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Some New Integral Inequalities for Functions Whose Derivatives of Absolute Values Are s-Convex

M. Emin Özdemir[1,*], Alper Ekinci[2]

[1]Uludag University, Education Faculty, Bursa, Turkey
[2]Bandirma Onyedi Eylul University, Bandirma Vocational School, Balıkesir, Turkey
*Corresponding author: eminozdemir@uludag.edu.tr

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Abstract In this paper, we prove some new inequalities for the functions whose derivatives absolute values are s-convex by dividing the interval to equal even sub-intervals. We obtain some new results involving intermediate values of in by using some classical inequalities like Hermite-Hadamard, Hölder and Power-Mean.

Keywords: *s-convex functions, Hermite-Hadamard Inequality, power-mean inequality, hölder inequality*

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1. Introduction

The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Geometrically, this means that if P,Q and R are three distinct points on the graph of f with Q between P and R, then Q is on or below chord PR. A huge amount of the researchers interested in this definition and there are several papers based on convexity. See the papers [1-10].

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$ with $a<b$ The following double inequality;

$$f(\frac{a+b}{2})\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}$$

is known in the literature as Hadamard's inequality [5]. Both inequalities hold in the reversed direction if f is concave.

In [2] Hudzik and Maligranda introduced following definition:

Definition 1.1. Let $s\in(0,1]$ be fixed real number. A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$$

holds for all $x,y\in[0,\infty)$ with $\alpha+\beta=1,\ \alpha,\beta\ge 0$.

Orlicz gave the following definition of $s-$convexity in the first sense in [3]:

Definition 1.2. Let $s\in(0,1]$ be fixed real number. A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex (in the first sense), or that f belongs to the class K_s^1, if

$$f(\alpha x+\beta y)\le\alpha^s f(x)+\beta^s f(y)$$

holds for all $x,y\in[0,\infty)$ with $\alpha^s+\beta^s=1,\ \alpha,\beta\ge 0$.

It is clear that $s-$convexity mean just the convexity when $s=1$. In [11], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which hold for $s-$convex functions in the second sense:

Theorem 1.3. Suppose that $f:[0,\infty)\to[0,\infty)$ is an $s-$convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1([a,b])$, then the following inequalities hold:

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}\quad(1.1)$$

The constant $k=\dfrac{1}{s+1}$ is the best possible in the second inequality in (1.1).

In a recent paper [1], Latif and Dragomir proved following Theorems:

Theorem 1.4. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f'\right|$ is convex on $[a,b]$ then the following inequality holds:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{b-a}{96}\right)\left[\left|f'(a)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right| +2\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 1.5. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|^q$ is convex on $[a,b]$ for some fixed $q>1$, then the following inequality holds:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right) \times\left\{\left(\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}} +\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}} +\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} +\left(\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\}, \tag{1.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.6. Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|^q$ is convex on $[a,b]$ for some fixed $q \ge 1$, then the following inequality holds:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{1.4}$$

$$\le \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^{\frac{1}{q}}\left(\frac{b-a}{16}\right) \times\left\{\left(\left|f'(a)\right|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}} +\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}} +\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+2\left|f'\left(\frac{a+3b}{4}\right)\right|^q\right)^{\frac{1}{q}} +\left(2\left|f'\left(\frac{a+3b}{4}\right)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right\},$$

Theorem 1.7. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|$ is concave on $[a,b]$ for some fixed $q>1$, then the following inequality holds:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{1.5}$$

$$\le \left(\frac{q-1}{2q-1}\right)^{\frac{q-1}{q}}\left(\frac{b-a}{16}\right)\left[\left|f'\left(\frac{7a+b}{8}\right)\right|+\left|f'\left(\frac{5a+3b}{8}\right)\right| +\left|f'\left(\frac{3a+5b}{8}\right)\right|+\left|f'\left(\frac{a+7b}{8}\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.8. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'\right|^q$ is concave on $[a,b]$ for some fixed $q>1$, then the following inequality holds:

$$\left|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{3b+a}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \tag{1.6}$$

$$\le \left(\frac{b-a}{32}\right)\left[\left|f'\left(\frac{13a+3b}{12}\right)\right|+\left|f'\left(\frac{11a+5b}{12}\right)\right| +\left|f'\left(\frac{5a+13b}{12}\right)\right|+\left|f'\left(\frac{3a+13b}{12}\right)\right|\right].$$

The main aim of this paper is to establish some new inequalities involving values of $\left|f'\right|$ at intermediate points of $[a,b]$ interval for functions whose absolute values of derivatives are s-convex and s-concave.

