Turkish Journal of Analysis and Number Theory, 2019, Vol. 7, No. 4, 113-116
Available online at http://pubs.sciepub.com/tjant/7/4/3
Published by Science and Education Publishing
DOI:10.12691/tjant-7-4-3

Generating Functions of Modified Pell Numbers and Bivariate Complex Fibonacci Polynomials

Souhila Boughaba, Ali Boussayoud*, Khadidja Boubellouta

LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
*Corresponding author: aboussayoud@yahoo.fr

Received July 01, 2019; Revised August 04, 2019; Accepted August 10, 2019

Abstract In this paper, we introduce a operator in order to derive a new generating functions of modified $k-$ Pell numbers, Gaussian modified Pell numbers. By making use of the operator defined in this paper, we give some new generating functions for Bivariate Complex Fibonacci and Lucas Polynomials, modified Pell Polynomials and Gaussian modified Pell Polynomials.

2010 Mathematics Subject Classification. Primary 05E05; Secondary 11B39.

Keywords: *Symmetric functions, Generating functions, modified k- Pell numbers, Bivariate Complex Fibonacci Polynomials, Bivariate Complex Lucas Polynomials*

Cite This Article: Souhila Boughaba, Ali Boussayoud, and Khadidja Boubellouta, "Generating Functions of Modified Pell Numbers and Bivariate Complex Fibonacci Polynomials." *Turkish Journal of Analysis and Number Theory*, vol. 7, no. 4 (2019): 113-116. doi: 10.12691/tjant-7-4-3.

1. Introduction

The modified $k-$ Pell numbers and Gaussian modified Pell numbers are the numbers of positive integers that have been studied over several years. These numbers are examples of a numbers defined by a recurrence relation of second order. It is well known that the modified $k-$ Pell numbers $\{q_{k,n}\}_{n\in\mathbb{N}}$ is defined in [1] by the following recurrence relation $q_{k,n} = 2q_{k,n-1} + kq_{k,n-2}, n \geq 2$ with initial conditions $q_{k,0} = q_{k,1} = 1$.

In [2] Tulay Yagmur and Nusret Karaaslan are defined the Gaussian modified Pell numbers by the recurrence relation $Gq_n = 2Gq_{n-1} + Gq_{n-2}$ for $n \geq 2$ with initial conditions $Gq_0 = 1-i$, $Gq_1 = 1+i$ and then they give the definition of the Gaussian modified Pell polynomials, for $n \geq 2$, by the relation $Gq_n(x) = 2xGq_{n-1}(x) + Gq_{n-2}(x)$, with initial conditions $Gq_0(x) = 1 - xi$ and $Gq_1(x) = x + i$.

Mustafa Asci and Esref Gurel are define and study the Bivariate Complex Fibonacci and Lucas Polynomials in [3]. They give generating function, Binet formula, explicit formula and partial derivation of these polynomials. By defining these Bivariate Polynomials for special cases $F_n(x,1)$ is the complex Fibonacci polynomials defined in [4] and $F_n(1,1)$ is the complex Fibonacci numbers, and give the divisibility properties of Bivariate Complex Fibonacci Polynomials.

The Bivariate Complex Fibonacci Polynomials $\{F_n(x,y)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$F_{n+1}(x,y) = ixF_n(x,y) + yF_{n-1}(x,y),$$

with initial conditions $F_0(x;y) = 0$ and $F_1(x;y) = 1$.

The bivariate complex Lucas polynomials $\{L_n(x,y)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$L_{n+1}(x,y) = ixL_n(x,y) + yL_{n-1}(x,y),$$

with initial conditions $L_0(x,y) = 2$ and $L_1(x,y) = ix$.

In [5], the modified Pell polynomials are defined recursively by $q_n(x) = 2xq_{n-1}(x) + q_{n-2}(x)$, with initial conditions $q_0(x) = 1$ and $q_1(x) = x$.

In this contribution, we are going to define an operator denoted by $\delta^k_{e_1e_2}$ that formulates, extends and proves results based on our previous ones, see [6-25]. In order to determine generating functions of modified k-Pell numbers, Gaussian modified Pell numbers, Bivariate Complex Fibonacci and Lucas Polynomials, modified Pell Polynomials and Gaussian modified Pell Polynomials, we use analytical means and series manipulation methods. In the sequel, we derive new symmetric functions and some new properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. Definitions and some Properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series

$$\lambda_z(A) = \Pi_{a\in A}(1+za),\ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1-za)}.$$

The expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete symmetric functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{\infty}\Lambda_n(A)z^n,\ \sigma_z(A) = \sum_{n=0}^{\infty}S_n(A)z^n.$$

Let us now start at the following definition.

