Turkish Journal of Analysis and Number Theory, 2019, Vol. 7, No. 4, 117-123
Available online at http://pubs.sciepub.com/tjant/7/4/4
Published by Science and Education Publishing
DOI:10.12691/tjant-7-4-4

Common Fixed Point Theorems in *b*-Metric Spaces

Yan Hao, Hongyan Guan*

School of Mathematics and Systems Science, Shenyang Normal University, Shenyang 110034, China
*Corresponding author: guanhy8010@163.com

Received June 03, 2019; Revised July 06, 2019; Accepted August 07, 2019

Abstract In this paper, we prove some common fixed point results for two mappings satisfying contraction conditions in complete *b*-metric spaces. Meanwhile, two examples are presented to support our results.

2010 *Mathematics Subject Classification*. Primary 47H10.

Keywords: *common fixed point, b-metric space, contraction condition.*

Cite This Article: Yan Hao, and Hongyan Guan, "Common Fixed Point Theorems in *b*-Metric Spaces." *Turkish Journal of Analysis and Number Theory*, vol. 7, no. 4 (2019): 117-123. doi: 10.12691/tjant-7-4-4.

1. Introduction

In 1922, Banach [1] proved the Banach contraction principle. Since then, several works have been done about fixed point theory regarding different classes of contractive conditions in some spaces such as: quasi-metric spaces [2,3], cone metric spaces [4,5], partially order metric spaces [6,7,8], G-metric spaces [9].

The concept of $b-$ metric space was introduced by Czerwik in [10]. After that, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in $b-$ metric spaces (see [2,11,12]). Aydi et al. in [13] proved common fixed point results for single-valued and multi-valued mappings satisfying a weak $\phi-$contraction in $b-$ metric spaces. Starting from the results of Berinde [14], Pacurar [15] proved the existence and uniqueness of fixed point of $\phi-$ contractions on *b*- metric spaces. Using a contraction condition defined by means of a comparison function, [16] established results regarding the common fixed points of two mappings. Hussain and Shah in [17] introduced the notion of a cone *b*- metric spaces, generalizing both the notions of *b*- metric spaces and cone metric spaces, they considered topological properties of cone $b-$metric spaces and results on KKM mappings in the setting of cone *b*- metric spaces.

The aim of this paper is to consider and establish some common fixed point results for two mappings satisfying contraction conditions in complete *b*-metric spaces. Meanwhile, two examples are presented to support our results.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of all real numbers and nonnegative numbers respectively. N denotes the set of positive integers and $N_0 = N\bigcup\{0\}$. Suppose

$\Phi = \{\phi \mid \phi:(\mathbb{R}^+)^7 \to \mathbb{R}^+$ is upper semicontinuous and nondecreasing in each coordinate variable satisfying condition $\phi(t,t,t,t,t,t,t) = \overline{\phi}(t) < t\}$

and

$\Psi = \{\psi \mid \psi:(\mathbb{R}^+)^9 \to \mathbb{R}^+$ is upper semicontinuous and nondecreasing in each coordinate variable satisfying condition $\psi(t,t,t,t,t,t,t,t,t) = \overline{\psi}(t) < t\}$.

In order to obtain our main results, we need to introduce some definitions and lemmas.

Definition. Let X be a nonempty set and $d: X\times X \to [0,+\infty)$. A function d is called a $b-$metric with constant $s \ge 1$ if

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,y)\le s(d(x,z)+d(y,z))$ for all $x,y,z\in X$.

The pair (X,d) is called a $b-$metric space.

It is obvious a $b-$metric space with $s=1$ is a metric space. There are examples of $b-$metric spaces which are not metric spaces. (see [18])

Definition. Let $\{x_n\}$ be a sequence in a $b-$metric space (X,d).

(1) A sequence $\{x_n\}$ is called convergent if and only if there is $x\in X$ such that $d(x_n,x)\to 0$ when $n\to+\infty$;

(2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_m)\to 0$ when $n,m\to+\infty$.

As usual, a *b*-metric space is said to be complete if and only if each Cauchy sequence in this space is convergent.

Lemma 2.1. [19] *Let* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ *be nondecreasing and upper semicontinuous. Then for each* $t>0$, $\psi(t)<t$ *if and only if* $\lim\limits_{n\to\infty}\psi^n(t)=0$.

3. Main Results

Now we are ready to prove our main results.

Theorem 3.1. *Let* (X,d) *be a complete* $b-$*metric space with constant* $s \geq 1$. *Suppose* A *and* $B: X \to X$ *are two mappings and one of them is continuous. If there exists* $\phi \in \Phi$ *such that*

$$s^4 d(Ax, By) \leq \phi\begin{pmatrix} d(Ax,x), d(By,y), d(x,y), \\ \frac{d(Ax,x)+d(By,y)}{2}, \frac{d(Ax,x)+d(x,y)}{2}, \\ \frac{d(By,y)+d(x,y)}{2}, \frac{d(Ax,y)+d(By,x)}{2} \end{pmatrix} \quad (1)$$

for all $x, y \in X$ *, then* A *and* B *have a unique common fixed point* $x^* \in X$.

