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Some Fixed Point Results on Multiplicative (b)-metric-like Spaces

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Abstract We give the concept of multiplicative partial metric space, multiplicative metric-like space, multiplicative b-metric space and multiplicative b-metric-like space. Then we build the existence and uniqueness of fixed points in a multiplicative b-metric-like space as well as in a partially ordered multiplicative b-metric-like space. We derive some fixed point results in multiplicative partial metric spaces, multiplicative metric-like spaces and multiplicative b-metric spaces as an application, some examples and an application to existence of solution of integral equations.

Keywords: *Partial metric space, metric-like space, b-metric space, b-metric-like space, fixed point, integral equation*

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1. Introduction

The idea of b-metric space and partial metric space were introduced by S. Czerwik [4] and S. G. Matthews [12], respectively. S. Shukla [15] introduced another generalization which is called a partial b-metric space. Amini Harandi [9] introduced a new extension of the concept of partial metric space, called a metric-like space. After that, A. Alghamdi [1] introduce the concept of b-metric-like space which generalizes the idea of partial metric space, metric-like space, and b-metric space. They established the existence and uniqueness of fixed points in a b-metric-like space as well as in a partially ordered b-metric-like space.

In 2008, Bashirov et al. [3] studied the usefulness of a new calculus, called multiplicative calculus due to Michael Grossman and Robert Katz in the period from 1967 till 1970. By using the concepts of multiplicative absolute values, Bashirov et al. defined a new distance so called multiplicative distance. Also, Ozavsar and Cevikel [6] introduced the concept of multiplicative contraction mapings and derive some fixed point results on these mappings on a complete multiplicative metric space.

In this paper, by using the concept of multiplicity we first introduce concept of multiplicative partial metric space, multiplicative metric-like space, multiplicative b-metric space and then we introduce a new generalization of these spaces which is called multiplicative b-metric-like space. Then, we derive some fixed point results. Also, some examples and an application to integral equations are provided for the support of our constructed results.

2. Multiplicative (b)-metric-like Space

We shall begin this section with the introduction to the concept of multiplicative partial metric space.

Definition 2.1. A mapping $\mathcal{D}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$, where $\mathcal{X}\neq\emptyset$, is called a multiplicative partial metric on $\mathcal{X}$ provided that, for all $x,y,z\in\mathcal{X}$,

($\mathcal{D}1$) $x=y\Leftrightarrow\mathcal{D}(x,x)=\mathcal{D}(y,y)=\mathcal{D}(x,y)$;

($\mathcal{D}2$) $\mathcal{D}(x,x)\le\mathcal{D}(x,y)$;

($\mathcal{D}3$) $\mathcal{D}(x,y)=\mathcal{D}(y,x)$;

($\mathcal{D}4$) $\mathcal{D}(x,z)\le\frac{[\mathcal{D}(x,y).\mathcal{D}(y,z)]}{\mathcal{D}(y,y)}$

$$\text{or } \mathcal{D}(x,z).\mathcal{D}(y,y)\le\mathcal{D}(x,y).\mathcal{D}(y,z).$$

The pair $(\mathcal{X},\mathcal{D})$ is a *multiplicative partial metric space.*

Definition 2.2. A mapping $\Psi:\mathcal{X}\times\mathcal{X}\to[1,\infty)$, where $\mathcal{X}\neq\emptyset$, is called a multiplicative metric-like on $\mathcal{X}$ provided that, for all $x,y,z\in\mathcal{X}$,

($\Psi1$) $\Psi(x,y)=1\Rightarrow x=y$;

($\Psi2$) $\Psi(x,y)=\Psi(y,x)$

($\Psi3$) $\Psi(x,z)\le\Psi(x,y).\Psi(y,z)$.

The pair $(\mathcal{X},\Psi)$ is a *multiplicative metric-like space.*

S. Czerwik [4] give idea of multiplicative b-metric-like space. Now we introduce concept of multiplicative b-metric space.

Definition 2.3. A mapping $\mathcal{N}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$, $\mathcal{X}\neq\emptyset$, is called a multiplicative b-metric on $\mathcal{X}$ provided that, for all $x,y,z\in\mathcal{X}$ and a constant $\mathbb{K}\ge1$,

($\mathcal{N}1$) $\mathcal{N}(x,y)=1\Leftrightarrow x=y$;

($\mathcal{N}2$) $\mathcal{N}(x,y)=\mathcal{N}(y,x)$;

($\mathcal{N}3$) $\mathcal{N}(x,z)\le[\mathcal{N}(x,y).\mathcal{N}(y,z)]^{\mathbb{K}}$.

The pair $(\mathcal{X},\mathcal{N})$ *is a multiplicative b-metric space.*

Definition 2.4. A mapping $\mathcal{M}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$, $\mathcal{X}\neq\emptyset$, is called a multiplicative b-metric-like on $\mathcal{X}$ provided that, for all $x,y,z\in\mathcal{X}$ and a constant $\mathbb{K}\geq 1$,

($\mathcal{M}$1) $\mathcal{M}(x,y)=1\Leftrightarrow x=y$;

($\mathcal{M}$2) $\mathcal{M}(x,y)=\mathcal{N}(y,x)$;

($\mathcal{M}$3) $\mathcal{M}(x,z)\leq[\mathcal{M}(x,y).\mathcal{M}(y,z)]^{\mathbb{K}}$.

The pair $(\mathcal{X},\mathcal{M},\mathbb{K})$ *is a multiplicative b-metric-like space.*

Example 2.5. Suppose $\mathcal{X}=[0,\infty)$, and let $\mathcal{M}:\mathcal{X}^2\to[1,\infty)$ be defined by $\mathcal{M}(x,y)=a^{(x+y)^2}$, where $a>1$ be a fixed real number, is a multiplicative b-metric-like for all $x,y,z\in\mathcal{X}$ with $\mathbb{K}=2$,

$$\begin{aligned}\mathcal{M}(x,y)&=a^{(x+y)^2}\\ &\leq a^{(x+z+z+y)^2}\\ &=a^{(x+z)^2+(z+y)^2+2(x+z)(z+y)}\\ &=a^{2\left[(x+z)^2+(z+y)^2\right]}\\ &=\left[a^{\left[(x+z)^2+(z+y)^2\right]}\right]^2\\ &=\left[a^{(x+z)^2}.a^{(z+y)^2}\right]^2\\ &\leq[\mathcal{M}(x,z).\mathcal{M}(z,y)]^2\end{aligned}$$

$$\mathcal{M}(x,y)\leq[\mathcal{M}(x,z).\mathcal{M}(z,y)]^2,$$

and so ($\mathcal{M}$3) holds. Clearly, ($\mathcal{M}$1) and ($\mathcal{M}$2) hold.

Similarly.

Example 2.6. Suppose $\mathcal{X}=[0,\infty)$ and let $\mathcal{M}:\mathcal{X}\to[1,\infty)$ be defined by $\mathcal{M}(x,y)=a^{(\max\{x,y\})^2}$, where $a>1$ be a fixed real number, is a multiplicative b-metric-like space with $\mathbb{K}=2$.

