \frac{sh((n+1)x)}{sh(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (2sh(x))^{2k} M_{2n}^{2k} = \frac{ch((2n+1)x)}{ch(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n \left(2sh\left(\frac{x}{2}\right)\right)^{2k} M_{2n+1}^{2k+1} = \frac{sh((n+1)x)}{sh(x)}$$

$$\forall n \in \mathbb{N} \quad \sum_{k=0}^n (-1)^{n-k} C_{n+k}^{n-k} = \frac{2}{\sqrt{3}} \sin\left(\frac{2n+1}{3}\pi\right)$$

$$\forall n \in \mathbb{N}$$

$$\sum_{k=0}^n (-1)^{n-k} 2^k C_{n+k}^{n-k} = \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) = (-1)^{\lfloor \frac{n}{2} \rfloor}$$

Another expression of the Dirichlet kernel [7]:

$$D_n(x) = \sum_{k=0}^n (-1)^{n-k} M_{2n}^{2k} \left[2\cos\left(\frac{x}{2}\right)\right]^{2k}$$

A formula verified by the Dirichlet kernels: $\forall k \in \llbracket 1; n \rrbracket$

$$\sum_{p=0}^{n-1} p D_p\left(\frac{2k\pi}{2n+1}\right) = \frac{1}{\sin\left(\frac{k\pi}{2n+1}\right)^2} \left[\frac{n \sin\left(\frac{nk\pi}{2n+1}\right)^2}{-\frac{2n+1}{4}} \right].$$

4. Kernels Matrix

4.1. Definitions

4.1.1. Trigonometric Kernels

We define trigonometric kernel by:

$$\forall (p, q) \in (\llbracket 1; n \rrbracket)^2 \quad \tau_n(p, q) = (-1)^{p-1} \frac{\sin\left(\frac{pq\pi}{n+1}\right)}{\sin\left(\frac{q\pi}{n+1}\right)}$$

4.1.2. Normalized Trigonometric Kernel

We define trigonometric kernel by:

$$\forall (p, q) \in (\llbracket 1; n \rrbracket)^2, \eta_n(p, q) = (-1)^{p-1} \frac{\sin\left(\frac{pq\pi}{n+1}\right)}{\sqrt{\sum_{k=1}^n \sin\left(\frac{k\pi}{n+1}\right)^2}}$$

Proving that $\sum_{k=1}^n \sin\left(\frac{k\pi}{n+1}\right)^2 = \frac{n+1}{2}$ by using formulas

(a) and (b), we deduce that:

$$\forall (p, q) \in (\llbracket 1; n \rrbracket)^2$$

$$\eta_n(p, q) = (-1)^{p-1} \sqrt{\frac{2}{n+1}} \sin\left(\frac{pq\pi}{n+1}\right).$$

4.2. Property 3

We have the following properties:

$$\eta_n(p, q) = (-1)^{p+q} \eta_n(q, p)$$

$$\eta_n(p, q) = 0 \Leftrightarrow pq \equiv 0 \pmod{n+1}$$

$$\eta_n(n+1-p, q) = (-1)^{n-p-q} \eta_n(p, q).$$

4.3. Kernels Matrix

We define the Kernels matrix of size n by:

$$K_n = \begin{pmatrix} \eta_n(1,1) & \eta_n(1,2) & \cdots & \eta_n(1,k) & \cdots & \eta_n(1,n) \\ \eta_n(2,1) & \eta_n(2,2) & \ddots & \eta_n(p,k) & \ddots & \eta_n(p,n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_n(k,1) & \eta_n(k,2) & \ddots & \eta_n(k,k) & \ddots & \eta_n(k,n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_n(n,1) & \eta_n(n,2) & \cdots & \eta_n(n,k) & \cdots & \eta_n(n,n) \end{pmatrix}.$$

4.4. Theorem

Let $n \in \mathbb{N}^*$. Kernel matrix K_n is a orthogonal matrix [8].