2. Main Results

We need following lemma to prove our main Theorems:

Lemma 2.1. [4]Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b \in I$ with $a<b$. If $f' \in L[a,b]$ and n is an odd number then the following equality holds:

$$\left(\frac{b-a}{n+1}\right)\sum_{k=0}^{(n-1)/2}\left[\int_0^1 tf'\left(\begin{array}{l} t\dfrac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\dfrac{a(n-2k+1)+b.(2k)}{n+1}\end{array}\right)dt\right.$$
$$\left.+\int_0^1 (1-t).f'\left(\begin{array}{l} t\dfrac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\dfrac{a(n-2k)+b.(2k+1)}{n+1}\end{array}\right)dt\right] \quad (2.1)$$
$$=\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx.$$

Theorem 2.2. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b \in I$ with $a<b$. If $\left|f'\right|$ is $s-$convex on $[a,b]$ in the second sense and n is an odd number then the following inequality holds:

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{(n+1)(s+1)(s+2)}$$
$$\times\sum_{k=0}^{(n-1)/2}\left(2(s+1)\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right. \quad (2.2)$$
$$+\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|$$
$$\left.+\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right).$$

Proof. By using Lemma 2.1 and properties of modulus, we have

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\int_0^1 \left|tf'\left(\begin{array}{l} t\dfrac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\dfrac{a(n-2k+1)+b2k}{n+1}\end{array}\right)\right|dt\right.$$
$$\left.+\int_0^1\left|(1-t)f'\left(\begin{array}{l} t\dfrac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\dfrac{a(n-2k)+b(2k+1)}{n+1}\end{array}\right)\right|dt\right).$$

By using the $s-$convexity of $\left|f'\right|$, we obtain

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\int_0^1 t^{s+1}dt\right.$$
$$+\left|f'\left(\frac{a(n-2k+1)+b.2k}{n+1}\right)\right|\int_0^1 t(1-t)^s dt$$
$$+\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|$$
$$\left.\times\int_0^1 (1-t)t^s dt+\left|f'\left(\frac{a(n-2k)+b.(2k+1)}{n+1}\right)\right|\int_0^1 (1-t)^{s+1}dt\right)$$
$$=\frac{b-a}{(n+1)(s+1)(s+2)}$$
$$\times\sum_{k=0}^{(n-1)/2}\left[2(s+1)\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|\right.$$
$$+\left|f'\left(\frac{a(n-2k+1)+b.2k}{n+1}\right)\right|$$
$$\left.+\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|\right].$$

Which completes the proof.

Corollary 2.3. If we choose $n=1$ in (2.2) we obtain the following result:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{2(s+1)(s+2)}\left(\begin{array}{l} 2(s+1)\left|f'\left(\dfrac{a+b}{2}\right)\right| \\ +\left|f'(a)\right|+\left|f'(b)\right|\end{array}\right).$$

Remark 2.4. If we choose $n=3$ and $s=1$ in (2.2) , this inequality reduces to (1.2).

Theorem 2.5. Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ where $a,b \in I$ with $a<b$. If $\left|f'\right|^q$ is $s-$convex on $[a,b]$ in the second sense for some fixed $p>1$ and n is an odd number, then the following inequality holds:

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \left(\frac{b-a}{n+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$

$$\times \sum_{k=0}^{(n-1)/2}\left\{\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^{q} + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^{q}\right]^{\frac{1}{q}} \right. \tag{2.3}$$

$$\left. + \left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^{q} + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^{q}\right]^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 2.1 and by using the Hölder inequality, we have

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left\{\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1\left|f'\left(\begin{array}{l} t\frac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\end{array}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int_0^1\left|f'\left(\begin{array}{l} t\frac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\end{array}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \tag{2.4}$$

Since $|f|^q$ is $s-$convex on $[a,b]$, we have

$$\int_0^1\left|f'\left(\begin{array}{l} t\frac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\end{array}\right)\right|^q dt$$

$$\le \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\int_0^1 t^s dt$$

$$+\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\int_0^1 (1-t)^s dt$$

$$=\frac{1}{(s+1)}\left[\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\right].$$

Similarly,

$$\left|\int_0^1\left|f'\left(\begin{array}{l} t\frac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\end{array}\right)\right|^q dt\right|$$

$$\le\frac{1}{s+1}\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q + \left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right].$$

By using the last two inequalities in (2.4), we obtain the desired result.

Corollary 2.6. If we choose $n=1$ in (2.3), we obtain the following result:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$

$$\times\left\{\left[\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right]^{\frac{1}{q}} + \left[\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Remark 2.7. If we choose $n=3$ and $s=1$ in (2.3), this inequality reduces to (1.3).

Theorem 2.8. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f'\right|^q$ is $s-$convex on $[a,b]$ in the second sense for some fixed $q\ge 1$ and n is an odd number, then the following inequality holds:

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{n+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\times\sum_{k=0}^{(n-1)/2}\left\{\left[(s+1)\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right.\right.$$

$$+\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\Bigg]^{\frac{1}{q}}$$

$$+\left[\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q+(s+1)\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}. \quad (2.5)$$

Proof. From Lemma 2.1 and by using Power mean inequality, we have

$$\left|\sum_{k=0}^{(n-1)/2}2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left\{\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1 t\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-t)dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1(1-t)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (2.6)$$