Example 2.7. Let $C_d(\mathcal{X})=\{p:\mathcal{X}\to\mathbb{R}:\sup_{x\in\mathcal{X}}|p(x)|<+\infty\}$. The function $\mathcal{M}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$, defined by

$$\mathcal{M}(p,r)=a\sqrt[3]{\sup_{x\in\mathcal{X}}\left(|p(x)|+|r(x)|\right)^3}$$
for all $p,r\in C_d(\mathcal{X})$,

where $a>1$ be a fixed real number, is a multiplicative b-metric-like with constant $\mathbb{K}=\sqrt[3]{4}$, , and so $(\mathcal{X},\mathcal{M},\sqrt[3]{4})$ is a multiplicative b-metric like space.

As we know, if $c,d\in\mathbb{R}_+$, then

$$(c+d)^3\leq 4\left(c^3+d^3\right) \text{ and } \sqrt[3]{c+d}\leq\sqrt[3]{c}+\sqrt[3]{d}.$$

This implies that

$$\mathcal{M}(p,r)\leq(\mathcal{M}(p,q).\mathcal{M}(q,r))^{\sqrt[3]{4}}$$
for all $p,q,r\in C_d(\mathcal{X})$.

Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ be a multiplicative b-metric-like space. Then multiplicative open ball with center at $x\in\mathcal{X}$ and radius $r>1$ is,

$$\mathcal{B}(x,r)=\left\{y\in\mathcal{X}:\left|\frac{\mathcal{M}(x,y)}{\mathcal{M}(x,x)}\right|<r\right\}.$$

Definition 2.8. Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ is multiplicative b-metric-like space, and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then point $x\in\mathcal{X}$ is the limit of the sequence $\{x_n\}$ if $\lim_{n\to+\infty}\mathcal{M}(x,x_n)=\mathcal{M}(x,x)$, and we say that the sequence $\{x_n\}$ is convergent to x.

Definition 2.9. Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ is multiplicative b-metric-like space.

($\mathbb{R}$1) A sequence $\{x_n\}$ is a Cauchy $\Leftrightarrow$ $\lim_{m,n\to+\infty}\mathcal{M}(x_m,x_n)$ exists.

($\mathbb{R}$2) A multiplicative b-metric-like space $(\mathcal{X},\mathcal{M})$ is complete if every cauchy sequence $\{x_n\}$ in $\mathcal{X}$ is convergent. That is,

$$\lim_{m,n\to+\infty}(x_m,x_n)=\mathcal{M}(x,x).$$

Proposition 2.10. *Let a multiplicative b-metric-like space* $(\mathcal{X},\mathcal{M},\mathbb{K})$, *and let a sequence* $\{x_n\}$ *in* $\mathcal{X}$ *such that* $\lim_{n\to+\infty}\mathcal{M}(x_n,x)=1$. *Then*

1. x is unique;

2. $(\mathcal{M}(x,y))^{1/\mathbb{K}}\leq\lim_{n\to\infty}\mathcal{M}(x_n,y)\leq(\mathcal{M}(x,y))^{\mathbb{K}}$,

Proof. (1) Suppose there exists $y\in\mathcal{X}$ such that $\lim_{n\to\infty}\mathcal{M}(x_n,y)=1$. Then

$$1\leq\mathcal{M}(y,x)\leq\left(\lim_{n\to\infty}(x_n,y).\lim_{n\to\infty}(x_n,x)\right)^{\mathbb{K}}=1.$$

Hence ($\mathcal{M}$1) gives $y=x$.

Proof. (2) As we know from ($\mathcal{M}$3)

$$\begin{aligned}(\mathcal{M}(x,y))^{1/\mathbb{K}}&\leq\lim_{n\to\infty}\mathcal{M}(x_n,y).\lim_{n\to\infty}\mathcal{M}(x_n,x)\\ \frac{(\mathcal{M}(x,y))^{1/\mathbb{K}}}{\lim_{n\to\infty}\mathcal{M}(x_n,x)}&\leq\lim_{n\to\infty}\mathcal{M}(x_n,y)\\ &\leq\left(\mathcal{M}(x,y).\lim_{n\to\infty}\mathcal{M}(x_n,x)\right)^{\mathbb{K}},\end{aligned}$$

and so

$$(\mathcal{M}(x,y))^{1/\mathbb{K}}\leq\lim_{n\to\infty}\mathcal{M}(x_n,y)\leq(\mathcal{M}(x,y))^{\mathbb{K}}.$$

Definition 2.11. Let a multiplicative b-metric-like space $(\mathcal{X},\mathcal{M},\mathbb{K})$, let $P\subseteq\mathcal{X}$. We say P is an open subset of $\mathcal{X}$ if for all $x\in P$ there exists $r>1$ such that $\mathcal{B}(x,r)\subseteq P$. Also, $\mathcal{F}\subseteq\mathcal{X}$ is closed subset of $\mathcal{X}$ if $\mathcal{X}\setminus\mathcal{F}$ is open subset of $\mathcal{X}$.

Proposition 2.12. *Let $(\mathcal{X},\mathcal{M})$ be a multiplicative b-metric-like space and let $\mathcal{F}\subseteq\mathcal{X}$. Then $\mathcal{F}$ is closed if and only if for any sequence $\{x_n\}\to x$, we have $\{x_n\}$, $x\in\mathcal{F}$.*

Proof. Let $\mathcal{F}$ is closed and $x\notin\mathcal{F}$, $\mathcal{X}\setminus\mathcal{F}$ is an open set. Then there exists $r>1$ such that $\mathcal{B}(x,r)\subseteq\mathcal{X}\setminus\mathcal{F}$.

Since $x_n\to x$ as $n\to\infty$, then

$$\lim_{n\to\infty}\left|\frac{\mathcal{M}(x_n,x)}{\mathcal{M}(x,x)}\right|=1.$$

Hence, there exists $n_0\in N$ such that for all $n\ge n_0$ we have

$$\left|\frac{\mathcal{M}(x_n,x)}{\mathcal{M}(x,x)}\right|<r.$$

So

$$\forall n\ge n_0,\{x_n\}\subseteq\mathcal{B}(x,r)\subseteq\mathcal{X}\setminus\mathcal{F}.$$

which yield to contradiction, since $\forall n\in N$, $\{x_n\}\subseteq\mathcal{F}$.

Conversely, let we have $x\in\mathcal{F}$, for any sequence $\{x_n\}$ in $\mathcal{F}$, such that $\{x_n\}\to x$. Let $y\notin\mathcal{F}$. Let us prove that for $\mathcal{B}(y,r_0)\cap\mathcal{F}\neq\varnothing$ there exist $r_0>1$. Suppose to the contrary that for $r>1$, we have $\mathcal{B}(y,r)\cap\mathcal{F}\neq\varnothing$. Then, for all $n\in N$, choose $x_n\in\mathcal{B}(y,1/n)\cap\mathcal{F}\neq\varnothing$. Therefore, $\left|\frac{\mathcal{M}(x_n,y)}{\mathcal{M}(y,y)}\right|<1/n$ for all $n\in N$. Hence, $x_n\to y$ as $n\to\infty$. So supposition on $\mathcal{F}$ implies $y\in\mathcal{F}$, which is wrong. Then, for all $y\notin\mathcal{F}$, there exists $r_0>1$ such that $\mathcal{B}(y,r_0)\cap\mathcal{F}\neq\varnothing$. That is, $\mathcal{F}$ is closed.