Proof. *Let* $x_0 \in X$ *be arbitrary. We define a sequence* $\{x_n\}$ *as follows*:

$$x_{2n+1} = Ax_{2n}, x_{2n+2} = Bx_{2n+1}, n \in N.$$

We now suppose that $d(x_n, x_{n+1}) > 0$ for every n. If not, there exists some $n \in N$ such that $x_n = x_{n+1}$. If $n = 2k$, then $x_{2k} = x_{2k+1}$ and from the contraction condition (1) with $x = x_{2k}$ and $y = x_{2k+1}$, we have

$$\begin{aligned} s^4 d(x_{2k+1}, x_{2k+2}) &= s^4 d(Ax_{2k}, Bx_{2k+1}) \\ &\leq \phi\begin{pmatrix} d(Ax_{2k}, x_{2k}), d(Bx_{2k+1}, x_{2k+1}), \\ d(x_{2k}, x_{2k+1}), \\ \frac{d(Ax_{2k}, x_{2k}) + d(Bx_{2k+1}, x_{2k+1})}{2}, \\ \frac{d(Ax_{2k}, x_{2k}) + d(x_{2k}, x_{2k+1})}{2}, \\ \frac{d(Bx_{2k+1}, x_{2k+1}) + d(x_{2k}, x_{2k+1})}{2}, \\ \frac{d(Ax_{2k}, x_{2k+1}) + d(Bx_{2k+1}, x_{2k})}{2} \end{pmatrix} \\ &= \phi\begin{pmatrix} d(x_{2k+1}, x_{2k}), d(x_{2k+2}, x_{2k+1}), \\ d(x_{2k}, x_{2k+1}), \\ \frac{d(x_{2k+1}, x_{2k}) + d(x_{2k+2}, x_{2k+1})}{2}, \\ \frac{d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k+1})}{2}, \\ \frac{d(x_{2k+2}, x_{2k+1}) + d(x_{2k}, x_{2k+1})}{2}, \\ \frac{d(x_{2k+1}, x_{2k+1}) + d(x_{2k+2}, x_{2k})}{2} \end{pmatrix}. \end{aligned}$$

Suppose that $d(x_{2k+1}, x_{2k+2}) > d(x_{2k}, x_{2k+1}) = 0$. It follows from the definition of ϕ that

$$\begin{aligned} &s^4 d(x_{2k+1}, x_{2k+2}) \\ &\leq \phi\begin{pmatrix} d(x_{2k+1}, x_{2k+2}), d(x_{2k+1}, x_{2k+2}), \\ d(x_{2k+1}, x_{2k+2}), \\ d(x_{2k+1}, x_{2k+2}), d(x_{2k+1}, x_{2k+2}), \\ d(x_{2k+1}, x_{2k+2}), sd(x_{2k+1}, x_{2k+2}) \end{pmatrix} \\ &\leq \phi\begin{pmatrix} s^4 d(x_{2k+1}, x_{2k+2}), s^4 d(x_{2k+1}, x_{2k+2}), \\ s^4 d(x_{2k+1}, x_{2k+2}), \\ s^4 d(x_{2k+1}, x_{2k+2}), s^4 d(x_{2k+1}, x_{2k+2}), \\ s^4 d(x_{2k+1}, x_{2k+2}), s^4 d(x_{2k+1}, x_{2k+2}) \end{pmatrix} \\ &= \bar{\phi}(s^4 d(x_{2k+1}, x_{2k+2})) < s^4 d(x_{2k+1}, x_{2k+2}), \end{aligned}$$

which is a contradiction. Therefore, $d(x_{2k+1}, x_{2k+2}) = 0$. By the definition of the sequence $\{x_n\}$, it means that $x_{2k} = Ax_{2k} = Bx_{2k}$. That is, x_{2k} is a common fixed point of A and B.

If $n = 2k+1$, then using the same arguments in the case $x_{2k} = x_{2k+1}$, it can be shown that x_{2k+1} is a common fixed point of A and B.

From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in N_0$. Now we shall prove that

$$s^4 d(x_n, x_{n+1}) \leq \bar{\phi}(s^4 d(x_n, x_{n+1})), \text{for each } n \in N_0. \quad (2)$$

We consider two cases:

Case I: $n = 2k, k \in N$. From the contraction condition (1) with $x = x_{2k}$ and $y = x_{2k-1}$, we get

$$\begin{aligned} &s^4 d(x_{2k+1}, x_{2k}) = s^4 d(Ax_{2k}, Bx_{2k-1}) \\ &\leq \phi\begin{pmatrix} d(Ax_{2k}, x_{2k}), d(Bx_{2k-1}, x_{2k-1}), \\ d(x_{2k}, x_{2k-1}), \frac{d(Ax_{2k}, x_{2k}) + d(Bx_{2k-1}, x_{2k-1})}{2}, \\ \frac{d(Ax_{2k}, x_{2k}) + d(x_{2k}, x_{2k-1})}{2}, \\ \frac{d(Bx_{2k-1}, x_{2k-1}) + d(x_{2k}, x_{2k-1})}{2}, \\ \frac{d(Ax_{2k}, x_{2k-1}) + d(Bx_{2k-1}, x_{2k})}{2}) \end{pmatrix} \\ &= \phi\begin{pmatrix} d(x_{2k+1}, x_{2k}), d(x_{2k}, x_{2k-1}), d(x_{2k}, x_{2k-1}), \\ \frac{d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k-1})}{2}, \\ \frac{d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k-1})}{2}, \\ \frac{d(x_{2k}, x_{2k-1}) + d(x_{2k}, x_{2k-1})}{2}, \\ \frac{d(x_{2k+1}, x_{2k-1}) + d(x_{2k}, x_{2k})}{2} \end{pmatrix}. \end{aligned}$$