Proof. We consider the matrix defined by:

$$T_n(\alpha, \beta, \lambda) = \begin{pmatrix} \beta & \alpha & 0 & 0 & \cdots & \cdots & 0 \\ \lambda & \beta & \alpha & 0 & \ddots & \ddots & 0 \\ 0 & \lambda & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \beta & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \alpha & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda & \beta & \alpha \\ 0 & 0 & \cdots & 0 & 0 & \lambda & \beta \end{pmatrix}$$

Where $(\alpha, \beta, \lambda) \in \mathbb{C}^3$. Let's calculate the determinant of this matrix noted $\Delta_n(\alpha, \beta, \lambda)$ for $n \geq 2$ such that:

$$\Delta_0(\alpha, \beta, \lambda) = 1 \text{ and } \Delta_1(\alpha, \beta, \lambda) = \beta$$

If we develop this determinant from the last column then the resulting determinant from the last row, we get:

$$\Delta_n(\alpha, \beta, \lambda) = \beta \begin{vmatrix} \beta & \alpha & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda & \beta & \alpha & 0 & \ddots & \cdots & \vdots \\ 0 & \lambda & \beta & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \beta & \alpha & 0 \\ \vdots & \cdots & \ddots & 0 & \lambda & \beta & \alpha \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda & \beta \end{vmatrix}$$

$$- \alpha \begin{vmatrix} \beta & \alpha & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda & \beta & \alpha & 0 & \ddots & \cdots & \vdots \\ 0 & \lambda & \beta & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \beta & \alpha & 0 \\ \vdots & \cdots & \ddots & 0 & \lambda & \beta & \alpha \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \lambda \end{vmatrix}$$

$$= \beta \Delta_{n-1}(\alpha, \beta, \lambda) - \alpha \lambda \Delta_{n-2}(\alpha, \beta, \lambda)$$

we easily notice that:

$$\forall n \geq 0 \quad \Delta_n(\alpha, \beta, \lambda) = A_n(\beta, \sqrt{\alpha\lambda})$$

In particular, if $\alpha\lambda$ is a real number then:

$$\Delta_n(\alpha, \beta, \lambda) = \begin{cases} A_n(\beta, \sqrt{\alpha\lambda}) & \text{if } \alpha\lambda \geq 0 \\ B_n(\beta, \sqrt{-\alpha\lambda}) & \text{if } \alpha\lambda \leq 0 \end{cases}$$

By setting $\beta = 2b$ and $\langle a \rangle = \sqrt{\alpha\lambda}$, we obtain:

$$\Delta_n(\alpha, \beta, \lambda) = \begin{cases} \beta^n & \text{if } a = 0 \\ \frac{\langle a \rangle^n}{2I_\delta\left(\frac{b}{\langle a \rangle}\right)} \begin{bmatrix} \delta\left(\frac{b}{\langle a \rangle}\right)^{n+1} \\ -\bar{\delta}\left(\frac{b}{\langle a \rangle}\right)^{n+1} \end{bmatrix} & \text{if } a \neq 0 \end{cases}$$

The matrix is invertible if and only if $\Delta_n(\alpha, \beta, \lambda) \neq 0$. This is a condition that is checked if:

$$b^2 \neq \left[a \cos\left(\frac{k\pi}{n+1}\right) \right]^2 \quad k \in \llbracket 1; n \rrbracket.$$

Using the same previous approach, the characteristic polynomial of $T_n(\alpha, \beta, \lambda)$, after development, verify:

$$\begin{cases} \Delta_0(a, b)(X) = 1 \text{ et } \Delta_1(a, b)(X) = 2b - X \\ \Delta_n(a, b)(X) = (2b - X) \Delta_{n-1}(a, b)(X) \\ \quad - a^2 \Delta_{n-2}(a, b)(X). \end{cases}$$