Lemma 2.13. *Suppose $(\mathcal{X},\mathcal{M},\mathbb{K})$ be a multiplicative b-metric-like space, and let $\{x_m\}_{m=0}^{n}\subset\mathcal{X}$. Then*

$$\mathcal{M}(x_n,x_0)\le(\mathcal{M}(x_0,x_1))^{\mathbb{K}}\ldots\ldots(\mathcal{M}(x_{n-2},x_{n-1}))^{\mathbb{K}^{n-1}}.(\mathcal{M}(x_{n-1},x_n))^{\mathbb{K}^{n-1}}.$$

For Lemma (2.13), we deduce the following result.

Lemma 2.14. *Suppose a sequence $\{y_n\}$ in multiplicative b-metric-like space $(\mathcal{X},\mathcal{M},\mathbb{K})$ such that*

$$\mathcal{M}(y_n,y_{n+1})\le\mathcal{M}(x_{n-1},x_n)^{\omega}$$

for some ω, $0<\omega<1/\mathbb{K}$, and each $n\in N$. Then $\lim_{m,n\to\infty}\mathcal{M}(y_m,y_n)=1$.

Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ be a multiplicative b-metric-like space. Define $\mathcal{M}^s:\mathcal{X}^2\to[1,\infty)$ by

$$\mathcal{M}^s(x,y)=\frac{|\mathcal{M}(x,y)|^2}{|\mathcal{M}(x,x).\mathcal{M}(y,y)|}$$

$\mathcal{M}^s(x,x)=1$ for all $x\in\mathcal{X}$.

3. Fixed Point of Expansion Mapping in Multiplicative (b)-metric-like Spaces

Many papers have been appeared on the work of expansive mapping see, e.g., ([1,5,11]). In this paper we drive fixed point results for expansive mappings in multiplicative b-metric-like space to the corresponding results of A. Alghamdi(see - [1]).

Theorem 3.1. *Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ ba a complete multiplicative b-metric-like space. Suppose $J:\mathcal{X}\to\mathcal{X}$ is onto mapping, such that*

$$\mathcal{M}(Jx,Jy)\ge[\mathcal{M}(x,y)]^{\left[S+I\min\left\{\begin{array}{l}\mathcal{M}^s(x,Jx),\mathcal{M}^s(y,Jy),\\ \mathcal{M}^s(x,Jy),\mathcal{M}^s(y,Jx)\end{array}\right\}\right]},\qquad(3.1)$$

$\forall x,y\in\mathcal{X}$, where $S>\mathbb{K}$, $I\ge0$. Then J has a fixed point.

Proof. Let $x_0\in\mathcal{X}$, as J is onto, so there exists $x_1\in\mathcal{X}$ such that $x_0=Jx_1$. Similarly, there exists $x_n\in\mathcal{X}$ such that $x_n=Jx_{n+1}$ for all $n\in N\cup\{0\}$. In case $x_{n_0}=x_{n+1}$ for some $n_0\in N\cup\{0\}$, then x_{n_0} is a fixed point of J. Now let $x_n\neq x_{n+1}$ for all n. Then from (3.1) with $x=x_n$ and $y=x_{n+1}$ we have

$$\mathcal{M}(Jx_n,Jx_{n+1})\ge[\mathcal{M}(x_n,x_{n+1})]^{\left[S+I\min\left\{\begin{array}{l}\mathcal{M}^s(x_n,Jx_n),\mathcal{M}^s(x_{n+1},Jx_{n+1}),\\ \mathcal{M}^s(x_n,Jx_{n+1}),\mathcal{M}^s(x_{n+1},Jx_n)\end{array}\right\}\right]},$$

which implies

$$\mathcal{M}(x_{n-1},x_n)\ge[\mathcal{M}(x_n,x_{n+1})]^{\left[S+I\min\left\{\begin{array}{l}\mathcal{M}^s(x_n,x_{n-1}),\mathcal{M}^s(x_{n+1},x_n),\\ \mathcal{M}^s(x_n,x_n),\mathcal{M}^s(x_{n+1},x_{n-1})\end{array}\right\}\right]}=[\mathcal{M}(x_n,x_{n+1})]^{S+I},$$

and so

$$\mathcal{M}(x_n,x_{n+1})\le[\mathcal{M}(x_{n-1},x_n)]^t \text{ where } t=\frac{1}{S+I}<\frac{1}{\mathbb{K}}.$$

Then by Lemma(2.14) we get $\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1$ exists (and is finite), so $\{x_n\}$ is a Cauchy sequence. Since $(\mathcal{X},\mathcal{M},\mathbb{K})$ is a complete multiplicative b-metric-like space, the sequence $\{x_n\}\to z\in\mathcal{X}$ so that

$$\lim_{m,n\to\infty}\mathcal{M}(x_n,z)=\mathcal{M}(z,z)=\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$$

Since J is onto, there exists $\hat{w}\in\mathcal{X}$ such that $z=J\hat{w}$. From (3.1) we have

$$\mathcal{M}(x_n,z)=\mathcal{M}(Jx_{n+1},J\hat{w})$$
$$\geq\left[\mathcal{M}(x_{n+1},\hat{w})\right]^{\left[S+I\min\left\{\begin{matrix}\mathcal{M}^S(x_{n+1},Jx_{n+1}),\mathcal{M}^S(\hat{w},J\hat{w}),\\ \mathcal{M}^S(x_{n+1},J\hat{w}),\mathcal{M}^S(\hat{w},Jx_{n+1})\end{matrix}\right\}\right]}$$
$$\geq\left[\mathcal{M}(x_{n+1},\hat{w})\right]^{\left[S+I\min\left\{\begin{matrix}\mathcal{M}^S(x_{n+1},x_n),\mathcal{M}^S(\hat{w},z),\\ \mathcal{M}^S(x_{n+1},z),\mathcal{M}^S(\hat{w},x_n)\end{matrix}\right\}\right]}.$$

By taking limit $n\to\infty$ in the above, we get

$$1=\lim_{n\to\infty}\mathcal{M}(x_n,z)\geq\left[\lim_{n\to\infty}\mathcal{M}(x_{n+1},\hat{w})\right]^{S+I}$$

which implies $\lim_{n\to\infty}\mathcal{M}(x_{n+1},\hat{w})=1.$ From Proposition 2.10(1), we have $z=\hat{w}$. That is, $z=Jz$.

If we take I = 0 in theorem (3.1), then we have the following corollary.

Corollary 3.2. *Let* $(\mathcal{X},\mathcal{M},\mathbb{K})$ *ba a complete multiplicative b-metric-like space. Suppose* $J:\mathcal{X}\to\mathcal{X}$ *is onto mapping and satisfies*

$$\mathcal{M}(Jx,Jy)\geq\left[\mathcal{M}(x,y)\right]^S,$$

$\forall x,y\in\mathcal{X}$, *where* $S>\mathbb{K}$. *Then* J *has a fixed point*.

Example 3.3. Let $\mathcal{X}=[0,\infty)$ and let a multiplicative b-metric-like $\mathcal{M}^s:\mathcal{X}^2\to[1,\infty)$ be defined by

$$\mathcal{M}(x,y)=a^{(x+y)^2},$$

where $a>1$ be a fixed real number.