If $d(x_{2k+1}, x_{2k}) > d(x_{2k}, x_{2k-1})$, by virtue of the definition of ϕ, one can obtain

$$s^4 d(x_{2k+1},x_{2k}) \le \phi\begin{pmatrix} d(x_{2k+1},x_{2k}), d(x_{2k+1},x_{2k}), d(x_{2k+1},x_{2k}), \\ d(x_{2k+1},x_{2k}), d(x_{2k+1},x_{2k}), \\ d(x_{2k+1},x_{2k}), sd(x_{2k+1},x_{2k}) \end{pmatrix} \le \phi\begin{pmatrix} s^4 d(x_{2k+1},x_{2k}), s^4 d(x_{2k+1},x_{2k}), \\ s^4 d(x_{2k+1},x_{2k}), \\ s^4 d(x_{2k+1},x_{2k}), s^4 d(x_{2k+1},x_{2k}), \\ s^4 d(x_{2k+1},x_{2k}), s^4 d(x_{2k+1},x_{2k}) \end{pmatrix} = \overline{\phi}(s^4 d(x_{2k+1},x_{2k})) < s^4 d(x_{2k+1},x_{2k}),$$

a contradiction. It follows that

$$d(x_{2k+1},x_{2k}) \le d(x_{2k},x_{2k-1}).$$

Hence,

$$s^4 d(x_{2k+1},x_{2k}) \le \overline{\phi}(s^4 d(x_{2k},x_{2k-1})), \text{for each } k \in N. \quad (3)$$

Case II: $n = 2k+1, k \in N_0$. Using the same technique in proving the case I, it can be proved that (2) holds for $n = 2k+1$. That is,

$$s^4 d(x_{2k+1},x_{2k+2}) \le \overline{\phi}(s^4 d(x_{2k+1},x_{2k})), \text{ for each } k \in N_0. \quad (4)$$

From (3) and (4), we conclude that (2) holds for all $n \in N_0$.

Since $\overline{\phi}(t) < t$ for all $t > 0$, using Lemma 2.3, we obtain that $\lim_{n\to+\infty} \overline{\phi}^n(t) = 0$ for all $t > 0$. It follows that

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0. \quad (5)$$

Now we prove that $\{x_n\}$ is a Cauchy sequence. To do this, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence in X. Assume on the contrary that $\{x_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequence $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ so that $2m_k$ is the smallest index for which $2m_k > 2n_k > k$,

$$d(x_{2m_k},x_{2n_k}) \ge \varepsilon \quad (6)$$

and

$$d(x_{2m_k-2},x_{2n_k}) < \varepsilon. \quad (7)$$

Using the triangle inequality in b-metric space and (6), we have

$$\begin{aligned} \varepsilon &\le d(x_{2m_k},x_{2n_k}) \\ &\le sd(x_{2m_k-2},x_{2n_k}) + s^2 d(x_{2m_k-2},x_{2m_k-1}) \\ &\quad + s^2 d(x_{2m_k-1},x_{2m_k}) \\ &\le \varepsilon s + s^2 d(x_{2m_k-2},x_{2m_k-1}) + s^2 d(x_{2m_k-1},x_{2m_k}). \end{aligned}$$

Taking the upper limit as $k \to +\infty$, one can obtain

$$\varepsilon \le \limsup_{k\to+\infty} d(x_{2m_k},x_{2n_k}) \le \varepsilon s. \quad (8)$$

Also,

$$\begin{aligned} \varepsilon &\le \limsup_{k\to+\infty} d(x_{2m_k},x_{2n_k}) \\ &\le sd(x_{2m_k},x_{2m_k-1}) + sd(x_{2m_k-1},x_{2n_k}), \end{aligned}$$

hence,

$$\frac{\varepsilon}{s} \le \limsup_{k\to+\infty} d(x_{2m_k-1},x_{2n_k}).$$

On the other hand, we get

$$d(x_{2m_k-1},x_{2n_k}) \le sd(x_{2m_k-1},x_{2m_k}) + sd(x_{2m_k},x_{2n_k}).$$

It follows from (5) and (8) that

$$\limsup_{k\to+\infty} d(x_{2m_k-1},x_{2n_k}) \le s \limsup_{k\to+\infty} d(x_{2m_k},x_{2n_k}) \le s^2\varepsilon.$$

Consequently,

$$\frac{\varepsilon}{s} \le \limsup_{k\to+\infty} d(x_{2m_k-1},x_{2n_k}) \le s^2\varepsilon. \quad (9)$$

Similarly, we deduce that

$$\frac{\varepsilon}{s^2} \le \limsup_{k\to+\infty} d(x_{2m_k-1},x_{2n_k+1}) \le s^3\varepsilon. \quad (10)$$