By setting $U = 2b - X$ and

$$P_n(U, a) = \Delta_n(a, b)(2b - U),$$

we find:

$$\begin{cases} P_0(U, a) = 1 \text{ et } P_1(U, a) = U \\ P_n(U, a) = UP_{n-1}(U, a) - a^2 P_{n-2}(U, a) \quad n \geq 2 \end{cases}$$

Thus, the polynomial P_n satisfies the same recurrence relation as the polynomial A_n : $\forall n \geq 0$,

$$P_n(\alpha, b)(U) = A_n(U, a) = \prod_{k=1}^n \left[U - 2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) \right]$$

$$= \prod_{k=1}^n \left[2b - X - 2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) \right].$$

It is then easy to determine the spectrum of $T_n(\alpha, \beta, \lambda)$:

$$\text{Sp}[T_n(\alpha, \beta, \lambda)] = \left\{ 2b - 2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) / k \in \llbracket 1; n \rrbracket \right\}.$$

Now, we will look for the eigenvectors to diagonalize the matrix $T_n(\alpha, \beta, \lambda)$.

Let

$$X_{k,n} [x_1(n, k), \dots, x_n(n, k)]$$

the eigenvectors associated with the eigenvalues:

$$\nu_{k,n}(a, b) = 2b - 2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right).$$

We proceed to determine the eigenvectors. For that, we consider the numerical sequence defined by:

$$u_p(n, k) = \frac{x_p(n, k)}{x_1(n, k)}.$$

We can easily prove that $x_1(n, k) \neq 0$ and therefore is well defined. We obtain the following system of equations:

$$\begin{cases} u_1(n, k) = 1 \text{ et } \alpha u_2(n, k) = -2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) \\ \lambda u_p(n, k) + \alpha u_{p+2}(n, k) \\ = -2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) u_{p+1}(n, k) \\ \lambda u_{n-1}(n, k) = -2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) u_n(n, k) \end{cases}$$

The characteristic equation of this sequence is given by:

$$\alpha r^2 + 2\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) r + \lambda = 0$$

The reduced discriminant of this equation is:

$$\Delta' = \left[\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) \right]^2 - \alpha \lambda = -\langle a \rangle^2 \sin^2\left(\frac{k\pi}{n+1}\right).$$

And the two solutions are:

$$\begin{cases} r_1 = \frac{-\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) - i \langle a \rangle \sin\left(\frac{k\pi}{n+1}\right)}{\alpha} = -\sqrt{\frac{\lambda}{\alpha}} e^{i \frac{k\pi}{n+1}} \\ r_1 = \frac{-\langle a \rangle \cos\left(\frac{k\pi}{n+1}\right) + i \langle a \rangle \sin\left(\frac{k\pi}{n+1}\right)}{\alpha} = -\sqrt{\frac{\lambda}{\alpha}} e^{-i \frac{k\pi}{n+1}} \end{cases}.$$

The sequence is therefore given by:

$$u_p(n, k) = \left(-\sqrt{\frac{\lambda}{\alpha}} \right)^p \begin{pmatrix} a_{n,k} e^{i \frac{pk\pi}{n+1}} \\ + b_{n,k} e^{-i \frac{pk\pi}{n+1}} \end{pmatrix} (a_{n,k}, b_{n,k}) \in \mathbb{C}^2.$$

Using the initial conditions, after solving the system:

$$\begin{aligned} u_p(n, k) &= \left(-\sqrt{\frac{\lambda}{\alpha}} \right)^p \frac{i \sqrt{\frac{\alpha}{\lambda}}}{2 \sin\left(\frac{k\pi}{n+1}\right)} \begin{pmatrix} e^{i \frac{pk\pi}{n+1}} - e^{-i \frac{pk\pi}{n+1}} \end{pmatrix} \\ &= (-u)^{p-1} \frac{\sin\left(\frac{pk\pi}{n+1}\right)}{\sin\left(\frac{k\pi}{n+1}\right)} \quad \left(u = \sqrt{\frac{\lambda}{\alpha}} \right) \end{aligned}$$

