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On Generating Functions of Quadruple Hypergeometric Function $X_8^{(4)}$

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Abstract In this work, by using Laplace integral representation of quadruple function $X_8^{(4)}$ defined in [1], we introduce new generating functions involving some triple hyper-geometric functions and $X_8^{(4)}$ itself. Some particular cases and consequences of our main results are also considered.

Keywords: *Laplace transforms, triple hyper-geometric functions, quadruple hypergeometric functions, generating functions*

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1. Introduction

Recently, Bin-Saad et al. [1] introduced five new quadruple hyper-geometric functions whose names are $X_6^{(4)}$ $X_7^{(4)}$ $X_8^{(4)}$ $X_9^{(4)}$ $X_{10}^{(4)}$ to investigate their five Laplace integral representations which include the confluent hyper-geometric function ${}_0F_1$, ${}_1F_1$, a Humbert functions Φ_2, Φ_3 and Ψ_2 in their kernels. Very recently Bin-Saad and Younis [2] established new integral representations of Euler type for some hyper-geometric functions of four variables, whose kernels include the quadruple hyper-geometric functions $X_6^{(4)}$ $X_7^{(4)}$ $X_8^{(4)}$ $X_9^{(4)}$, of which $X_7^{(4)}$ $X_8^{(4)}$ is defined as follows

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2.c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \left[\frac{(a_1)_{2m+2q+n+p}\,(a_2)_n\,(a_3)_p}{(c_1)_{m+n}\,(c_2)_p\,(c_3)_q} \times \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \right] \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol given by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1)$ $(n \in N := \{1,2,3,...\})$ and $(a)_0 = 1$.

The following is the Laplace integral representation of the function $X_8^{(4)}$ (see [1]):

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u) = \frac{1}{\Gamma(a_1)} \int_0^{\infty} e^{-s}\, s^{a_1-1} \left[\begin{array}{l} \Phi_3\left(a_2;c_1;,sy,s^2x\right) \\ \times {}_1F_1(a_3;c_2;sz) \\ \times {}_1F_1(a_3;c_3;sz) \\ \times {}_0F_1\left(-;c_3;s^2u\right) \end{array} \right] ds, \tag{1.2}$$

$(\mathrm{Re}(a_1) > 0)$.

In diverse areas in applied mathematics and mathematical physics, generating functions play an important role in the investigation of various useful properties of the sequences which they generate. These are used to find certain properties and formulas for numbers and polynomials in a wide range of research subjects such as modern combinatory. One can refer the extensive work of Srivastava and Manocha [3] for a systematic introduction to, and several interesting and useful applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of hyper-geometric functions and polynomials in one, two or more variables, among much abundant literature. In fact, a remarkable large number of generating functions involving a variety of hyper-geometric functions have been developed by many authors (for example [4,5,6,7] and the related references therein). Here, we use the integral representation of the hyper-geometric function of four variables $X_8^{(4)}$ o obtain new generating functions involving Exton's functions X_1 ,, X_2 X_6 , X_8 of three variables, the Lauricella functions of three variables $F_A^{(3)}, F_C^{(3)}$ and the quadruple functions $X_8^{(4)}$ itself. Some special cases of the main results here are also considered.

2. Generating Functions

For our purpose, we begin by recalling Exton's functions of three variables X_1 , X_2 , X_6 and X_8 defined by

$$X_1(a_1,a_2;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+2n+p}\,(a_2)_p}{(c_1)_{n+p}\,(c_2)_m}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.1}$$

$$X_2(a_1,a_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+2n+p}\,(a_2)_p}{(c_1)_m\,(c_2)_n\,(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.2}$$

$$X_6(a_1,a_2,a_3;c_1,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}\,(a_2)_n(a_3)_p}{(c_1)_{m+n}\,(c_2)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.3}$$

and

$$X_1(a_1,a_2,a_3;c_1,c_2.c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}\,(a_2)_n(a_3)_p}{(c_1)_m\,(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{2.4}$$

respectively (see [8]). Lauricella hyper-geometric functions of three variables $F_A^{(3)}, F_C^{(3)}$ are as below (see [9])

$$F_A^{(3)}(a,a_1,a_2,a_3;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p+}\,(a_1)_m\,(a_2)_n\,(a_3)_p}{(c_1)_m\,(c_2)_n\,(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.5}$$

$$F_C^{(3)}(a_1,a_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}\,(a_2)_{m+n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}. \tag{2.6}$$