Using the triangle inequality in b-metric space and contraction condition (1), we have

$$\begin{aligned} &d(x_{2n_k},x_{2m_k}) \\ &\le sd(x_{2n_k},x_{2n_k+1}) + sd(x_{2n_k+1},x_{2m_k}) \\ &\le sd(x_{2n_k},x_{2n_k+1}) + sd(Ax_{2n_k},Bx_{2m_k-1}) \\ &\le sd(x_{2n_k},x_{2n_k+1}) \\ &+\frac{1}{s^3}\phi\begin{pmatrix} d(Ax_{2n_k},x_{2n_k}), d(Bx_{2m_k-1},x_{2m_k-1}), d(x_{2n_k},x_{2m_k-1}), \\ \frac{d(Ax_{2n_k},x_{2n_k}) + d(Bx_{2m_k-1},x_{2m_k-1})}{2}, \\ \frac{d(Ax_{2n_k},x_{2n_k}) + d(x_{2n_k},x_{2m_k-1})}{2}, \\ \frac{d(Bx_{2m_k-1},x_{2m_k-1}) + d(x_{2n_k},x_{2m_k-1})}{2}, \\ \frac{d(Ax_{2n_k},x_{2m_k-1}) + d(Bx_{2m_k-1},x_{2n_k})}{2} \end{pmatrix} \\ &= sd(x_{2n_k},x_{2n_k+1}) \\ &+\frac{1}{s^3}\phi\begin{pmatrix} d(x_{2n_k+1},x_{2n_k}), d(x_{2m_k},x_{2m_k-1}), d(x_{2n_k},x_{2m_k-1}), \\ \frac{d(x_{2n_k+1},x_{2n_k}) + d(x_{2m_k},x_{2m_k-1})}{2}, \\ \frac{d(x_{2n_k+1},x_{2n_k}) + d(x_{2n_k},x_{2m_k-1})}{2}, \\ \frac{d(x_{2m_k},x_{2m_k-1}) + d(x_{2n_k},x_{2m_k-1})}{2}, \\ \frac{d(x_{2n_k+1},x_{2m_k-1}) + d(x_{2m_k},x_{2n_k})}{2} \end{pmatrix}. \end{aligned}$$

In view of above inequality and (5), (9), (10), one can obtain that

$$\begin{aligned}\varepsilon &\le \limsup_{k\to+\infty} d(x_{2n_k}, x_{2m_k})\\ &\le 0+\frac{1}{s^3}\phi(0,0,\varepsilon s^2,0,\varepsilon s^2,\frac{\varepsilon s^2}{2},\frac{\varepsilon s^3+\varepsilon s}{2})\\ &\le \frac{1}{s^3}\phi(\varepsilon s^3,\varepsilon s^3,\varepsilon s^3,\varepsilon s^3,\varepsilon s^3,\varepsilon s^3,\varepsilon s^3)\\ &= \frac{1}{s^3}\overline{\phi}(\varepsilon s^3)\\ &< \varepsilon.\end{aligned}$$

It is a contradiction and it follows that $\{x_{2n}\}$ is a Cauchy sequence in X. Since X is complete, there exists x^* such that

$$\begin{aligned}\lim_{n\to+\infty} x_{2n+1} &= \lim_{n\to+\infty} Ax_{2n}\\ &= \lim_{n\to+\infty} x_{2n+2} = \lim_{n\to+\infty} Bx_{2n+1} = x^*.\end{aligned}$$

Without loss of generality, we suppose A is continuous. It follows that

$$x^* = \lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} Ax_{2n} = A\lim_{n\to+\infty} x_{2n} = Ax^*.$$

This implies that x^* is a fixed point of A.

Next, we show that x^* is a fixed point of \$B.\$ In view of the contraction condition (1), we get that

$$\begin{aligned}&s^4 d(Bx^*,x^*)\\ &= s^4 d(Bx^*,Ax^*)\\ &\le \phi\begin{pmatrix} d(Ax^*,x^*), d(Bx^*,x^*), d(x^*,x^*),\\ \dfrac{d(Ax^*,x^*)+d(Bx^*,x^*)}{2},\\ \dfrac{d(Ax^*,x^*)+d(x^*,x^*)}{2},\\ \dfrac{d(Bx^*,x^*)+d(x^*,x^*)}{2},\\ \dfrac{d(Ax^*,x^*)+d(Bx^*,x^*)}{2}\end{pmatrix},\\ &= \phi(0,d(Bx^*,x^*),0,\frac{d(Bx^*,x^*)}{2},0,\frac{d(Bx^*,x^*)}{2},\frac{d(Bx^*,x^*)}{2}).\end{aligned}$$

If suppose that $d(Bx^*,x^*)>0,$ then we have

$$\begin{aligned}&s^4 d(Bx^*,x^*)\\ &\le \phi\begin{pmatrix} d(Bx^*,x^*), d(Bx^*,x^*),\\ d(Bx^*,x^*), d(Bx^*,x^*),\\ d(Bx^*,x^*), d(Bx^*,x^*), d(Bx^*,x^*)\end{pmatrix}\\ &= \overline{\phi}(d(Bx^*,x^*)) < d(Bx^*,x^*),\end{aligned}$$

a contradiction. It follows that $d(Bx^*,x^*)=0.$ That is, x^* is also a fixed point of *B*.