Then, we divide by the norm $\| \cdot \|_2$ in order to normalize the eigenvector:

$$\begin{aligned} \eta_{pq}(u) &= \frac{u_p(n, q)}{\|X_{k,n}\|_2} \\ &= (-u)^{p-1} \frac{\sin\left(\frac{pq\pi}{n+1}\right)}{\sqrt{\sum_{k=1}^n \left[(-u)^{k-1} \sin\left(\frac{kq\pi}{n+1}\right) \right]^2}}. \end{aligned}$$

But for $u = 1$, we have $\alpha = \lambda$ and more:

$$\begin{aligned} \eta_{pq}(1) &= \frac{u_p(n, q)}{\|X_{k,n}\|_2} \\ &= (-1)^{p-1} \frac{\sin\left(\frac{pq\pi}{n+1}\right)}{\sqrt{\sum_{k=1}^n \sin^2\left(\frac{kq\pi}{n+1}\right)}} = \eta_n(p, q) \end{aligned}$$

It is the normalized kernel which is the component of index (p, q) of the kernels matrix.

Now, let $(\alpha, \beta, \lambda) \in \mathbb{R}^3$. Given that $u = 1$, $T_n(\alpha, \beta, \lambda)$ will be symmetric real matrix [8], therefore its passage matrix will be orthogonal. Moreover, the eigenvectors are normalized; which allows us to conclude that the kernels matrix is orthonormal.

4.5. Kernels Determinant

The determinant of the kernels matrix is called kernels determinant. It's noted $\chi(n)$:

$$\chi(n) = \begin{vmatrix} \eta_n(1,1) & \eta_n(1,2) & \cdots & \eta_n(1,k) & \cdots & \eta_n(1,n) \\ \eta_n(2,1) & \eta_n(2,2) & \ddots & \eta_n(p,k) & \ddots & \eta_n(p,n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_n(k,1) & \eta_n(k,2) & \ddots & \eta_n(k,k) & \ddots & \eta_n(k,n) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_n(n,1) & \eta_n(n,2) & \cdots & \eta_n(n,k) & \cdots & \eta_n(n,n) \end{vmatrix}.$$

4.6. Corollary

We can easily deduce that: $\forall n \in \mathbb{N}, |\chi(n)| = 1$.

5. Applications

a) If we note $\phi(n)$ the following determinant:

$$\begin{vmatrix} \sin\left(\frac{\pi}{n+1}\right) & \cdots & \sin\left(\frac{k\pi}{n+1}\right) & \cdots & \sin\left(\frac{n\pi}{n+1}\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{k-1} \sin\left(\frac{k\pi}{n+1}\right) & \cdots & (-1)^{k-1} \sin\left(\frac{k^2\pi}{n+1}\right) & \cdots & (-1)^{k-1} \sin\left(\frac{kn\pi}{n+1}\right) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{n-1} \sin\left(\frac{n\pi}{n+1}\right) & \cdots & (-1)^{n-1} \sin\left(\frac{nk\pi}{n+1}\right) & \cdots & (-1)^{n-1} \sin\left(\frac{n^2\pi}{n+1}\right) \end{vmatrix}$$

So: $\forall n \in \mathbb{N}, |\phi(n)| = \sqrt{\left(\frac{n+1}{2}\right)^n}$.

b) For n sufficiently large, we have the following equivalence: $|\phi_n| \sim \left(\frac{n}{2}\right)^{\frac{n}{2}} \sqrt{e}$

c) We have the following special case [9]:

$$\forall n \in \mathbb{N}, A_n(2X, 1) = U_n(X)$$

Where U_n is the polynomial of Chebyshev of the 2nd species. We can thus deduce several results on the factorial means using the polynomials of Chebyshev. We can also easily deduce a recurring relation verified by U_n :

$$\begin{cases} U_0(X, 1) = 1 \text{ and } U_1(X, 1) = 2X \\ U_{n+2}(X) = 2XU_{n+1}(X) - U_n(X) \end{cases}$$

d) Example for kernels matrix ($n = 4$):