Definition 1: *Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:*

$$\frac{\Pi_{b\in B}(1-zb)}{\Pi_{a\in A}(1-za)} = \sum_{n=0}^{\infty}S_n(A-B)z^n = \sigma_z(A-B), \quad (2.1)$$

with the condition $S_n(A-B)=0$ for $n<0$ (see [26]).

Corollary 1: *Taking $A=0$ in* (2.1) *gives*

$$\Pi_{b\in B}(1-zb) = \sum_{n=0}^{\infty}S_n(-B)z^n = \lambda_z(-B). \quad (2.2)$$

Further, in the case $A=0$ or $B=0$, we have

$$\sum_{n=0}^{\infty}S_n(A-B)z^n = \sigma_z(A)\times\lambda_z(-B). \quad (2.3)$$

Thus,

$$S_n(A-B) = \sum_{k=0}^{n}S_{n-k}(A)S_k(-B) \text{ (see [26])}.$$

Definition 2: [27] *Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form*

$$\partial_{x_ix_{i+1}}(g) = \frac{\begin{bmatrix} g(x_1,\cdots,x_i,x_{i+1},\cdots x_n) \\ -g(x_1,\cdots x_{i-1},x_{i+1},x_i,x_{i+2}\cdots x_n)\end{bmatrix}}{x_i - x_{i+1}}.$$

Definition 3: [6] *Given an alphabet $E=\{e_1,e_2\}$, the symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{e_1e_2}(e_1^n) = \frac{e_1^{k+n}-e_2^{k+n}}{e_1-e_2} = S_{k+n-1}(e_1+e_2)\ \left(k,n\in\mathbb{N}\right).$$

3. Construction of Generating Functions of Some Numbers and Polynomials

The following proposition is one of the key tools of the proof of our main result. It has been proved in [7] for the completeness of the paper we state its proof here.

Proposition 1: *Given an alphabet $A=\{a_1,a_2\}$, then*

$$\sum_{n=0}^{+\infty}S_n(a_1,[-a_2])z^n = \frac{1}{(1-a_1z)(1+a_2z)}. \quad (3.1)$$

Based on the relationship (3.1) we have

$$\sum_{n=0}^{+\infty}S_{n-1}(a_1+[-a_2])z^n = \frac{z}{(1-a_1z)(1+a_2z)}. \quad (3.2)$$

The substitutions $\begin{cases} a_1-a_2=2, \\ a_1a_2=k, \end{cases}$ in (3.1) and (3.2) we obtain

$$\sum_{n=0}^{+\infty}S_n(a_1+[-a_2])z^n = \frac{1}{1-2z-kz^2}. \quad (3.3)$$

$$\sum_{n=0}^{\infty}S_{n-1}\left(a_1+\left[-a_2\right]\right)z^n = \frac{z}{1-2z-kz^2}, \quad (3.4)$$

and we have the following results.

Corollary 2 : *For $n\in\mathbb{N}$, the new generating function of modified $k-$ Pell numbers is given by*

$$\sum_{n=0}^{+\infty}q_{k,n}z^n = \frac{1-z}{1-2z-kz^2}, \quad (3.5)$$

with $q_{k,n} = S_n(a_1+[-a_2]) - S_{n-1}(a_1+\left[-a_2\right])$.

- Put $k=1$ in the relationship (3.5) we get the following corollary

Corollary 3: *For $n\in\mathbb{N}$, the new generating function of modified Pell numbers is given by*

$$\sum_{n=0}^{+\infty}q_nz^n = \frac{1-z}{1-2z-z^2}.$$

The substitutions $\begin{cases} a_1-a_2=2, \\ a_1a_2=1, \end{cases}$ in (3.1) and (3.2) we obtain

$$\sum_{n=0}^{+\infty}S_n(a_1+[-a_2])z^n = \frac{1}{1-2z-z^2}. \quad (3.6)$$

$$\sum_{n=0}^{\infty}S_{n-1}(a_1+\left[-a_2\right])z^n = \frac{z}{1-2z-z^2}. \quad (3.7)$$

Multiplying the equation (3.6) by $(1-i)$ and (3.7) by $(3i-1)$, we obtain

$$\sum_{n=0}^{+\infty}((1-i)S_n(a_1+[-a_2])+(3i-1)S_{n-1}(a_1+\left[-a_2\right]))z^n = \frac{1-i+(3i-1)z}{1-2z-z^2}.$$

Accordingly, we conclude the following Corollary.