Assume that y^* is another common fixed point of A and B, that is, $d(x^*,y^*)>0.$ Then

$$\begin{aligned}&s^4 d(x^*,y^*)\\ &= s^4 d(Ax^*,By^*)\\ &\le \phi\begin{pmatrix} d(Ax^*,x^*), d(By^*,y^*), d(x^*,y^*),\\ \dfrac{d(Ax^*,x^*)+d(By^*,y^*)}{2},\\ \dfrac{d(Ax^*,x^*)+d(x^*,y^*)}{2},\\ \dfrac{d(By^*,y^*)+d(x^*,y^*)}{2},\\ \dfrac{d(Ax^*,y^*)+d(By^*,x^*)}{2}\end{pmatrix},\\ &= \phi(0,0,d(x^*,y^*),0,\frac{d(x^*,y^*)}{2},\frac{d(x^*,y^*)}{2},d(x^*,y^*))\\ &\le \overline{\phi}(d(x^*,y^*)) < d(x^*,y^*),\end{aligned}$$

which is a contradiction. It follows that x^* is a unique common fixed point in *X*. This completes the proof.

If $A=B$ in Theorem 1, then we get that:

Corollary 3.2. *Let* (X,d) *be a complete* $b-$*metric space with constant* $s\ge 1$ *and* $A: X\to X$ *be a continuous mapping. If there exists* $\phi\in\Phi$ *such that*

$$\begin{aligned}&s^4 d(Ax,Ay)\\ &\le \phi\begin{pmatrix} d(Ax,x), d(Ay,y), d(x,y),\\ \dfrac{d(Ax,x)+d(Ay,y)}{2}, \dfrac{d(Ax,x)+d(x,y)}{2},\\ \dfrac{d(Ay,y)+d(x,y)}{2}, \dfrac{d(Ax,y)+d(Ay,x)}{2}\end{pmatrix}\end{aligned}$$

for all $x,y\in X$, *then* A *has a unique fixed point* $x^*\in X$.

Theorem 3.3. *Let* (X,d) *be a complete* $b-$*metric space with constant* $s\ge 1$. *Suppose* A *and* $B: X\to X$ *are two mappings and one of them is continuous. If there exists* $\psi\in\Psi$ *such that*

$$\begin{aligned}&s^6 d^2(Ax,By)\\ &\le \psi\begin{pmatrix} d(Ax,x)d(By,y), d(Ax,x)d(x,y),\\ d(By,y)d(x,y), d(Ax,By)d(x,y),\\ d(Ax,By)d(Ax,x), d(Ax,By)d(By,y),\\ d^2(Ax,x), d^2(By,y), d^2(x,y)\end{pmatrix}\end{aligned} \tag{11}$$

for all $x,y\in X$, *then* A *and* B *have a unique common fixed point* $x^*\in X$.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Ax_{2n}, x_{2n+2} = Bx_{2n+1}, n \in N_0.$$

We now suppose that $d(x_n, x_{n+1}) > 0$ for every \$n.\$ Otherwise, there exists some $n \in N_0$ such that $x_n = x_{n+1}$. If $n = 2k$, from the contraction condition (11) with $x = x_{2k}$ and $y = x_{2k+1}$, one can obtain

$$\begin{aligned}
&s^6 d^2(x_{2k+1}, x_{2k+2}) \\
&= s^6 d^2(Ax_{2k}, Bx_{2k+1}) \\
&\le \psi\begin{pmatrix} d(Ax_{2k}, x_{2k})d(Bx_{2k+1}, x_{2k+1}), \\ d(Ax_{2k}, x_{2k})d(x_{2k}, x_{2k+1}), \\ d(Bx_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+1}), \\ d(Ax_{2k}, Bx_{2k+1})d(x_{2k}, x_{2k+1}), \\ d(Ax_{2k}, Bx_{2k+1})d(Ax_{2k}, x_{2k}), \\ d(Ax_{2k}, Bx_{2k+1})d(Bx_{2k+1}, x_{2k+1}), \\ d^2(Ax_{2k}, x_{2k}), d^2(Bx_{2k+1}, x_{2k+1}), d^2(x_{2k}, x_{2k+1}) \end{pmatrix} \\
&= \psi\begin{pmatrix} d(x_{2k+1}, x_{2k})d(x_{2k+2}, x_{2k+1}), \\ d(x_{2k+1}, x_{2k})d(x_{2k}, x_{2k+1}), \\ d(x_{2k+2}, x_{2k+1})d(x_{2k}, x_{2k+1}), \\ d(x_{2k+1}, x_{2k+2})d(x_{2k}, x_{2k+1}), \\ d(x_{2k+1}, x_{2k+2})d(x_{2k+1}, x_{2k}), \\ d(x_{2k+1}, x_{2k+2})d(x_{2k+2}, x_{2k+1}), \\ d^2(x_{2k+1}, x_{2k}), d^2(x_{2k+2}, x_{2k+1}), \\ d^2(x_{2k}, x_{2k+1})). \end{pmatrix}.
\end{aligned}$$