$$K_4 = \frac{1}{2} \begin{pmatrix} \sqrt{\sqrt{5}-1} & \sqrt{\sqrt{5}+1} & \sqrt{\sqrt{5}+1} & \sqrt{\sqrt{5}-1} \\ -\sqrt{\sqrt{5}+1} & -\sqrt{\sqrt{5}-1} & \sqrt{\sqrt{5}-1} & \sqrt{\sqrt{5}+1} \\ \sqrt{\sqrt{5}+1} & -\sqrt{\sqrt{5}-1} & -\sqrt{\sqrt{5}+1} & \sqrt{\sqrt{5}+1} \\ -\sqrt{\sqrt{5}-1} & \sqrt{\sqrt{5}+1} & -\sqrt{\sqrt{5}+1} & \sqrt{\sqrt{5}-1} \end{pmatrix}$$

e) Resolution of a recurrence sequence

Let $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{R}^*$. We consider the recurrent sequence defined by:

$$\begin{cases} (u_1, u_2) \in \mathbb{C}^2 \\ \alpha u_{p-1} + \beta u_p + \bar{\alpha} u_{p+1} = v_p \quad p \geq 2 \end{cases}$$

We complete this system by defining v_1 :

$$\begin{cases} \beta u_1 + \bar{\alpha} u_2 = v_1 \\ \alpha u_{p-1} + \beta u_p + \bar{\alpha} u_{p+1} = v_p \quad 2 \leq p \leq n-1 \\ \alpha u_{n-1} + \beta u_n = v_n - \bar{\alpha} u_{n+1} \end{cases}$$

By writing this linear system in its matrix form, we get:

$$T_n(\alpha, \beta, \bar{\alpha}) U_n = V_n$$

Where U_n and V_n are the column vectors of u_p and v_p (Last component of V_n is $v_n - \bar{\alpha} u_{n+1}$). We notice that

$T_n(\alpha, \beta, \bar{\alpha})$ is a Hermitian matrix [8]. In this case, we have $\beta = 2b$ and $\langle a \rangle = \sqrt{\alpha \bar{\alpha}} = \|\alpha\|$. Also, the eigenvalues of the matrix are given by:

$$v_{k,n}(\alpha, b) = 2 \left[b - \|\alpha\| \cos\left(\frac{k\pi}{n+1}\right) \right]$$

If we suppose that $b^2 \neq \left[\|\alpha\| \cos\left(\frac{k\pi}{n+1}\right) \right]^2 \quad k \in \llbracket 1; n \rrbracket$, then the matrix $T_n(\alpha, \beta, \bar{\alpha})$ will be invertible. We note the following trigonometric kernels:

$$\eta_{pq} = (-1)^{p-1} e^{i(p-1)\arg(\alpha)} \frac{\sin\left(\frac{pq\pi}{n+1}\right)}{\sqrt{\sum_{k=1}^n \left[e^{i(k-1)\arg(\alpha)} \sin\left(\frac{kq\pi}{n+1}\right) \right]^2}}$$

We use diagonalization formula to reverse $T_n^{-1}(\alpha, \beta, \bar{\alpha})$:

$$T_n^{-1}(\alpha, \beta, \bar{\alpha}) = K_n(\alpha) D_n^{-1}(\alpha, \beta, \bar{\alpha}) K_n^*(\alpha)$$

Where $D_n(\alpha, \beta, \bar{\alpha})$ is the diagonal matrix similar to $T_n(\alpha, \beta, \bar{\alpha})$. We obtain then:

$$T_n^{-1} = \begin{pmatrix} \sum_{p=1}^n \frac{\|\eta_{1p}\|^2}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\eta_{1p} \bar{\eta}_{kp}}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\eta_{1p} \bar{\eta}_{np}}{v_{n,p}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sum_{p=1}^n \frac{\eta_{kp} \bar{\eta}_{1p}}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\|\eta_{kp}\|^2}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\eta_{kp} \bar{\eta}_{np}}{v_{n,p}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \sum_{p=1}^n \frac{\eta_{np} \bar{\eta}_{1p}}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\eta_{np} \bar{\eta}_{kp}}{v_{n,p}} & \cdots & \sum_{p=1}^n \frac{\|\eta_{np}\|^2}{v_{n,p}} \end{pmatrix}$$