Clearly $(\mathcal{X},\mathcal{M},2)$ is a complete multiplicative b-metric-like space. $J:\mathcal{X}\to\mathcal{X}$ be defined by

$$J(x)=\begin{cases}7x & \text{if } x\in[0,1),\\ 4x+1 & \text{if } x\in[1,2),\\ 9x+2 & \text{if } x\in[2,\infty).\end{cases}$$

Clearly, as J onto, so consider the following cases:

- Let $x,y\in[0,1)$, then

$$\mathcal{M}(Jx,Jy)=a^{(7x+7y)^2}=a^{49(x+y)^2}=\left(a^{(x+y)^2}\right)^{49}$$
$$\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

- Let $x,y\in[1,2)$, then

$$\mathcal{M}(Jx,Jy)=a^{(4x+4y+2)^2}\geq a^{(4x+4y)^2}=a^{16(x+y)^2}$$
$$=\left(a^{(x+y)^2}\right)^{16}\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

- Let $x,y\in[2,\infty)$, then

$$\mathcal{M}(Jx,Jy)=a^{(9x+9y+4)^2}\geq a^{(9x+9y)^2}=a^{81(x+y)^2}$$
$$=\left(a^{(x+y)^2}\right)^{81}\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

- Let $x\in[0,1)$, and let $y\in[1,2)$, then

$$\mathcal{M}(Jx,Jy)=a^{(7x+4y+1)^2}\geq a^{(4x+4y)^2}=a^{16(x+y)^2}$$
$$=\left(a^{(x+y)^2}\right)^{16}\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

- Let $x\in[0,1)$, and let $y\in[2,\infty)$, then

$$\mathcal{M}(Jx,Jy)=a^{(7x+9y+2)^2}\geq a^{(7x+7y)^2}=a^{49(x+y)^2}$$
$$=\left(a^{(x+y)^2}\right)^{49}\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

- Let $x\in[1,2)$, and let $y\in[2,\infty)$, then

$$\mathcal{M}(Jx,Jy)=a^{(4x+9y+3)^2}\geq a^{(4x+4y)^2}=a^{16(x+y)^2}$$
$$=\left(a^{(x+y)^2}\right)^{16}\geq\left(a^{(x+y)^2}\right)^3=\left(\mathcal{M}(x,y)\right)^3.$$

That is, $\mathcal{M}(Jx,Jy)\geq\left[\mathcal{M}(x,y)\right]^S$ for all $x,y\in\mathcal{X}$, where $S=3>2=\mathbb{K}$. The conditions of Corollary 3.2 are satisfied and J has a fixed point $x=0$.

Let $\sqcup_\xi^L$ be the class of functions $\zeta:(1,\infty)\to\left(L^2,\infty\right)$ such that it satisfy the condition $\xi(\hat{t}_n)\to\left(L^2\right)\Rightarrow\hat{t}_n\to 1,$ where $L>0$.

Theorem 3.4. *Let* $(\mathcal{X},\mathcal{M},\mathbb{K})$ *be a complete multiplicative b-metric-like space. Suppose the mapping* $J:\mathcal{X}\to\mathcal{X}$ *is onto and satisfies*

$$\mathcal{M}(Jx,Jy)\geq\left[\mathcal{M}(x,y)\right]^{\xi(\mathcal{M}(x,y))}\tag{3.2}$$

$\forall x,y\in\mathcal{X}$, *where* $\xi\in\sqcup_\xi^{\mathbb{K}}$ *. Then J has a fixed point.*

Proof. Let $x_0\in\mathcal{X}$, as J is onto, so there exists $x_1\in\mathcal{X}$ such that $x_0=Jx_1$. Similarly, there exists $x_n\in\mathcal{X}$ such that $x_n=Jx_{n+1}$ for all $n\in N\cup\{0\}$. In case $x_{n_0}=x_{n+1}$ for some $n_0\in N\cup\{0\}$, then it is clear that x_{n_0} is a fixed point of J. Now suppose that $x_n\neq x_{n+1}$ for all n. From (3.2) with $x=x_n$ and $y=x_{n+1}$ we get

$$\begin{aligned}\mathcal{M}(x_{n-1},x_n)&=\mathcal{M}(Jx_n,Jx_{n+1})\\ &\geq\mathcal{M}(x_n,x_{n+1})^{\xi(\mathcal{M}(x_n,x_{n+1}))}\\ &\geq\mathcal{M}(x_n,x_{n+1})^{\mathbb{K}^2}\geq\mathbb{K}\mathcal{M}(x_n,x_{n+1}).\end{aligned}\tag{3.3}$$

Then the sequence $\{\mathcal{M}(x_n,x_{n+1})\}$ is decreasing in $[1,\infty)$ and so there exists $m\geq 1$ such that

$\lim_{n\to\infty}\mathcal{M}(x_n,x_{n+1})=m.$ Let us prove that $m=1.$ Suppose to the contrary that $m\geq 0$ by (3.3) with $x=x_n$ and $y=x_{n+1},$ we get

$$\mathcal{M}(x_{n-1},x_n)^{\mathbb{K}^2}\geq\mathcal{M}(x_{n-1},x_n)$$
$$\geq\mathcal{M}(x_n,x_{n+1})^{\xi(\mathcal{M}(x_n,x_{n+1}))}\geq\mathcal{M}(x_n,x_{n+1})^{\mathbb{K}^2}.$$

Taking limit $n\to\infty$ in the above, we get $\lim_{n\to\infty}\xi(\mathcal{M}(x_n,x_{n+1}))=\mathbb{K}^2.$ Hence

$$m=\lim_{n\to\infty}\mathcal{M}(x,y)=1,$$

which is contradiction. That is, $m = 0$. We shall show that

$$\lim_{m,n\to\infty}\sup\mathcal{M}(x_n,x_m)=1.$$

Suppose to the contrary that

$$\lim_{m,n\to\infty}\sup\mathcal{M}(x_n,x_m)\geq 1.$$

By (3.2) we have

$$\mathcal{M}(x_n,x_m)=\mathcal{M}(Jx_{n+1},Jx_{m+1})$$
$$\geq\mathcal{M}(x_{n+1},x_{m+1})^{\xi(\mathcal{M}(x_{n+1},x_{m+1}))}.$$

That is,

$$\mathcal{M}(x_n,x_m)^{\frac{1}{\xi(\mathcal{M}(x_{n+1},x_{m+1}))}}\geq\mathcal{M}(x_{n+1},x_{m+1}).$$

Then by ($\mathcal{M}3$) we get

$$\mathcal{M}(x_n,x_m)$$
$$\leq\mathcal{M}(x_n,x_{n+1})^{\mathbb{K}}.\mathcal{M}(x_{n+1},x_{m+1})^{\mathbb{K}^2}.\mathcal{M}(x_{m+1},x_m)^{\mathbb{K}^2},$$
$$\leq\mathcal{M}(x_n,x_{n+1})^{\mathbb{K}}.(\mathcal{M}(x_n,x_m))^{\frac{\mathbb{K}^2}{\xi(\mathcal{M}(x_{n+1},x_{m+1}))}}$$
$$.\mathcal{M}(x_{m+1},x_m)^{\mathbb{K}^2}.$$

therefore,

$$\mathcal{M}(x_n,x_m)$$
$$\leq\begin{pmatrix}\mathcal{M}(x_n,x_{n+1})^{\mathbb{K}}\\ .\mathcal{M}(x_{m+1},x_m)^{\mathbb{K}^2}\end{pmatrix}^{\left(1-\frac{\mathbb{K}^2}{\xi(\mathcal{M}(x_{n+1},x_{m+1}))}\right)^{-1}}.$$