Corollary 4: *For* $n \in \mathbb{N}$ *, the new generating function of Gaussian modified Pell numbers is given by*

$$\sum_{n=0}^{+\infty} Gq_n z^n = \frac{1-i+(3i-1)z}{1-2z-z^2},$$

with $G\,q_n = (1-i)S_n(a_1+[-a_2]) + (3i-1)S_{n-1}(a_1+[-a_2])$.

The substitutions $\begin{cases} a_1 - a_2 = 2x, \\ a_1 a_2 = 1, \end{cases}$ in (3.1) and (3.2) we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])z^n = \frac{1}{1-2xz-z^2}. \quad (3.8)$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])z^n = \frac{z}{1-2xz-z^2}, \quad (3.9)$$

and we have the following results.

Corollary 5: *For* $n \in \mathbb{N}$*, the new generating function of modified Pell Polynomial is given by*

$$\sum_{n=0}^{+\infty} q_n(x)z^n = \frac{1-xz}{1-2xz-z^2},$$

with $q_n(x) = S_n(a_1+[-a_2]) - xS_{n-1}(a_1+[-a_2])$.

Multiplying the equation (3.8) by $(1-ix)$ and (3.9) by $\left(2ix^2+i-x\right)$, we obtain

$$\sum_{n=0}^{+\infty}\left((1-ix)S_n(a_1+[-a_2]) + \left(2ix^2+i-x\right)S_{n-1}(a_1+[-a_2])\right)z^n$$
$$= \frac{1-ix+\left(2ix^2+i-x\right)z}{1-2xz-z^2}.$$

Accordingly, we conclude the following Corollary

Corollary 6: *For* $n \in \mathbb{N}$*, the new generating function of Gaussian modified Pell Polynomial is given by*

$$\sum_{n=0}^{+\infty} Gq_n(x)z^n = \frac{1-ix+\left(2ix^2+i-x\right)z}{1-2xz-z^2},$$

with

$$G\,q_n(x) = (1-ix)S_n(a_1+[-a_2]) + \left(2ix^2+i-x\right)S_{n-1}(a_1+[-a_2]).$$

Choosing a_1 and a_2 such that $\begin{cases} a_1 - a_2 = ix \\ a_1 a_2 = y \end{cases}$ and substituting in (3.1) and (3.2), we obtain

$$\sum_{n=0}^{+\infty} S_n(a_1+[-a_2])z^n = \frac{1}{1-ixz-yz^2}. \quad (3.10)$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])z^n = \frac{z}{1-ixz-yz^2}, \quad (3.11)$$

and we have the following Corollary.

Corollary 7: *For* $n \in \mathbb{N}$*, the new generating function of Bivariate Complex Fibonacci is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)z^n = \frac{z}{1-ixz-yz^2},$$

with $F_n(x,y) = S_{n-1}(a_1+[-a_2])$.

Multiplying the equation (3.10) by 2 and (3.11) by ix we obtain

$$\sum_{j=0}^{+\infty}\left(2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])\right)z^j$$
$$= \frac{2-ixz}{1-ixz-yz^2}.$$

Accordingly, we conclude the following Corollary.

Corollary 8: *For* $n \in \mathbb{N}$*, the new generating function of Bivariate Complex Lucas is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y)z^n = \frac{2-ixz}{1-ixz-yz^2},$$

with $L_n(x,y) = 2S_n(a_1+[-a_2]) - ixS_{n-1}(a_1+[-a_2])$.

4. Conclusion

In this paper, by making use of Eq. (3.1), we have derived some new generating functions for the modified *k*- Pell numbers, Gaussian modified Pell numbers, Bivariate Complex Fibonacci and Lucas Polynomials, modified Pell Polynomials and Gaussian modified Pell Polynomials. The derived proposition and corollaries are based on symmetric functions and these numbers and polynomials.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

References

[1] P. Catarino, H. Campos, Incomplete *k* -Pell, *k* -Pell Lucas and modified *k* -Pell numbers, Hacet. J. Math. Stat. 46(3), 361-372, 2017.