The following sums are called arms:

$$m_{pq}(\alpha, \beta) = \sum_{k=1}^n \frac{\eta_{pk} \bar{\eta}_{qk}}{v_{n,k}}$$

We easily obtain the u_k which are given by:

$$\begin{aligned} u_k &= \sum_{j=1}^n m_{kj}(\alpha, \beta) v_j - \bar{\alpha} m_{kn}(\alpha, \beta) u_{n+1} \\ &= \sum_{j=1}^n \sum_{p=1}^n \frac{\eta_{kp} \bar{\eta}_{jp}}{v_{n,p}} v_j - \bar{\alpha} \sum_{p=1}^n \frac{\eta_{kp} \bar{\eta}_{np}}{v_{n,p}} u_{n+1} \end{aligned}$$

For $k = 1$, we deduce the expression of u_{n+1} :

$$u_{n+1} = \frac{1}{\bar{\alpha} m_{1n}(\alpha, \beta)} \left(\sum_{j=1}^n m_{1j}(\alpha, \beta) v_j - u_1 \right)$$

Then, we replace u_{n+1} by its expression in order to obtain all the u_k using the known data of the problem:

$$\begin{aligned} u_k &= \sum_{j=1}^n m_{kj}(\alpha, \beta) v_j - \bar{\alpha} m_{kn}(\alpha, \beta) u_{n+1} \\ &= \sum_{j=1}^n \left(m_{kj}(\alpha, \beta) - \frac{m_{kn}(\alpha, \beta) m_{1j}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} \right) v_j \\ &\quad + \frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} u_1. \end{aligned}$$

f) Particular case

If we suppose that $v_p = 0 \quad 2 \leq p \leq n$:

$$\begin{cases} (u_1, u_2) \in \mathbb{C}^2 \\ \alpha u_{p-1} + \beta u_p + \bar{\alpha} u_{p+1} = 0 \quad p \geq 2 \end{cases}$$

Using the previous formula, we can write:

$$\begin{aligned} u_k &= \left(m_{k1}(\alpha, \beta) - \frac{m_{kn}(\alpha, \beta) m_{11}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} \right) v_1 \\ &\quad + \frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} u_1 \\ &= \left(\frac{m_{k1}(\alpha, \beta) - \frac{m_{kn}(\alpha, \beta) m_{11}(\alpha, \beta)}{m_{1n}(\alpha, \beta)}}{m_{1n}(\alpha, \beta)} \right) (\beta u_1 + \bar{\alpha} u_2) \\ &\quad + \frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} u_1 \\ &= \left[\left(\beta m_{k1}(\alpha, \beta) + \frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} (1 - \beta m_{11}(\alpha, \beta)) \right) u_1 \right. \\ &\quad \left. + \bar{\alpha} \left(m_{k1}(\alpha, \beta) - \frac{m_{kn}(\alpha, \beta) m_{11}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} \right) u_2 \right]. \end{aligned}$$

Now we use the classic method of the characteristic equation ($\alpha \in \mathbb{C}^*$):

$$\bar{\alpha} r^2 + 2br + \alpha = 0$$

The reduced discriminant: $\Delta' = b^2 - \|\alpha\|^2$. The two solutions of this equation are:

$$\begin{aligned} r_1 &= \frac{-b - \sqrt{b^2 - \|\alpha\|^2}}{\bar{\alpha}} \\ &= -\frac{\|\alpha\|}{\bar{\alpha}} \left(\frac{b}{\|\alpha\|} + \sqrt{\left(\frac{b}{\|\alpha\|} \right)^2 - 1} \right) \\ &= -\sqrt{\frac{\alpha}{\bar{\alpha}}} \delta \left(\frac{b}{\|\alpha\|} \right) = -e^{i \arg(\alpha)} \delta \left(\frac{b}{\|\alpha\|} \right) \\ r_1 &= -e^{i \arg(\alpha)} \bar{\delta} \left(\frac{b}{\|\alpha\|} \right) \end{aligned}$$