We suppose that $d(x_{2k+1}, x_{2k+2}) > d(x_{2k}, x_{2k+1}) = 0.$ By the definition of ψ, we have

$$\begin{aligned}
&s^6 d^2(x_{2k+1}, x_{2k+2}) \\
&\le \psi(0,0,0,0,0, d^2(x_{2k+1}, x_{2k+2}), 0, d^2(x_{2k+1}, x_{2k+2}), 0) \\
&\le \psi\begin{pmatrix} d^2(x_{2k+1}, x_{2k+2}), d^2(x_{2k+1}, x_{2k+2}), \\ d^2(x_{2k+1}, x_{2k+2}), d^2(x_{2k+1}, x_{2k+2}), \\ d^2(x_{2k+1}, x_{2k+2}), d^2(x_{2k+1}, x_{2k+2}), \\ d^2(x_{2k+1}, x_{2k+2}), d^2(x_{2k+1}, x_{2k+2}), \\ d^2(x_{2k+1}, x_{2k+2}) \end{pmatrix} \\
&= \overline{\psi}(d^2(x_{2k+1}, x_{2k+2})) < d^2(x_{2k+1}, x_{2k+2}),
\end{aligned}$$

a contradiction. Hence, $d(x_{2k+1}, x_{2k+2}) = 0$. It follows from the definition of the sequence $\{x_n\}$ that

$$x_{2k} = Ax_{2k} = Bx_{2k}.$$

That is, x_{2k} is a common fixed point of A and B.

Similarly, if $n = 2k+1$, we can prove that x_{2k+1} is a common fixed point of A and B.

From now on, we suppose that $x_n \ne x_{n+1}$ for all $n \in N_0$. Using the similar argument in the proof of Theorem 3.1, one can deduce that

$$d^2(x_n, x_{n+1}) \le \overline{\psi}(d^2(x_n, x_{n-1})), \text{for each } n \in N. \quad (12)$$

It follows from Lemma 2.3 that $\lim_{n\to+\infty} \overline{\psi}^n(t) = 0$ for all $t > 0$, which implies that

$$\text{S} \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \quad (13)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. Obviously, it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence in X. As in the proof of Theorem 3.1, we obtain that inequalities (9),(10) hold, and

$$\frac{\varepsilon}{s} \le \limsup_{k\to+\infty} d(x_{2m_k}, x_{2n_k+1}) \le s^2\varepsilon. \quad (14)$$

The triangle inequality in b-metric space and contraction condition (11) ensure that

$$\begin{aligned}
&d^2(x_{2n_k}, x_{2m_k}) \\
&\le (sd(x_{2n_k}, x_{2n_k+1}) + sd(x_{2n_k+1}, x_{2m_k}))^2 \\
&= s^2 d(x_{2n_k}, x_{2n_k+1}) + 2s^2 d(x_{2n_k}, x_{2n_k+1})d(x_{2n_k+1}, x_{2m_k}) \\
&\quad + s^2 d(Ax_{2n_k}, Bx_{2m_k-1}) \\
&\le s^2 d(x_{2n_k}, x_{2n_k+1}) \\
&\quad + 2s^2 d(x_{2n_k}, x_{2n_k+1})d(x_{2n_k+1}, x_{2m_k}) \\
&\quad + \frac{1}{s^4}\psi\begin{pmatrix} d(Ax_{2n_k}, x_{2n_k})d(Bx_{2m_k-1}, x_{2m_k-1}), \\ d(Ax_{2n_k}, x_{2n_k})d(x_{2n_k}, x_{2m_k-1}), \\ d(Bx_{2m_k-1}, x_{2m_k-1})d(x_{2n_k}, x_{2m_k-1}), \\ d(Ax_{2n_k}, Bx_{2m_k-1})d(x_{2n_k}, x_{2m_k-1}), \\ d(Ax_{2n_k}, Bx_{2m_k-1})d(Ax_{2n_k}, x_{2n_k}), \\ d(Ax_{2n_k}, Bx_{2m_k-1})d(Bx_{2m_k-1}, x_{2m_k-1}), \\ d^2(Ax_{2n_k}, x_{2n_k}), d^2(Bx_{2m_k-1}, x_{2m_k-1}), \\ d^2(x_{2n_k}, x_{2m_k-1})) \end{pmatrix} \\
&\le s^2 d(x_{2n_k}, x_{2n_k+1}) \\
&\quad + 2s^2 d(x_{2n_k}, x_{2n_k+1})d(x_{2n_k+1}, x_{2m_k})) \\
&\quad + \frac{1}{s^4}\psi\begin{pmatrix} d(x_{2n_k+1}, x_{2n_k})d(x_{2m_k}, x_{2m_k-1}), \\ d(x_{2n_k+1}, x_{2n_k})d(x_{2n_k}, x_{2m_k-1}), \\ d(x_{2m_k}, x_{2m_k-1})d(x_{2n_k}, x_{2m_k-1}), \\ d(x_{2n_k+1}, x_{2m_k})d(x_{2n_k}, x_{2m_k-1}), \\ d(x_{2n_k+1}, x_{2m_k})d(x_{2n_k+1}, x_{2n_k}), \\ d(x_{2n_k+1}, x_{2m_k})d(x_{2m_k}, x_{2m_k-1}), \\ d^2(x_{2n_k+1}, x_{2n_k}), d^2(x_{2m_k}, x_{2m_k+1}), \\ d^2(x_{2n_k}, x_{2m_k-1})). \end{pmatrix}
\end{aligned}$$

In light of above inequality and (9), (10), (13) and (14), we have

$$\begin{aligned}\varepsilon^2 &\le \limsup_{k\to+\infty} d^2(x_{2m_k}, x_{2n_k})\\ &\le 0+0+\frac{1}{s^4}\psi(0,0,0,\varepsilon^2 s^4,0,0,0,0,\varepsilon^2 s^4)\\ &\le \frac{1}{s^4}\psi\begin{pmatrix}\varepsilon^2 s^4,\varepsilon^2 s^4,\varepsilon^2 s^4,\varepsilon^2 s^4,\\ \varepsilon^2 s^4,\varepsilon^2 s^4,\varepsilon^2 s^4,\varepsilon^2 s^4,\varepsilon^2 s^4\end{pmatrix}\\ &= \frac{1}{s^4}\overline{\psi}(\varepsilon^2 s^4)\\ &< \varepsilon^2.\end{aligned}$$