Now, we begin the following theorem:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1,c_2,a_1+k;c_1,c_1,2c_2,c_3;x,y,2z,u\right)\\&=(1-z)^{-a_1}\sum_{k,q=0}^{\infty}\frac{(a_1+k)_{2q}}{(c_3)_q q!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{u}{(1-z)^2}\right)^k X_1\left(a_1+k+q,\,a_2;c_2+\tfrac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right);\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_1,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u\right)\\&=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_1\left(a_1+k,\,a_2;c_3,c_1;\frac{u}{(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right);\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u\right)\\&=(1-z)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{u}{1-z}\right)^n X_2\left(a_1+k+n,\,c_2-a_3;c_1+n,c_3,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right);\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)\\&=(1+2u)^{-a_1}\sum_{k,p=0}^{\infty}\frac{(a_1+k)_p(a_3)_p}{(c_2)_p p!k!}\left(\frac{w}{1+2u}\right)^k\left(\frac{z}{1-z}\right)^p X_6\left(a_1+k+p,\,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1+2u)^2},\frac{y}{1+2u},\frac{4u}{1+2u}\right);\end{aligned} \tag{2.10}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k=0}^{\infty}\left(\frac{w}{1+2u-z}\right)^k X_6\left(a_1+k,\,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{\left(1-2u-z\right)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right);\tag{2.11}$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1+2u-z}\right)^k\left(\frac{y}{1+2u-z}\right)^n\tag{2.12}$$
$$X_8\left(a_1+k+n,\,c_2-a_3,c_3-\tfrac{1}{2};c_1+n,c_2,2c_3-1;\frac{x}{\left(1-2u-z\right)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u^2\right)$$
$$=\left(1+2u\right)^{-a_1}\sum_{k,m=0}^{\infty}\frac{(a_1+k)_{2m}}{(c_1)_m m!k!}\left(\frac{w}{1+2u}\right)^k\left(\frac{x}{\left(1+2u\right)^2}\right)^m\tag{2.13}$$
$$F_A^{(3)}\left(a_1+k+2m,\,a_2,a_3,c_3-\tfrac{1}{2};c_1+m,c_2,2c_3-1;\frac{x}{1+2u},\frac{y}{1+2u},\frac{4u}{1+2u}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,2c_2,c_3;x,y,2z,u\right)$$
$$=\left(1-z\right)^{-a_1}\sum_{k,n=0}^{\infty}\frac{(a_1+k)_n(a_2)_n}{(c_1)_n n!k!}\left(\frac{w}{1-z}\right)^k\left(\frac{y}{1-z}\right)^n\tag{2.14}$$
$$F_C^{(3)}\left(\frac{a_1+k+n}{2},\frac{a_1+k+n+1}{2},c_1+n,c_2+\tfrac{1}{2},c_3;\frac{4x}{\left(1-z\right)^2},\frac{z^2}{\left(1-z\right)^2},\frac{4u}{\left(1-z\right)^2}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,a_3,a_1+k;c_1,c_1,c_2,c_3;x,y,z,u\right)$$
$$=\left(1-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k\tag{2.15}$$
$$X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2-a_3,a_1+k;c_1,c_1,c_2,c_3;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1},\frac{u}{(1-z)^2}\right);$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_8^{(4)}(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_2,a_1+k;c_1,c_1,c_2,c_3;x,y,2z,u^2)$$
$$=\left(1+2u-z\right)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1+2u-z}\right)^k\tag{2.16}$$
$$X_8^{(4)}\left(a_1+k,a_1+k,a_1+k,a_1+k,a_1+k,a_2,c_3-\tfrac{1}{2},a_1+k;c_1,c_1,c_2+\tfrac{1}{2};\frac{x}{(1+2u-z)^2},\frac{y}{1+2u-z},\frac{4u}{1+2u-z},\frac{z^2}{(1+2u-z)^2}\right);$$

