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Fixed Point of Surface Transformation and a Fixed Point Theorem for Triangular Surface Mapping

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Abstract In this study, it was shown that the existence of fixed points of some surface transformations was defined as an example according to the theorems in functional analysis in differential geometry. It was shown that f_i real-valued coordinate functions of F triangular space mapping defined from E^n to E^m has a single fixed point, if $\sup\left\{\left|\frac{\partial f_i}{\partial x_i}(X)\right| : i = 1,2,\ldots,n \in R \text{ ve } X \in U\right\} < 1.$

Keywords: *fixed point, triangular mapping, triangular space mapping, surface transformation*

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1. Introduction

In this study, we defined the triangular mapping in addition to surface mapping which exist in differential geometry. We proof that this triangular surface mapping has a unique fixed point and we examine the condition of convergence to this fixed point. I define a triangular surface mapping and examine the conditions of existence and uniqueness of this surface map's fixed point by originating the study "A Fixed Point Theorem For Triangular Mappings" made by F. Sart [2].

2. Definitions and Theorems

2.1. Definition: Let X be any space and f a map of X, or of a subset of X into . X .A point $x \in X$ is called a fixed point for f if $x = f(x)$. The set of all fixed point of f is denoted by Fix (f) [5].

2.2. Definition: Let X be a normed space, let K be a non-empty subset of X and $F: K \to K$ be a transformation. If $p \in F_T \neq \varnothing$ and for $\forall x \in K$ and

$$\|Fx - p\| \leq \|x - p\|.$$

Then, F is called a quasi-convergence transformation [7].

2.1. Example:

Let $S^2 = \{(x,y,z) : z = \sqrt{r^2 - x^2 - y^2}, x,y,z,r \in R\}$ be a sphere surface. Let's define a surface transformation $F(x,y,z) = (x,y,\sqrt{r^2 - x^2 - y^2})$ at $F: S^2 \to S^2$. In this case, since $\forall (x,y,z) \in S^2$ and $F(x,y,z) = (x,y,z)$, then $\forall (x,y,z)$ becomes a fixed point at the transformation of F (See Figure 1).

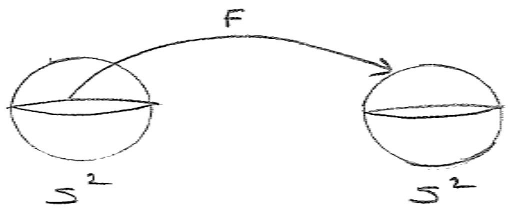

Figure 1. Sphere Surface

F surface transformation is a quasi-convergence transformation because for $\forall x, y \in S^2$ $\|Fx - y\| = \|x - y\|$.

Theorem 2.1: Let X be a Banach space, let K be a non-empty, compact convex subset of X and let $F: K \to K$ be a continuous transformation. In such case, F has at least one fixed point [6].

2.2. Example: Let $F(x,y,z) = (x,y,xy)$ be a transformation defined on a saddle surface $M = \{(x,y,z) : z = xy, x,y,z \in R\}$.

This transformation has at least one fixed point according to the Theorem 2.1.

For $\forall (x,y,z) \in M$ $F(x,y,z) = (x,y,z)$. Under the F transformation, $\forall (x,y,z) \in M$ is a fixed point and F has infinite fixed points.

Moreover, F surface transformation is a quasi-convergence transformation because for

$$\forall x, y \in M \quad \|Fx - y\| = \|x - y\|$$

2.3. Definition: Giving a function $F: R^n \to R^m$, let $f_1, f_2, f_3, \ldots f_m$, denote the real-valued functions on R^n such that

$$F(p) = (f_1(p), f_2(p), \ldots, f_m(p))$$

for all points p in R^n. These functions are called the Euclidean coordinate functions of F, and we write

$$F = (f_1, f_2, f_3, \ldots f_m).$$

The function F is differentiable provided its coordinate functions are differentiable in the unusal sense. A differentiable function $F: R^n \to R^m$ is called a mapping from R^n to R^m [1].

2.4. Definition: Let $F:\ E^n \to\ E^m$ be a mapping. If v is a tangent vector to E^n at P, let $F_*(v)$ be a initial velocity of the curve $t \to F(p + t v)$. The resulting function F_* send tangent vectors to E^n to tangent vectors to E^m, and is called the tangent map of F [1].

2.5. Definition: A function $F:\ M \to N$ from one surface to another is differentiable provided that for each patch X in M and Y in N the composite function $Y^{-1}MX$ is Euclidean differentiable (and defined on an open set of R^2). F is then called a mapping of surfaces [1].