Since $\left|f'\right|^q$ is $s-$convex on $[a,b]$ in the second sense, we have

$$\int_0^1 t\left|f'\left(t\frac{a(n-2k)+b(2k+1)}{n+1}+(1-t)\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q dt$$

$$\le\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q\int_0^1 t^{s+1}dt$$

$$+\left|f'\left(\frac{a(n-2k+1)+b(2k)}{n+1}\right)\right|^q\int_0^1 t(1-t)^{s+1}dt$$

$$=\frac{1}{s+2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q$$

$$+\frac{1}{(s+1)(s+2)}\left|f'\left(\frac{a(n-2k+1)+b2k}{n+1}\right)\right|^q.$$

Similarly,

$$\left|\int_0^1(1-t)\left|f'\left(t\frac{a(n-2k-1)+b(2k+2)}{n+1}+(1-t)\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q dt\right|$$

$$\le\frac{1}{(s+1)(s+2)}\left|f'\left(\frac{a(n-2k-1)+b(2k+2)}{n+1}\right)\right|^q$$

$$+\frac{1}{s+2}\left|f'\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)\right|^q$$

Using the last two inequalities in (2.6), we get the result.

Corollary 2.9. If we choose $n=1$ in (2.5), we obtain the following result:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\times\left\{\left[(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right]^{\frac{1}{q}}+\left[(s+1)\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

Remark 2.10. If we choose $n=3$ and $s=1$ in (2.3), this inequality reduces to (1.4).

Theorem 2.11. Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on $\overset{\circ}{I}$ such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f'\right|^q$ is $s-$concave on $[a,b]$ in the second sense for some fixed $q>1$, and n is an odd number, then the following inequality holds:

$$\left|\sum_{k=0}^{(n-1)/2}2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{n+1}\right)\left(\frac{q-1}{2q-1}\right)2^{s-1}$$

$$\times\sum_{k=0}^{(n-1)/2}\left(\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q+\left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q\right) \quad (2.7)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and using the Hölder inequality for $q>1$ and $p=\frac{q}{q-1}$, we have

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le \frac{b-a}{n+1}\sum_{k=0}^{(n-1)/2}\left(\left(\int_0^1 t^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}\right.$$

$$\times\left(\int_0^1\left|f'\left(\begin{array}{l} t\dfrac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\dfrac{a(n-2k+1)+b2k}{n+1}\end{array}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{\frac{q-1}{q}}$$

$$\left.\times\left(\int_0^1\left|f'\left(\begin{array}{l} t\dfrac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\dfrac{a(n-2k)+b(2k+1)}{n+1}\end{array}\right)\right|^q dt\right)^{\frac{1}{q}}\right) \quad (2.8)$$

Since $\left|f'\right|^q$ is s-concave on $[a,b]$ and by using the Hadamard inequality for $s-$ concave functions, we have

$$\int_0^1\left|f'\left(\begin{array}{l} t\dfrac{a(n-2k)+b(2k+1)}{n+1} \\ +(1-t)\dfrac{a(n-2k+1)+b2k}{n+1}\end{array}\right)\right|^q dt$$

$$\le 2^{s-1}\left|f'\left(\frac{\dfrac{a(n-2k)+b(2k+1)}{n+1}+\dfrac{a(n-2k+1)+b(2k)}{n+1}}{2}\right)\right|^q$$

$$= 2^{s-1}\left|f'\left(\frac{a(2n-4k+1)+b(4k+1)}{2(n+1)}\right)\right|^q$$

and similarly,

$$\int_0^1\left|f'\left(\begin{array}{l} t.\dfrac{a(n-2k-1)+b(2k+2)}{n+1} \\ +(1-t)\dfrac{a(n-2k)+b.(2k+1)}{n+1}\end{array}\right)\right|^q dt$$

$$\le 2^{s-1}\left|f'\left(\frac{\dfrac{a(n-2k-1)+b(2k+2)}{n+1}+\dfrac{a(n-2k)+b.(2k+1)}{n+1}}{2}\right)\right|^q$$

$$= 2^{s-1}\left|f'\left(\frac{a(2n-4k-1)+b(4k+3)}{2(n+1)}\right)\right|^q.$$

Using these two inequalities in (2.8), we get the desired result.

Corollary 2.12. If we choose $n=1$ in (2.7), we obtain the following result:

$$\left|2f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{2}\right)\left(\frac{q-1}{2q-1}\right)2^{s-1}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|^q+\left|f'\left(\frac{3b+a}{4}\right)\right|^q\right].$$

Corollary 2.13. Under the conditions of Theorem 2.11 and assume that $\left|f'\right|$ is a linear function, the following inequality holds:

$$\left|\sum_{k=0}^{(n-1)/2} 2f\left(\frac{a(n-2k)+b(2k+1)}{n+1}\right)-\frac{n+1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\left(\frac{b-a}{n+1}\right)\left(\frac{q-1}{2q-1}\right)2^s$$

$$\times\sum_{k=0}^{(n-1)/2}\left[\left(\frac{n-k}{n+1}\right)\left|f'(a)\right|+\left(\frac{k+1}{n+1}\right)\left|f'(b)\right|\right].$$

Proof. It follows directly from Theorem 2.11 and linearity of $\left|f'\right|$.

Remark 2.14. If we choose $n=3$ and $s=1$ in (2.7), this inequality reduces to (1.5).

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