Proof. To prove the above relations, we need the following formulae (cf. [3,10,11,12]):

$$\Gamma\left(z\right)=s^z\int_0^{\infty}e^{-st}\,t^{z-1}\,dt,\ \ \mathrm{Re}\left(z\right)>0;\tag{2. 17}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a \neq 0,-1,-2,\ldots; \tag{2.18}$$

$$(a)_{n+m} = (a)_n + (a+n)_m; \tag{2.19}$$

$$(a)_{2m} = 2^{2m}\left(\frac{a}{2}\right)_m \left(\frac{a+1}{2}\right)_m, \quad m = 0,1,2\ldots; \tag{2.20}$$

$${}_0F_0(-;-;x) = e^{-x}; \tag{2.21}$$

$${}_0F_1\left(-;a;x^2\right) = e^{-2x}\,{}_1F_1\left(a-\tfrac{1}{2};\,2a-1;\,4x\right); \tag{2.22}$$

$${}_0F_1\left(-;a+\tfrac{1}{2};\frac{x^2}{4}\right) = e^{-x}\,{}_1F_1(a;\,2a;\,2x); \tag{2.23}$$

$${}_1F_1(a;b;x) = e^{x}\,{}_1F_1(b-a;\,b;\,-x); \tag{2.24}$$

For the convenience, we denote the left hand side of (2.7) with δ, using (1.2)

$$\delta = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(a_1+k)} \int_0^{\infty} e^{-s} s^{a_1+k-1}\, \Phi_3\left(a_2;c_1;,sy,s^2x\right) {}_1F_1(c_2;2c_2;2sz)\, {}_1F_1(a_3;c_3;sz)\, {}_0F_1\left(-;c_3;s^2u\right) ds,$$

by using (2.23), we have

$$\delta = \sum_{k,m,n,q=0}^{\infty} \frac{(a_2)_n w^k x^m y^n u^q}{(c_1)_{m+n}(c_3)_q k!m!n!q!\Gamma(a_1+k)} \int_0^{\infty} e^{-s(1-z)}\, s^{a_1+k+2m+2q+n-1}\, {}_0F_1\left(-;c_2+\tfrac{1}{2};\tfrac{1}{4}s^2z^2\right) ds.$$

The function ${}_0F_1$ which appears in above equation can be replaced by its series form and then interchanging the order of the summation and integral sign which is permissible here, we get

$$\delta = \sum_{k,m,n,p,q=0}^{\infty} \frac{(a_2)_n w^k x^m y^n (\frac{1}{4}z^2)^p u^q}{(c_1)_{m+n}(c_2+\frac{1}{2})_p (c_3)_q k!m!n!p!q!\Gamma(a_1+k)} \int_0^{\infty} e^{-s(1-z)}\, s^{a_1+k+2m+2p+2q+n-1}.$$

Now, use of (2.17), (2.18) and (2.19), in above equation and then simplified with series manipulation completes the proof of relation (2.7). From the relations (2.17) to (2. 24), one can easily obtain the other generating functions.

3. Special Cases

It is easy to observe that the main results (2.7) to (2.16) gave a number generating functions and transformations for the hhyper-geometric series of four variables $X_8^{(4)}$. In the present section, we will mention only some special cases. setting $k=0$ in (2.7) to (2.16), we obtain the following relations:

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,c_2,a_1;c_1,c_1,2c_2,c_3;x,y,2z,u)$$
$$= (1-z)^{-a_1} \sum_{q=0}^{\infty} \frac{(a_1)_{2q}}{(c_3)_q q!} \left(\frac{u}{(1-z)^2}\right)^q X_1\left(a_1+q,a_2;c_2+\tfrac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \tag{3.1}$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,c_2,a_1;c_1,c_1,c_2,c_3;x,y,z,u) = (1-z)^{-a_1} X_1\left(a_1,a_2;c_3,c_1;\frac{u}{(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \tag{3.2}$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$= (1-z)^{-a_1} \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(c_1)_n n!} \left(\frac{u}{1-z}\right)^n X_2\left(a_1+n,c_2-a_3;c_1+n,c_3,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right); \tag{3.3}$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u)^{-a_1}\sum_{p=0}^{\infty}\frac{(a_1)_p(a_3)_p}{(c_2)_p\,p!}\left(\frac{z}{1-z}\right)^p X_6\left(a_1+k+p,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1+2u)^2},\frac{y}{1+2u},\frac{4u}{1+2u}\right); \quad (3.4)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u-z)^{-a_1}X_6\left(a_1,a_2,c_3-\tfrac{1}{2};c_1,2c_3-1;\frac{x}{(1-2u-z)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right); \quad (3.5)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u-z)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(c_1)_n\,n!}\left(\frac{y}{1+2u-z}\right)^n \quad (3.6)$$
$$X_8\left(a_1+n,c_2-a_3,c_3-\tfrac{1}{2};c_1+n,c_2,2c_3-1;\frac{x}{(1-2u-z)^2},\frac{y}{1-2u-z},\frac{4u}{1-2u-z}\right);$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u^2)$$
$$=(1+2u)^{-a_1}\sum_{m=0}^{\infty}\frac{(a_1)_{2m}}{(c_1)_m\,m!}\left(\frac{x}{(1+2u)^2}\right)^m F_A^{(3)}\left(a_1+2m,a_2,a_3,c_3-\tfrac{1}{2};c_1+m,c_2,2c_3-1;\frac{x}{1+2u},\frac{y}{1+2u},\frac{4u}{1+2u}\right); \quad (3.7)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,2c_2,c_3;x,y,2z,u)$$
$$=(1-z)^{-a_1}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(c_1)_n\,n!}\left(\frac{y}{1-z}\right)^n F_C^{(3)}\left(\frac{a_1+n}{2},\frac{a_1+n+1}{2},c_1+n,c_2+\tfrac{1}{2},c_3;\frac{4x}{(1-z)^2},\frac{z^2}{(1-z)^2},\frac{4u}{(1-z)^2}\right); \quad (3.8)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_1,c_2,c_3;x,y,z,u)$$
$$=(1-z)^{-a_1}X_8^{(4)}(a_1,a_1+k,a_1,a_1,a_1,a_2,c_2-a_3,a_1;c_1,c_1,c_2,c_3;\tfrac{x}{(1-z)^2},\tfrac{y}{1-z},\tfrac{z}{z-1},\tfrac{u}{(1-z)^2}); \quad (3.9)$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,c_2,a_1;c_1,c_1,c_2,c_3;x,y,2,zu^2)$$
$$=(1+2u-z)^{-a_1}X_8^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,c_3-\tfrac{1}{2},a_1;c_1,c_1,c_2+\tfrac{1}{2};\tfrac{x}{(1+2u-z)^2},\tfrac{y}{1+2u-z},\tfrac{4u}{1+2u-z},\tfrac{z^2}{(1+2u-z)^2}\right); \quad (3.10)$$

Equations (3.2), (3.7) and (3.9) with $x=0$, yield Exton's results [8]. Equations (3.4), (3.5) and (3.10) with $z=0$, yields the known results [8]. If we put $y=0$ in (3.8), we get

$$X_2(a_1,c_2;c_1,c_3 2c_2;x,u,2z)=(1-z)^{-a_1}F_C^{(3)}\left(\frac{a_1}{2},\frac{a_1+1}{2},c_1,c_2+\tfrac{1}{2},c_3;\frac{4x}{(1-z)^2},\frac{z^2}{(1-z)^2},\frac{4u}{(1-z)^2}\right); \quad (3.11)$$

Now, if in (2.7) and (2.9), we take $u=0$, we shall obtain the following generating functions :

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_6(a_1+k,a_2,c_2;c_1,2c_2,c_3;x,y,2z)$$
$$=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_1\left(a_1+k,a_2;c_2+\tfrac{1}{2},c_1;\frac{z^2}{4(1-z)^2},\frac{x}{(1-z)^2},\frac{y}{1-z}\right); \quad (3.12)$$

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}X_6(a_1+k,a_2,a_3;c_1,c_2;x,y,z)$$
$$=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{w}{1-z}\right)^k X_6\left(a_1+k,a_2,c_2-a_3;c_1,c_2;\frac{x}{(1-z)^2},\frac{y}{(1-z)^2},\frac{z}{1-z}\right). \quad (3.13)$$

A special case of (3.13) when $K = 0$, yields the well-known results (see [8]).

4. Conclusion and Observation

Based on Laplace integral representation of the quadruple function $X_8^{(4)}$ defined in [1], we introduce new generating functions for the function $X_8^{(4)}$ involving triple hyper-geometric functions. Some particular cases and consequences of our main results are also considered. We conclude this investigation by remarking that the schema suggested in the derivation of the results in this work can be applied to find other new generating functions of other four variables hyper-geometric series and study their particular cases.

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