Taking limit as $m,n\to\infty$ in the above, since $\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\geq 1$ and

$$m=\lim_{m,n\to\infty}\mathcal{M}(x_n,x_{n+1})=0,$$

then we obtain

$$\lim_{m,n\to\infty}\sup\left(1-\frac{\mathbb{K}^2}{\xi(\mathcal{M}(x_{n+1},x_{m+1}))}\right)^{-1}=\infty,$$

which implies

$$\lim_{m,n\to\infty}\sup\xi(\mathcal{M}(x_{n+1},x_{m+1}))=\mathbb{K}^2,$$

and so

$$\lim_{m,n\to\infty}\sup\mathcal{M}(x_{n+1},x_{m+1})=1,$$

which is contradiction. Hence,

$$\lim_{m,n\to\infty}\sup\mathcal{M}(x_n,x_m)=1.$$

Now, since $\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1$ exists (and finite), so $\{x_n\}$ is a Cauchy sequence. Since $(\mathcal{X},\mathcal{M},\mathbb{K})$ is complete multiplicative b-metric-like space, $\{x_n\}$ in $\mathcal{X}$ converges to $z\in\mathcal{X}$ so that

$$\lim_{m,n\to\infty}\mathcal{M}(x_n,z)=\mathcal{M}(z,z)=\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$$

As J is onto, so there exists $\hat{w}\in\mathcal{X}$ such that $z=J\hat{w}.$ Let us prove that $z=J\hat{w}.$ Suppose to the contrary that $z\neq\hat{w}.$ Then by (3.2) we have

$$\mathcal{M}(x_n,z)=\mathcal{M}(Jx_{n+1},J\hat{w})\geq\mathcal{M}(x_{n+1},\hat{w})^{\xi(\mathcal{M}(x_{n+1},\hat{w}))}.$$

Taking limit as $n\to\infty$ and by applying proposition 2.10(2), we have

$$1=\lim_{n\to\infty}\mathcal{M}(x_n,z)$$
$$\geq\left(\lim_{n\to\infty}\mathcal{M}(x_{n+1},\hat{w})\right)^{\lim_{n\to\infty}\xi(\mathcal{M}(x_{n+1},\hat{w}))},$$
$$\geq(\mathcal{M}(z,\hat{w}))^{\frac{\lim_{n\to\infty}\xi(\mathcal{M}(x_{n+1},z))}{\mathbb{K}}},$$

and hence

$$\lim_{n\to\infty}\xi(\mathcal{M}(x_{n+1},z))=0,$$

which is contradiction. Indeed $\lim_{n\to\infty}\xi(\mathcal{M}(x_{n+1},z))\geq\mathbb{K}^2.$ Since $\xi(\hat{t})\geq\mathbb{K}^2$ for all $\hat{t}\in[1,\infty),$ therefore $z=\hat{w}.$ That is, $z=J\hat{w}=Jz.$

Example 3.5. Let $\mathcal{X}=[0,\infty)$ and $\mathcal{M}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$ be defined by

$$\mathcal{M}(x,y)=\left(a^{\max\{x,y\}^2}\right)^2.$$

Clearly, $(\mathcal{X},\mathcal{M},4)$ is a complete multiplicative b-metric-like space. Let $J:\mathcal{X}\times\mathcal{X}$ be defined by

$$Jx=4x^{8\left(1+x^2\right)}.$$

Also define $\xi:(1,\infty)\to(16,\infty)$ by $\xi(t)=8(1+t).$ Mapping J is an onto.

Suppose $x\leq y.$ Now, since

$$Jy\geq y^{8\left(1+y^2\right)},$$

so

$$(Jy)^2\geq y^{2\left(8\left(1+8y^2\right)\right)},$$

equivalently,

$$\left(a^{\max\{x,y\}^2}\right)^2 \geq \left(a^{\max\{x,y\}^2}\right)^{2\left(8\left(1+\left(a^{\max\{x,y\}^2}\right)^2\right)\right)},$$

and hence

$$\mathcal{M}(Jy,Jx) \geq (\mathcal{M}(x,y))^{8(1+\mathcal{M}(x,y))}.$$

That is

$$\mathcal{M}(Jx,Jy) \geq (\mathcal{M}(x,y))^{\xi(\mathcal{M}(x,y))}.$$

The condition of theorem(3.4) hold and J has a fixed point ($x = 0$ is a fixed point of J).

NOTE. We can obtain the following corollaries because multiplicative b-metric-like spaces are extension of multiplicative partial metric, multiplicative metric-like and multiplicative b-metric spaces.

Corollary 3.6. *Let $(\mathcal{X},\mathcal{D})$ be a complete multiplicative partial metric space. Suppose $J:\mathcal{X}\times\mathcal{X}$ is onto and satisfies*

$$\mathcal{D}(Jx,Jy) \geq \left[\mathcal{D}(x,y)\right]^{\xi(\mathcal{D}(x,y))},$$

$\forall x,y\in\mathcal{X}$, *where $\xi \in \sqcup_{\xi}^{1}$. Then J has a fixed point.*

Corollary 3.7. *Let $(\mathcal{X},\Psi)$ be a complete multiplicative metric-like space. Suppose $J:\mathcal{X}\times\mathcal{X}$ is onto and satisfies*

$$\Psi(Jx,Jy) \geq \left[\Psi(x,y)\right]^{\xi(\Psi(x,y))},$$

$\forall x,y\in\mathcal{X}$, *where $\xi \in \sqcup_{\xi}^{1}$. Then J has a fixed point.*

Corollary 3.8. *Let $(\mathcal{X},\mathcal{N},\mathbb{K})$ be a complete multiplicative b-metric space. Suppose $J:\mathcal{X}\times\mathcal{X}$ is onto and satisfies*

$$\mathcal{N}(Jx,Jy) \geq \left[\mathcal{N}(x,y)\right]^{\xi(\mathcal{N}(x,y))}, \quad (3.4)$$

$\forall x,y\in\mathcal{X}$, *where $\xi \in \sqcup_{\xi}^{\mathbb{K}}$. Then J has a fixed point.*

4. Partially Ordered Multiplicative (b)-metric-like Spaces and Fixed Point Theorems

A. Alghamdi [1] proved fixed point point results which extend results of A. Harandi and someothers (see [2,9]). Now, we prove some fixed point results in partially ordered multiplicative b-metric-like space to the corresponding results of A. Alghamdi [1].

Let $\sqcup_{\zeta}^{L}$ be the class of functions $\zeta:(1,\infty)\to\left(0,\frac{1}{L^2}\right)$ such that it satisfy the condition $\zeta(\hat{t}_n)\to\left(\frac{1}{L^2}\right)$ $\Rightarrow \hat{t}_n \to 1$, where $L>0$.