2.6. Definition: Let U be a open-convex subset of E^n and C be nonempty compact subset of U. Let $f_1, f_2, \ldots, f_n :\to R$ be real-valued and differentiable functions. If mapping F is defined with $F: C \to C$ and

$$F(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f_1(x_1, 0, 0, \ldots, 0), \\ f_2(x_1, x_2, 0, \ldots, 0), \ldots, \\ f_n(x_1, x_2, x_3, \ldots, x_n) \end{pmatrix}$$

then F is called triangular mapping.

2.3. Example:

$$F(x,y,z) = (f_1(x), f_2(x,y), f_3(x,y,z)) = (x^2, xy - y^2, xyz).$$

If the Jakobian matrix of function F is stated as following:

$$JF = \begin{bmatrix} 2x & 0 & 0 \\ y & x-2y & 0 \\ yz & xz & yx \end{bmatrix}.$$

Lets take $V_p = (1,2,1)_{(1,1,1)}$. In this case,

$$JF(V_p) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = (2,-1,4)_{(1,0,1)}.$$

2.7. Definition: A square matrix is called upper triangular if all etries below the main diagonal are zones. Similarly, a square matrix is called lower triangular if all the entries above the main diagonal are zeros. A matrix that is either upper or lower triangular is called triangular [4].

Theorem 2.2. Let U be an open convex set of E^n, and C a non-empty compact subset U. Let F be a triangular mapping of U, i.e. of a type

$$(x_1, x_2, x_3, \ldots, x_n) \to \begin{pmatrix} f_1(x_1, x_2, x_3, \ldots x_n), \\ f_2(x_2, x_3, \ldots x_n), \ldots, f_n(x_n) \end{pmatrix},$$

which has the following properties:

1. F is a self-map on, C
2. F is continuous and differentiable on U;
3. F has a bounded derivative on U such that

$$\sup\left\{\left|\frac{\partial f_i}{\partial x_i}(X)\right| : i = 1, 2, \ldots, n \in R \ ve\ X \in U\right\} < 1.$$

Then;

- F has a unique fixed point in U;
- For any X_0 in C, the iterative sequence $X_{k+1} = F(X_k)$ converges to the unique fixed point.) [2].

Uniquenses proof. Suppose that X and Y are two different fixed points. By using fixed point definition

$$F(Y) - F(X) = Y - X.$$

On the other hand, by the mean-value theorem (Lagrange's theorem), there is an n-tuble of points W^*, such that

$$\frac{F(Y) - F(X)}{Y - X} = J_F(W^*)$$

where

$$J_F(W^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(W_1) & \frac{\partial f_1}{\partial x_2}(W_1)\ldots & \frac{\partial f_1}{\partial x_n}(W_1) \\ 0 & \frac{\partial f_2}{\partial x_2}(W_2)\ldots & \frac{\partial f_2}{\partial x_n}(W_2) \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \frac{\partial f_n}{\partial x_n}(W_n) \end{bmatrix}$$

$W_1, W_2, \ldots, W_n$ being each on the line segment joining X and Y. Therefore

$$J_F(W^*)(Y - X) = Y - X$$

and thus 1 is eigenvalue of the matrix $J_F(W^*)$. As it is triangular 1 belongs to diagonal, which contradicts the assumption on those elements [2].

2.4. Example: Lets state the **Jakobiyen** matrix $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ of transformation $= \left(\frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{2}x_3\right)$ as following

$$JF = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

From here,

$$\frac{F(X)-F(Y)}{X-Y} = \frac{1}{2} JF \left(JF \neq 1\right)$$

$$F(X)-F(Y) = \frac{1}{2}(X-Y)(JF \neq 1).$$

The number of fixed points of F transformation is one according to **Theorem 2.2**. This fixed point is the point of $(0,0,0)$.

Then, $F(0,0,0) = (0,0,0)$.

2.5. Example: For $F : E^3 \to E^3$ and

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3).$$

$$\lim_{x \to y} \frac{F(X)-F(Y)}{X-Y} = JF.$$

From here,

$$JF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$\lim_{x \to y} \frac{F(X)-F(Y)}{X-Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If

$$F(X) - F(Y) = X - Y,$$

for $\forall x \in E^3$ the $F(x) = x$ function has infinite fixed-points according to **Theorem 2.2**.