It is a contradiction. Hence, $\{x_{2n}\}$ is a Cauchy sequence in X. The completeness of X ensures that there exists x^* such that

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} Ax_{2n} = \lim_{n\to+\infty} x_{2n+2} = \lim_{n\to+\infty} Bx_{2n+1} = x^*.$$

Without loss of generality, we suppose A is continuous. It follows that

$$x^* = \lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} Ax_{2n} = A\lim_{n\to+\infty} x_{2n} = Ax^*.$$

That is, x^* is a fixed point of A.

Next, we shall prove that x^* is a fixed point of B. By the contraction condition (11), we obtain that

$$\begin{aligned}s^6 d^2(x^*, Bx^*) &= s^6 d^2(Ax^*, Bx^*)\\ &\le \psi\begin{pmatrix} d(Ax^*,x^*)d(Bx^*,x^*), d(Ax^*,x^*)d(x^*,x^*),\\ d(Bx^*,x^*)d(x^*,x^*), d(Ax^*,Bx^*)d(x^*,x^*),\\ d(Ax^*,Bx^*)d(Ax^*,x^*),\\ d(Ax^*,Bx^*)d(Bx^*,x^*),\\ d^2(Ax^*,x^*), d^2(Bx^*,x^*), d^2(x^*,x^*)\end{pmatrix}\\ &= \psi(0,0,0,0,0,d^2(Bx^*,x^*),0,d^2(Bx^*,x^*),0).\end{aligned}$$

If we suppose that $d(Bx^*,x^*)>0$, then one can get

$$s^6 d^2(Bx^*,x^*) \le \overline{\psi}(d^2(Bx^*,x^*)) < d^2(Bx^*,x^*),$$

which is a contradiction. Hence, we deduce that x^* is also a fixed point of B.

Suppose that x^* and y^* are different common fixed points of A and B, then we obtain that

$$\begin{aligned}s^6 d^2(x^*, y^*) &= s^6 d^2(Ax^*, By^*)\\ &\le \psi\begin{pmatrix} d(Ax^*,x^*)d(By^*,y^*), d(Ax^*,x^*)d(x^*,y^*),\\ d(By^*,y^*)d(x^*,y^*), d(Ax^*,By^*)d(x^*,y^*),\\ d(Ax^*,By^*)d(Ax^*,x^*), d(Ax^*,By^*)d(By^*,y^*),\\ d^2(Ax^*,x^*), d^2(By^*,y^*), d^2(x^*,y^*))\end{pmatrix}\\ &= \psi(0,0,0,d^2(x^*,y^*),0,0,0,0,d^2(x^*,y^*))\\ &\le \overline{\psi}(d^2(x^*,y^*)) < d^2(x^*,y^*),\end{aligned}$$

a contradiction. Consequently, x^* is a unique common fixed point in *X*. This completes the proof.

If $A=B$ in Theorem 3, we have the following result.

Corollary 3.4. Let (X,d) be a complete $b-$metric space with constant $s\ge 1$. Suppose $A: X\to X$ be a continuous mapping. If there exists $\psi\in\Psi$ such that

$$s^6 d^2(Ax, Ay) \le \psi\begin{pmatrix} d(Ax,x)d(Ay,y), d(Ax,x)d(x,y),\\ d(Ay,y)d(x,y), d(Ax,Ay)d(x,y),\\ d(Ax,Ay)d(Ax,x), d(Ax,Ay)d(Ay,y),\\ d^2(Ax,x), d^2(Ay,y), d^2(x,y)\end{pmatrix} \quad (15)$$

for all $x,y\in X$ *, then* A *has a unique common fixed point* $x^*\in X$.

4. Examples

Example 4.1. Let $X=[0,1]$ endowed with the $b-$metric:

$$d: X\times X\to[0,+\infty), d(x,y)=|x-y|^2$$

with constant $s=2$. Consider mappings $A,B: X\times X$ by $Ax=\frac{x}{16}$ and $Bx=\frac{x}{32}$. Define the mapping $\phi:(\mathbb{R}^+)^7\to\mathbb{R}^+$ by

$$\phi(x_1,x_2,x_3,x_4,x_5,x_6,x_7)=\frac{1}{7}(\sum_{i=1}^{7}\frac{x_i}{1+x_i}).$$

Clearly, (X,d) is a complete $b-$metric space and A is continuous with respect to d. So we verify the contraction condition (1).