Theorem 4.1. Let $(\mathcal{X},\mathcal{N},\mathbb{K},\preceq)$, be a partially ordered complete multiplicative b-metric-like space, and suppose the mapping $J:\mathcal{X}\times\mathcal{X}$ is non-decreasing such that

$$\mathcal{M}(Jx,Jy) \leq \mathcal{G}(x,y)^{\zeta(\mathcal{G}(x,y))}.H(x,y)^{J(H(x,y))}, \quad (4.1)$$

$\forall x,y\in\mathcal{X}$ with $x\preceq y$, where $\zeta\in\sqcup_{\zeta}^{\mathbb{K}}$, $J:[0,\infty)\to[0,\infty)$ is bounded function and

$$\mathcal{G}(x,y)=\max\left\{\begin{matrix}\mathcal{M}(x,y),\mathcal{M}(x,Jx),\mathcal{M}(y,Jy),\\ \left[\mathcal{M}(x,Jy).\mathcal{M}(y,Jx)^{\frac{1}{6\mathbb{K}}}\right]\end{matrix}\right\},$$

and

$$H(x,y)=\min\left\{\begin{matrix}\mathcal{M}^s(x,Jx),\mathcal{M}^s(y,Jy),\\ \mathcal{M}^s(x,Jy),\mathcal{M}^s(y,Jx)\end{matrix}\right\}.$$

Also, assume that the following assertions hold:
(1) for $x_0\preceq Jx_0$, there exists $x_0\in\mathcal{X}$;
(2) for an increasing sequence $\{x_n\}\to x\in\mathcal{X}$, where $\{x_n\}\subset\mathcal{X}$, we have $x_n\preceq Jx_n$ for all $n\in N$; then J has a fixed point.

Proof. Let $x_0\preceq Jx_0$. If $x_0=Jx_0$, then the result is proved. Now we assume that $x_0\prec fx_0$. Define a sequence $\{x_n\}$ by $x_n=J^n x_0=Jx_{n-1}$ for all $n\in N$. Since J is non-decreasing and $x_0\prec fx_0$, then

$$x_0\prec x_1\preceq x_2\preceq\ldots, \quad (4.2)$$

and hence the sequence $\{x_n\}$ is non-decreasing. If $x_n=x_{n+1}=Jx_n$ for some $n\in N$, then the result is satisfied as x_n is a fixed point of J. In what follows we will suppose that $x_n\neq x_{n+1}$ for all $n\in N$. From (4.1) and (4.2) we have

$$\begin{aligned}\mathcal{M}(x_n,x_{n+1})&=\mathcal{M}(Jx_{n-1},x_n)\\ &\leq\mathcal{G}(x_{n-1},x_n)^{\zeta(\mathcal{G}(x_{n-1},x_n))}\\ &\quad .H(x_{n-1},x_n)^{JH(x_{n-1},x_n)},\end{aligned}$$

where

$$H(x_{n-1},x_n)=\min\begin{pmatrix}\mathcal{M}^s(x_{n-1},Jx_{n-1}),\mathcal{M}^s(x_n,Jx_n),\\ \mathcal{M}^s(x_{n-1},Jx_n),\mathcal{M}^s(x_n,Jx_{n-1})\end{pmatrix}$$

$$\leq\min\begin{pmatrix}\mathcal{M}^s(x_{n-1},x_n),\mathcal{M}^s(x_n,x_{n+1}),\\ \mathcal{M}^s(x_{n-1},x_{n+1}),\mathcal{M}^s(x_n,x_n)\end{pmatrix},$$

$$=1.$$

Then

$$\mathcal{M}(x_n,x_{n+1})\leq\mathcal{G}(x_{n-1},x_n)^{\zeta(\mathcal{G}(x_{n-1},x_n))}. \quad (4.3)$$

On the other hand, from ($\mathcal{M}3$) we have

$$\mathcal{M}(x_{n-1},x_{n+1})\leq\left[\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})\right]^{\mathbb{K}},$$

and

$$\mathcal{M}(x_n,x_n)\le[\mathcal{M}(x_n,x_{n+1})]^{2\mathbb{K}}$$
$$\le[\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{2\mathbb{K}}.$$

Then

$$[\mathcal{M}(x_{n-1},x_{n+1}).\mathcal{M}(x_n,x_n)]^{\frac{1}{6\mathbb{K}}}$$
$$\le[\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{\frac{1}{2}},$$

and hence

$$\begin{aligned}
&\mathcal{G}(x_{n-1},x_n)\\
&=\max\left\{\begin{array}{l}\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_{n-1},Jx_{n-1}),\mathcal{M}(x_n,Jx_n),\\ [\mathcal{M}(x_{n-1},Jx_n).\mathcal{M}(x_n,Jx_{n-1})]^{\frac{1}{6\mathbb{K}}}\end{array}\right\}\\
&=\max\left\{\begin{array}{l}\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1}),\\ [\mathcal{M}(x_{n-1},x_{n+1}).\mathcal{M}(x_n,x_n)]^{\frac{1}{6\mathbb{K}}}\end{array}\right\},\\
&=\max\left\{\begin{array}{l}\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1}),\\ [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{\frac{1}{2}}\end{array}\right\},\\
&=\max\{\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1})\}\\
&\le\mathcal{M}(x_{n-1},x_n).
\end{aligned}$$

That is

$$\mathcal{M}(x_{n-1},x_n)=\max\{\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1})\}.$$

Now by (4.3) we get

$$\begin{aligned}
&\mathcal{M}(x_n,x_{n+1})\\
&\le\left[\max\left\{\begin{array}{l}\mathcal{M}(x_{n-1},x_n),\\ \mathcal{M}(x_n,x_{n+1})\end{array}\right\}\right]^{\zeta(\max\{\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1})\})}.
\end{aligned}$$

If $\max\{\mathcal{M}(x_{n-1},x_n),\mathcal{M}(x_n,x_{n+1})\}=\mathcal{M}(x_n,x_{n+1})$, then

$$\mathcal{M}(x_n,x_{n+1})\le[\mathcal{M}(x_n,x_{n+1})]^{\zeta(\mathcal{M}(x_n,x_{n+1}))}$$
$$\le[\mathcal{M}(x_n,x_{n+1})]^{\frac{1}{\mathbb{K}^2}}\le\mathcal{M}(x_n,x_{n+1}),$$

which is contradiction. Hence,

$$\begin{aligned}
\mathcal{M}(x_n,x_{n+1})&\le[\mathcal{M}(x_{n-1},x_n)]^{\zeta(\mathcal{M}(x_{n-1},x_n))}\\
&\le\mathcal{M}(x_{n-1},x_n)
\end{aligned}\tag{4.4}$$

and so $\{\mathcal{M}(x_n,x_{n+1})\}$ is decreasing sequence. Then there exists $m\ge1$ such that $\lim_{n\to\infty}\mathcal{M}(x_n,x_{n+1})=m.$ By (4.4) we have

$$[\mathcal{M}(x_n,x_{n+1})]^{\frac{1}{\mathbb{K}^2}}\le\mathcal{M}(x_n,x_{n+1})$$
$$\le[\mathcal{M}(x_{n-1},x_n)]^{\zeta(\mathcal{M}(x_{n-1},x_n))}\le[\mathcal{M}(x_{n-1},x_n)]^{\frac{1}{\mathbb{K}^2}}.$$