[2] T. Yagmur, N. Karaaslan, Gaussian Modified Pell Sequence and Gaussian Modified Pell Polynomial Sequence, *Aksaray J. Sci. Eng*. 2(1), 63-72, 2018.

[3] M. ASCI, E. GUREL, On Bivariate Complex Fibonacci and Lucas Polynomials, *Conference on Mathematical Sciences ICM 2012*, March 11-14, 2012.

[4] A.F Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J.* 32, 437-446, 1965.

[5] N. Karaaslan, A Note on Modified Pell Polynomials, *Aksaray J. Sci. Eng*. 3, 1-7, 2019.

[6] A. Boussayoud, M. Kerada , Symmetric and Generating Functions, *Int. Electron. J. Pure Appl. Math.* 7, 195-203, 2014.

[7] A. Boussayoud, M.kerada, M, Boulyer, A simple and accurate method for determination of some generalized sequence of numbers, *Int.J. Pure Appl Math*. 108, 503-511, 2016.

[8] A. Abderrezzak, Généralisation d'identité s de Carlitz. Howard et Lehmer. *Aequ. Math*. 49, 36-46, 1995.

[9] A. Boussayoud, M. Kerada, N. Harrouche, On the k-Lucas numbers and Lucas Polynomials, *Turkish Journal of Analysis and Number*. 5, 121-125, 2017.

[10] A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, *Online J. Anal. Comb*. 12, 1-10, 2017.

[11] A. Boussayoud, N.Harrouche, Complete symmetric functions and k-Fibonacci numbers. *Commun. Appl. Anal*. 20, 457-465, 2016.

[12] A. Boussayoud, M.Kerada, R.Sahali, Symmetrizing Operations on Some Orthogonal Polynomails, *Int. Electron. J. Pure Appl. Math*. 9, 191-199, 2015.

[13] A. Boussayoud, R.Sahali, The application of the operator $L_{b_1 b_2}^{-k}$ in the series $\sum_{j=0}^{+\infty} a_j b_1^j z^j$. *J. Adv. Res. Appl. Math.* 7, 68-75, 2015.

[14] A. Boussayoud, M. Kerada, Symmetric and Generating Functions, *Int. Electron. J. Pure Appl. Math.* 7, 195-203, 2014.

[15] A. Boussayoud, M. Kerada, R.Sahali, W.Rouibah, Some Applications on Generating Functions, *J. Concr. Appl. Math*. 12, 321-330, 2014.

[16] A. Boussayoud, L'action de l'opérateur $\delta_{e_1 e_2}^{k}$ sur la série $\sum_{n=0}^{\infty} S_n(A) e_1^n z^n$. (Doctoral dissertation), Mohamed Seddik Ben Yahia University, Jijel, Algeria (2017).

[17] A.F. Horadam, J.M. Mahon, Pell and Pell-Lucas Polynomials, *Fibonacci Quarterly*. 23, 7-20, 1985.

[18] A. Pintér, H. M. Srivastava, Generating functions of the incomplete Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo*. 48, 591-596, 1999.

[19] C. Bolat, H Kose, On the Properties of k-Fibonacci Numbers, *Int. J. Contemp. Math. Sciences*. 5, 1097-1105, 2010.

[20] D. Foata and G-N. Han, Nombres de Fibonacci et Polynômes Orthog onaux, Leonardo Fibonacci: il tempo, le opere, l'eredità scientifica, 179-200, 1994.

[21] D. Tasci and M. Cetin Firengiz, Incomplete Fibonacci and Lucas p-numbers, *Math. Comput. Modelling.* 52, 1763-1770, 2010.

[22] D.V Kruchinin, V.V. Kruchinin, Application of a composition of generating functions for obtaining explicit formulas of polynomials, *J. Math. Anal. Appl*. 404 , 161.171, 2013.

[23] G. B. Djordjevic, Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers, *Fibonacci Q*. 42, 106-113, 2004.

[24] G. B. Djordjevic and H. M. Srivastava, Incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers, *Math. Comput. Modelling.* 42, 1049-1056, 2005.

[25] I. Mezo, Several Generating Functions for second-order recurrence sequences, *J. Integer Seq*. 12, 1-16, 2009.

[26] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle. *Adv. Math*. 103, 180-195 , 1994.

[27] I.G.Macdonald, Symmetric Functions and Hall Polynomias. Oxford University Press, Oxford (1979).