The recurrent sequence is written in the form:

$$u_k = (-1)^k e^{ik \arg(\alpha)} \left(t \delta^k \left(\frac{b}{\|\alpha\|} \right) + s \bar{\delta}^k \left(\frac{b}{\|\alpha\|} \right) \right)$$

Where $(t, s) \in \mathbb{C}^2$. To determine these constants, we use the initial conditions. We find:

$$\begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{2e^{2i \arg(\alpha)} I_\delta \left(\frac{b}{\|\alpha\|} \right)} \begin{pmatrix} e^{i \arg(\alpha)} \bar{\delta} \left(\frac{b}{\|\alpha\|} \right)^2 u_1 \\ -\bar{\delta} \left(\frac{b}{\|\alpha\|} \right) u_2 \\ -e^{i \arg(\alpha)} \delta \left(\frac{b}{\|\alpha\|} \right)^2 u_1 \\ +\delta \left(\frac{b}{\|\alpha\|} \right) u_2 \end{pmatrix}.$$

Then, we obtain the expression of the recurrent sequence:

$$u_k = \frac{(-1)^k e^{i(k-2) \arg(\alpha)}}{2I_\delta \left(\frac{b}{\|\alpha\|} \right)} \begin{pmatrix} e^{i \arg(\alpha)} \begin{bmatrix} \delta^{k-2} \left(\frac{b}{\|\alpha\|} \right) \\ -\bar{\delta}^{k-2} \left(\frac{b}{\|\alpha\|} \right) \end{bmatrix} u_1 \\ - \begin{bmatrix} \delta^{k-1} \left(\frac{b}{\|\alpha\|} \right) \\ -\bar{\delta}^{k-1} \left(\frac{b}{\|\alpha\|} \right) \end{bmatrix} u_2 \end{pmatrix}.$$

Since u_1 and u_2 can be chosen arbitrarily, we can identify their coefficients in the two expressions obtained from the recurrent sequence in question. We thus obtain:

$$\begin{aligned} &\beta m_{k1}(\alpha, \beta) + \frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} (1 - \beta m_{11}(\alpha, \beta)) \\ &= \frac{(-1)^k e^{i(k-1) \arg(\alpha)}}{2I_\delta \left(\frac{b}{\|\alpha\|} \right)} \left[\delta^{k-2} \left(\frac{b}{\|\alpha\|} \right) - \bar{\delta}^{k-2} \left(\frac{b}{\|\alpha\|} \right) \right]. \quad (4) \end{aligned}$$

And also:

$$\begin{aligned} &\bar{\alpha} \left(m_{k1}(\alpha, \beta) - \frac{m_{kn}(\alpha, \beta) m_{11}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} \right) \\ &= \frac{(-1)^{k-1} e^{i(k-2) \arg(\alpha)}}{2I_\delta \left(\frac{b}{\|\alpha\|} \right)} \left[\delta^{k-1} \left(\frac{b}{\|\alpha\|} \right) - \bar{\delta}^{k-1} \left(\frac{b}{\|\alpha\|} \right) \right]. \quad (5) \end{aligned}$$