Taking $\lim_{n\to\infty}$ in the above inequality, we get

$$\lim_{n\to\infty}\zeta(\mathcal{M}(x_{n-1},x_n))=\frac{1}{\mathbb{K}^2},$$

and so $m=\lim_{n\to\infty}\mathcal{M}(x_{n-1},x_n)=1.$ Now we want to show that

$$\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$$

Suppose to the contrary that

$$\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)>1.$$

At first,

$$\begin{aligned}
&\limsup_{m,n\to\infty}H(x_n,x_m)\\
&=\limsup_{m,n\to\infty}\min\left\{\begin{array}{l}\mathcal{M}^s(x_n,x_{n+1}),\mathcal{M}^s(x_m,x_{m+1}),\\ \mathcal{M}^s(x_n,x_{m+1}),\mathcal{M}^s(x_m,x_{m+1})\end{array}\right\}.
\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}
&\limsup_{m,n\to\infty}\mathcal{G}(x_n,x_m)\\
&=\limsup_{m,n\to\infty}\max\left\{\begin{array}{l}\mathcal{M}(x_n,x_m),\mathcal{M}(x_n,x_{n+1}),\\ \mathcal{M}(x_m,x_{m+1}),\\ [\mathcal{M}(x_n,x_{m+1}).\mathcal{M}(x_m,x_{n+1})]^{\frac{1}{6\mathbb{K}}}\end{array}\right\}\\
&\le\limsup_{m,n\to\infty}\max\left\{\begin{array}{l}\mathcal{M}(x_n,x_m),\\ \mathcal{M}(x_n,x_{n+1}),\\ \mathcal{M}(x_m,x_{m+1}),\\ \left[\begin{array}{l}(\mathcal{M}(x_n,x_m).\mathcal{M}(x_m,x_{m+1}))^{\mathbb{K}}\\ (\mathcal{M}(x_m,x_n).\mathcal{M}(x_n,x_{n+1}))^{\mathbb{K}}\end{array}\right]^{\frac{1}{6\mathbb{K}}}\end{array}\right\},\\
&=\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\le\limsup_{m,n\to\infty}\mathcal{G}(x_n,x_m).
\end{aligned}$$

That is,

$$\limsup_{m,n\to\infty}\mathcal{G}(x_n,x_m)=\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m).\tag{4.6}$$

Now by (4.1) we have

$$\begin{aligned}
&\limsup_{m,n\to\infty}\mathcal{M}(x_{n+1},x_{m+1})\\
&=\limsup_{m,n\to\infty}\mathcal{M}(Jx_n,Jx_m)\\
&\le\left[\limsup_{m,n\to\infty}\mathcal{G}(x_n,x_m)\right]^{\limsup_{m,n\to\infty}\zeta(G(x_n,x_m))}.\\
&\quad\left[\limsup_{m,n\to\infty}H(x_n,x_m)\right]^{\limsup_{m,n\to\infty}J(H(x_n,x_m))},
\end{aligned}$$

and so from (4.5) and (4.6) we get

$$\begin{aligned}
&\limsup_{m,n\to\infty}\mathcal{M}(x_{n+1},x_{m+1})\\
&\le\left[\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\right]^{\limsup_{m,n\to\infty}\zeta(\mathcal{G}(x_n,x_m))}.
\end{aligned}\tag{4.7}$$

By ($\mathcal{M}3$) we have

$$\mathcal{M}(x_n,x_m) \le \mathcal{M}(x_n,x_{n+1})^{\mathbb{K}}.\mathcal{M}(x_{n+1},x_{m+1})^{\mathbb{K}^2}.\mathcal{M}(x_{m+1},x_m)^{\mathbb{K}^2}.$$

Taking $\limsup_{n\to\infty}$ in the above inequality, we have

$$\left[\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\right]^{\frac{1}{\mathbb{K}^2}} \le \lim\sup_{m,n\to\infty}\mathcal{M}(x_{n+1},x_{m+1}).$$

Then by (4.7) we deduce

$$\left[\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\right]^{\frac{1}{\mathbb{K}^2}} \le \left[\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)\right]^{\limsup_{m,n\to\infty}\zeta(\mathcal{G}(x_n,x_m))}.$$

Now, since $\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)>1,$ then

$$\frac{1}{\mathbb{K}^2}\le \lim\sup_{m,n\to\infty}\zeta(\mathcal{G}(x_n,x_m)).$$

On the other hand, since

$$\limsup_{m,n\to\infty}\zeta(\mathcal{G}(x_n,x_m))\le\frac{1}{\mathbb{K}^2},$$

hence

$$\lim\sup_{m,n\to\infty}\zeta(\mathcal{G}(x_n,x_m))=\frac{1}{\mathbb{K}^2}.$$

This implies that

$$\lim\sup_{m,n\to\infty}\mathcal{M}(x_n,x_m)=\lim\sup_{m,n\to\infty}\mathcal{G}(x_n,x_m)=1,$$

which is contradiction. Thus, $\limsup_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$ Now, since $\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1$ exists (and finite), so $\{x_n\}$ is a cauchy sequence. Since $(\mathcal{X},\mathcal{M},\mathbb{K})$ is complete multiplicative b-metric-like space, the sequence $\{x_n\}$ in $\mathcal{X}$ converges to $z\in\mathcal{X}$ so that

$$\lim_{m,n\to\infty}\mathcal{M}(x_n,z)=\mathcal{M}(z,z) =\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$$

From(2) and(4.1), with $x=x_n$ and $y=z,$ we obtain

$$\mathcal{M}(x_{n+1},Jz)=\mathcal{M}(Jx_n,Jz) \le\mathcal{G}(x_n,z)^{\zeta(\mathcal{G}(x_n,z))}.H(x_n,z)^{J(H(x_n,z))}. \tag{4.8}$$

On the other hand,

$$\lim_{n\to\infty}H(x_n,z) = \lim_{n\to\infty}\min\left\{\begin{matrix}\mathcal{M}^s(x_n,x_{n+1}),\mathcal{M}^s(z,Jz),\\ \mathcal{M}^s(x_n,Jz),\mathcal{M}^s(z,x_{n+1})\end{matrix}\right\}=1,$$

and

$$\lim_{n\to\infty}\mathcal{G}(x_n,z) = \lim_{n\to\infty}\min\left\{\begin{matrix}\mathcal{M}(x_n,z),\mathcal{M}(x_n,x_{n+1}),\mathcal{M}(z,Jz),\\ \left[\mathcal{M}(x_n,Jz).\mathcal{M}(z,x_{n+1})\right]^{\frac{1}{6\mathbb{K}}}\end{matrix}\right\}, \le\mathcal{M}(z,Jz)\,(\text{by using Proposition 2.10(2)}).$$

Then $\lim_{n\to\infty}\mathcal{G}(x_n,z)=\mathcal{M}(z,Jz).$ Again, from Proposition 2.10(2) and 4.8, we have

$$\left[\mathcal{M}(z,Jz)\right]^{\frac{1}{\mathbb{K}^2}}\le\left[\mathcal{M}(z,Jz)\right]^{\frac{1}{\mathbb{K}}} \le\lim_{n\to\infty}\mathcal{M}(x_{n+1},Jz)\le\left[\mathcal{M}(z,Jz)\right]^{\lim_{n\to\infty}\zeta(\mathcal{G}(x_n,Jz))}.$$

Now, if $\mathcal{M}(z,Jz)>1,$ then

$$\lim_{n\to\infty}\zeta(\mathcal{M}(x_n,z))=\frac{1}{\mathbb{K}^2}.$$

This implies

$$\mathcal{M}(z,Jz)=\lim_{n\to\infty}\mathcal{M}(x_n,z)=1,$$

which is contradiction. Hence, $\mathcal{M}(z,Jz)=1.$ That is, $z=Jz.$

Example 4.2. Let $\mathcal{X}=[0,\infty)$ and $\mathcal{M}:\mathcal{X}\times\mathcal{X}\to[1,\infty)$ be defined by

$$\mathcal{M}(x,y)=\left(a^{\max\{x,y\}^2}\right)^2.$$

Clearly, $(\mathcal{X},\mathcal{M},4)$ is a complete multiplicative b-metric-like space, let $J:\mathcal{X}\to\mathcal{X}$ be defined by

$$Jx=2x^{\frac{1}{8(1+x^2)}}.$$

Also define $\zeta:(1,\infty)\to\left(\frac{1}{16},\infty\right)$ by $\zeta(\hat{t})=\frac{1}{8(1+\hat{t})}$