By calculus, we have

$$\begin{aligned}s^4 d(Ax,By) &= 16\left|\frac{x}{16}-\frac{y}{32}\right|^2\\ &\le \frac{x^2}{16}+\frac{y^2}{64} < \frac{1}{7}\left(\frac{225x^2}{512}+\frac{961y^2}{2048}\right)\\ &\le \frac{1}{7}\left(\frac{d(Ax,x)}{1+d(Ax,x)}+\frac{d(By,y)}{1+d(By,y)}\right)\\ &\le \phi\begin{pmatrix} d(Ax,x), d(By,y), d(x,y), \frac{d(Ax,x)+d(By,y)}{2},\\ \frac{d(Ax,x)+d(x,y)}{2}, \frac{d(By,y)+d(x,y)}{2},\\ \frac{d(Ax,y)+d(By,x)}{2}\end{pmatrix}.\end{aligned}$$

Therefore, we show that the contraction condition (1) is satisfied. It follows that we can apply Theorem 3.1 and A and B have a unique common fixed point $x^*=0$.

Example 4.2. Let $X=[0,1]$ endowed with the $b-$metric:

$$d: X\times X\to[0,+\infty), d(x,y)=|x-y|^2$$

with constant $s=2$. Define mappings $A,B: X\times X$ by

$Ax = \frac{x}{16}$ and $Bx = \frac{x}{8}$. Consider the mapping $\psi : (\mathbb{R}^+)^9 \to \mathbb{R}^+$ by

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8.x_9) = \frac{1}{9}(\sum_{i=1}^{9} \frac{x_i}{1+x_i}).$$

It is easy to verify that (X, d) is a complete $b-$metric space and A is continuous with respect to *d*. By calculus, we obtain that

$$s^6 d^2(Ax, By) = 64(|\frac{x}{16} - \frac{y}{8}|^2)^2$$
$$\leq 64\,(\frac{x^2}{256} + \frac{y^2}{64})^2$$
$$\leq \frac{7x^4}{1536} + \frac{7y^4}{256}$$
$$\leq \frac{1}{9}(\frac{d^2(Ax,x)}{1+d^2(Ax,x)} + \frac{d^2(By,y)}{1+d^2(By,y)})$$
$$\leq \psi \begin{pmatrix} d(Ax,x)d(By,y), d(Ax,x)d(x,y), \\ d(By,y)d(x,y), d(Ax,By)d(x,y), \\ d(Ax,By)d(Ax,x), \\ d(Ax,By)d(By,y), \\ d^2(Ax,x), d^2(By,y), d^2(x,y) \end{pmatrix}.$$

That is, the contraction condition (11) holds. Theorem 3.3 ensures that A and B have a unique common fixed point $x^* = 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the pub-lication of this paper.

Authors Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

The second author would like to acknowledge the grant: Science and Research Project Foundation of Liaoning Province Education Department LQN201902 and the Research Foundation for the Doctoral Program of Shenyang Normal University BS201703, for financial support.

References

[1] S. Banach, *Surles operations dans ensembles abstraits et leur application aux equations integrales*, Fundamenta Mathematicae 3(1922), no.3, 51-57.

[2] J. Caristi, *Fixed point theorems for mapping satisfying inwardness conditions, Trans.* Am. Math. Soc. 215(1976), 241-251.

[3] T. L. Hicks, F*ixed point theorems for quasi-metric spaces*, Math. Jpn. 33(1998), no. 2, 231-236.

[4] I. Altun, G. Durmaz, *Some fixed point results in cone metric spaces*, Rend. Circ. Mal. Palermo 58(2009), 319-325.

[5] B. Choudhury, N. Metiya, *Coincidence point and fixed point theorems in odered cone metric spaces*, J. Adv. Math. Stud. 5(2012), no. 2, 20-31.

[6] I. Altun, H. Simsek, *Some fixed point theorems on dualistic partial metric spaces*, J.Adv. Math. Stud. 1(2008), 1-8.

[7] H. Aydi, *Fixed point results for weakly contractive mappings in ordered partial metric spaces*, J. Adv. Math. Stud. 4(2011), 1-12.

[8] A. Khan, M. Abbas, T. Nazi, C. Lonescu, *Fixed points of multivalued contractive mappings in partial metric spaces*, Abstr. Appl. Anal. 2014(2014), Article ID 230708.

[9] W. Shatanawi, A. Pitea, *Fixed and coupled fixed point theorems for omega-distance for nonlinear contraction*, Fixed Point Theory Appl. 2013(2013), Article ID 275.

[10] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta. Math. Inform. Univ. Ostrav. 1(1993), 5-11.

[11] M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Mod. Math. 4(2009), 285-301.

[12] M. Boriceanu, *Fixed point theory for multivalued generalized contraction on a set with two b-metrics*, Stud. Univ. Babes-Bolyai Math. LIV, 2009.

[13] H. Aydi, M. Bota, S.Moradi, *A common fixed points for weak b-contractions on b-metric spaces*, Fixed Point Theory 13(2012), 337-346.

[14] V. Berinde, *Generalized contractions in quasimetric space*s, Seminar on Fixed Point Theory Preprint 3(1993), 3-9.

[15] M. Pacurar, *A fixed point result for φ-contractions and fixed points on b-metric spaces without the boundness assumption*, Fasc. Math. 43(2010), 127-136.

[16] W. Shatanawi, A. Pitea, R. Lazovic, *Contraction conditions using comparision fuctions on b-metric spaces*, Fixed Point Theory Appl. 2014(2014), Article ID 135.

[17] N. Hussain, M. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math Appl. 61(2011), 1677-1684.

[18] S. Singh, B. Prasad, *Some coincidence theorems and stability of iterative proceders*, Comput. Math. Appl. 55(2008), 2512-2520.

[19] J. Matkkowski, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. 62(1977), 344-348.