Theorem 2.3. If a function $f(x, y, z)$ is continuously differentible in an open set of R^2 containing the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and the line segment connecting them, then an equation

$$\begin{aligned} & f(x_2, y_2, z_2) - f(x_1, y_1, z_1) \\ & = f_x(a,b,c)(x_2 - x_1) + f_y(a,b,c)(y_2 - y_1) \\ & \quad + f_z(a,b,c)(z_2 - z_1) \end{aligned}$$

Where (a, b, c) an interior point of the line segment, is valid. (PlaneMath: mean value theorem for several variables=3) [3]

3. A New Definition and Theorem

Following definition 3.1 is triangular space mapping which was defined by using definition 2.1 and 2.5, and 2.6.

Definition 3.1: Let M and N be surfaces in E^n Euclid space, and let $F : M \to N$ be a mapping of surface. In addition, if F is a triangular mapping, then F is called triangular space mapping.

Theorem 3.1: Let M be a surface and, let $F : U \subseteq M \to M$ be a triangular space mapping. If including the transformation of F to a single fixed point

$$\sup\left\{\left|\frac{\partial f_i}{\partial x_i}(x)\right| : i = 1,2,3,...,n \in R \ ve \ X \in U\right\} \langle 1$$

where $f_1, f_2, ..., f_n$ are real-valued coordinate functions of F.

Proof: Let F be triangular space mapping which has a single fixed point. Then, from **Theorem 2.2**

$$JF\left(W^*\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} W_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} W_n & \cdots & \frac{\partial f_n}{\partial x_n} W_n \end{pmatrix}.$$

Determinant value of this matrix can be 1. If this value is 1, then the determinant value belongs to the diagonal of this matrix. Because, this matrix is the lower triangular matrix. At that rate,

$$\sup\left\{\left|\frac{\partial f_i}{\partial x_i}(x)\right| : i = 1,2,3,...,n \in R \ ve \ X \in U\right\} \langle 1.$$

3.1. Example: Let's define a triangular surface transformation as $U \subset R^3$,

$$U = \left\{(x,y,z) : x^2 + y^2 + z^2 \le \frac{1}{4}, 0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}, 0 \le z \le \frac{1}{2}\right\}$$

and

$$F : U \to R, F(x,y,z) = (x^2, x+y, yz)$$

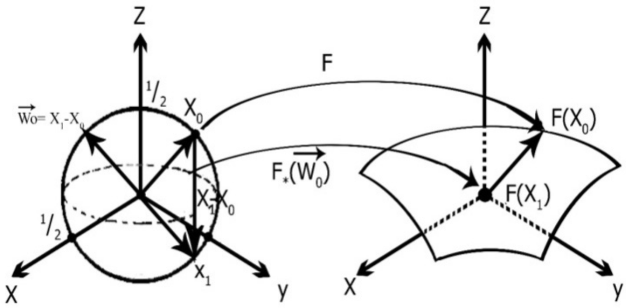

Figure 2. Triangular Surface Transformation

$$JF = \begin{pmatrix} 2x & 0 & 0 \\ 1 & 1 & 0 \\ 0 & z & y \end{pmatrix} \left(w_i\right)$$

For Jakobiyen matrix it is obvious that $\det JF \ \left(w_i\right) < 1$ is true for $\forall x_i \in U$

$$\det JF\left(w_i\right) = 2xy\left(w_i\right) \text{ and } \overrightarrow{W_i} = x_{p+k} - x_k.$$

Because each component is within the interval of $[0,1/2]$.

It can be told that $\text{JF}(w_i) = 2xy(w_i) < 1$.

Then, according to Theorem 4.2.2 transformation of F has limited derivative at U.

In this case;

1) F has one fixed point at U.

2) For any $x_0 \in V \subset U$ Picard iteration progression of $x_{k+1} = F(x_k)$ converges to this fixed point.

This point is $F^n(x_0) \to \bar{x}$.

$$x_0 = (1/2, 0, 0)$$

$$x_1 = F(x_0) = F(1/2, 0, 0) = \left(1/2^2, 0, 0\right)$$

$$x_2 = F(x_1) = F\left(1/2^2, 0, 0\right) = F\left(1/2^4, 0, 0\right)$$

$$x_3 = F(x_2) = F\left(1/2^4, 0, 0\right) = F\left(1/2^8, 0, 0\right).$$

The Picard iteration progression of

$$x_m = F(x_{m-1}) = F\left(1/2^{2^{m-1}}, 0, 0\right) = F\left(1/2^{2^m}, 0, 0\right)$$

converges to the point of $(0,0,0)$..

In this case $F^n(1/2, 0, 0) \to (0,0,0)$. Then $(0,0,0) \in U$ is a fixed point of transformation of F.

That means $F(0,0,0) = (0,0,0)$.

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