By making an adequate subtraction between (4) and (5), we can obtain:

$$\frac{m_{kn}(\alpha, \beta)}{m_{1n}(\alpha, \beta)} = \frac{(-1)^k e^{i(k-2)\arg(\alpha)}}{2I_\delta\left(\frac{b}{\|\alpha\|}\right)} \left\{ e^{i\arg(\alpha)} \begin{bmatrix} \delta^{k-2}\left(\frac{b}{\|\alpha\|}\right) \\ -\bar{\delta}^{k-2}\left(\frac{b}{\|\alpha\|}\right) \end{bmatrix} + \frac{\beta}{\bar{\alpha}} \begin{bmatrix} \delta^{k-1}\left(\frac{b}{\|\alpha\|}\right) \\ -\bar{\delta}^{k-1}\left(\frac{b}{\|\alpha\|}\right) \end{bmatrix} \right\}$$

$$\sum_{p=1}^n \frac{(-1)^p \sin\left(\frac{kp\pi}{n+1}\right) \sin\left(\frac{p\pi}{n+1}\right)}{1 - \cos\left(\frac{p\pi}{n+1}\right)} = (3k-4) \sum_{p=1}^n \frac{(-1)^{p-1} \sin\left(\frac{p\pi}{n+1}\right)^2}{1 - \cos\left(\frac{p\pi}{n+1}\right)}$$

Where:

$$m_{kn}(\alpha, \beta) = \frac{1}{2} \sum_{p=1}^n \frac{\left[(-1)^{n+k+p+1} e^{-i(n-k)\arg(\alpha)} \times \sin\left(\frac{kp\pi}{n+1}\right) \sin\left(\frac{p\pi}{n+1}\right) \right]}{\left\{ \left[b - \|\alpha\| \cos\left(\frac{p\pi}{n+1}\right) \right] \times \left\| \sum_{q=1}^n \left[e^{i(q-1)\arg(\alpha)} \sin\left(\frac{pq\pi}{n+1}\right) \right] \right\|^2 \right\}}$$

e) Examples.

Using the preceding relations, one can establish several formulas and values of sums. We will cite a few examples below, especially for cases $b = \alpha$ and $b = 2\alpha$ (case of the equation of the three moments in structural mechanics).

In the case where α is a real, its main argument is therefore either 0 (if it is positive) or π (if it is negative). We find in these two cases the following formula:

If we note $b = \lambda\alpha$, then:

$$\frac{\sum_{p=1}^n \frac{(-1)^p \sin\left(\frac{kp\pi}{n+1}\right) \sin\left(\frac{p\pi}{n+1}\right)}{\lambda - \cos\left(\frac{p\pi}{n+1}\right)}}{\sum_{p=1}^n \frac{(-1)^p \sin\left(\frac{p\pi}{n+1}\right)^2}{\lambda - \cos\left(\frac{p\pi}{n+1}\right)}} = -\frac{1}{2I_\delta(\lambda)} \left\{ \begin{bmatrix} \delta^{k-2}(\lambda) \\ -\bar{\delta}^{k-2}(\lambda) \end{bmatrix} + 2\lambda \begin{bmatrix} \delta^{k-1}(\lambda) \\ -\bar{\delta}^{k-1}(\lambda) \end{bmatrix} \right\}$$

And we get for $\lambda \rightarrow 1$ (using limited development):

get for $\lambda = 2$:

$$\frac{\sum_{p=1}^n \frac{(-1)^p \sin\left(\frac{kp\pi}{n+1}\right) \sin\left(\frac{p\pi}{n+1}\right)}{2 - \cos\left(\frac{p\pi}{n+1}\right)}}{\sum_{p=1}^n \frac{(-1)^p \sin\left(\frac{p\pi}{n+1}\right)^2}{2 - \cos\left(\frac{p\pi}{n+1}\right)}} = -\frac{1}{2\sqrt{3}} \left\{ \begin{bmatrix} \left[(2+\sqrt{3})^{k-2} - (2-\sqrt{3})^{k-2} \right] \\ +4 \left[(2+\sqrt{3})^{k-1} - (2-\sqrt{3})^{k-1} \right] \end{bmatrix} \right\}$$

6. Conclusion

The twin polynomials are well introduced and several formulas are established. Also, we have well defined the kernel matrix and we have shown that it is orthogonal. We have thus proposed a method for solving recurrent sequences of order 3. Finally, other future researches will be based on this article as a fundamental reference.

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