Let $x\preceq y\Leftrightarrow x\le y.$ That is for all $y\in\mathcal{X},$ we have

$$Jy\le y^{\frac{1}{8(1+y^2)}},$$

which implies

$$(Jy)^2\le y^{\frac{2}{8(1+y^2)}},$$

equivalently,

$$\left(a^{\max\{Jx,Jy\}^2}\right)^2\le\left(a^{\max\{x,y\}^2}\right)^{\frac{2}{8(1+(a^{\max\{x,y\}^2})^2)}},$$

and so

$$\mathcal{M}(Jx,Jy)\le\mathcal{M}(x,y)^{\frac{1}{8(1+\mathcal{M}(x,y))}} \le\mathcal{G}(x,y)^{\frac{1}{8(1+\mathcal{G}(x,y))}}=(\mathcal{G}(x,y))^{\zeta(\mathcal{G}(x,y))}.$$

So Theorem (4.1) hold and J has a fixed point.

Also we derive the following corollaries.

Corollary 4.3. *Let $(\mathcal{X},\mathcal{D},\preceq)$ be a partially ordered complete multiplicative partial metric space, and suppose the mapping $J:\mathcal{X}\to\mathcal{X}$ is non-decreasing such that*

$$\mathcal{D}(Jx,Jy)\le\mathcal{G}(x,y)^{\zeta(\mathcal{G}(x,y))}.H(x,y)^{J(H(x,y))},\ (4.9)$$

$\forall x,y\in\mathcal{X}$ with $x\preceq y$, where $J\in\sqcup_J^1$, $J:[0,\infty)\to[0,\infty)$ is bounded function and

$$\mathcal{G}(x,y)=\max\begin{Bmatrix}\mathcal{D}(x,y),\mathcal{D}(x,Jx),\mathcal{D}(y,Jy),\\ [\mathcal{D}(x,Jy),\mathcal{D}(y,Jx)]^{\frac{1}{6}}\end{Bmatrix},$$

and

$$H(x,y)=\min\begin{Bmatrix}\mathcal{D}^s(x,Jx),\mathcal{D}^s(y,Jy),\\ \mathcal{D}^s(x,Jy),\mathcal{D}^s(y,Jx)\end{Bmatrix}.$$

Also, assume that the following assertions hold:

(1) for $x_0\preceq Jx_0$, there exists $x_0\in\mathcal{X}$;

(2) for an increasing sequence $\{x_n\}\to x\in\mathcal{X}$, where $\{x_n\}\subset\mathcal{X}$, we have $x_n\preceq Jx_n$ for all $n\in N$; then J has a fixed point.

Corollary 4.4. *Let $(\mathcal{X},\mathcal{N},\mathbb{K},\preceq)$ be a partially ordered complete multiplicative b-metric-like space, and suppose the mapping $J:\mathcal{X}\to\mathcal{X}$ is non-decreasing such that*

$$\mathcal{M}(Jx,Jy)\le\mathcal{G}(x,y)^{\zeta(\mathcal{G}(x,y))}.H(x,y)^{J(H(x,y))},\ (4.10)$$

for all $x,y\in\mathcal{X}$ with $x\preceq y$, where $J\in\sqcup_J^{\mathbb{K}}$, $J:[0,\infty)\to[0,\infty)$ is bounded function and

$$\mathcal{G}(x,y)=\max\begin{Bmatrix}\mathcal{N}(x,y),\mathcal{N}(x,Jx),\mathcal{N}(y,Jy),\\ [\mathcal{N}(x,Jy).\mathcal{N}(y,Jx)]^{\frac{1}{4\mathbb{K}}}\end{Bmatrix},$$

and

$$H(x,y)=2\min\begin{Bmatrix}\mathcal{N}^s(x,Jx),\mathcal{N}^s(y,Jy),\\ \mathcal{N}^s(x,Jy),\mathcal{N}^s(y,Jx)\end{Bmatrix}.$$

Also, suppose that the following assertions hold:

(1) for $x_0\preceq Jx_0$, there exists $x_0\in\mathcal{X}$;

(2) for an increasing sequence $\{x_n\}\to x\in\mathcal{X}$, where $\{x_n\}\subset\mathcal{X}$, we have $x_n\preceq Jx_n$ for all $n\in N$; then J has a fixed point.

5. Fixed Point Results for Cyclic Contraction

A.Alghamdi [1] proved some results which is the generalization of the results proved by Edelstein [7], Suzuki [16] and Kirk [10]. Ozavar and Cevikel [6] introduce the concepts of Banach-contraction in multiplicative metric spaces. By using the idea, in this section we derive some results to the setting of multiplicative b-metric-like spaces corresponding to the results proved by Alghamdi [1].

Theorem 5.1. *Let $(\mathcal{X},\mathcal{M},\mathbb{K})$ be a multiplicative b-metric-like space, and suppose the family $\{E_i\}_{i=1}^m$ of non-empty closed subsets of $\mathcal{X}$ with $\mathcal{Y}=\bigcup_{i=1}^m E_i$. Let $J:\mathcal{Y}\to\mathcal{Y}$ be a map satisfying*

$$J(E_i)\subseteq E_{i+1},\ i=1,2,\ldots,m \text{ where } E_{m+1}=E_1.\quad(5.1)$$

Assume that

$$[\mathcal{M}(x,Jx)]^{\frac{1}{2\mathbb{K}}}\le\mathcal{M}(x,y).$$

This implies,

$$\mathcal{M}(Jx,Jy)$$
$$\le[\mathcal{M}(x,y)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}.[\mathcal{M}(x,Jx).\mathcal{M}(y,Jy)]^{\beta_0}\quad(5.2)$$
$$.[\mathcal{M}(x,Jy).\mathcal{M}(y,Jx)]^{\frac{\gamma_0}{3\mathbb{K}}}.[\mathcal{M}(x,x).\mathcal{M}(y,y)]^{\frac{\delta_0}{4\mathbb{K}}},$$

for all $x\in E_i$ and $y\in E_{i+1}$, where $\alpha_0,\beta_0,\gamma_0,\delta_0\ge 0$ and $\alpha_0+\beta_0+\gamma_0+\delta_0<\frac{1}{\mathbb{K}+1}$. Then J has a fixed point in $\bigcap_{i=1}^m E_i$.

Proof. Let $x_0\in E_1$ and define a sequence $\{x_n\}$ in the following way:

$$x_n=Jx_{n-1},\ n=1,2,3,\ldots.\quad(5.3)$$

We have $x_0\in E_1$, $x_1\in Jx_0\in E_2$, $x_2=Jx_1\in E_3$... for some $n_0\in N_0$, then, clearly, the fixed point of the map J is x_n. Hence, we assume that $x_n\ne x_{n+1}$ for all $n\in N$.

Clearly, $[\mathcal{M}(x_{n-1},Jx_{n-1})]^{\frac{1}{2\mathbb{K}}}\le\mathcal{M}(x_{n-1},x_n)$. Now, from (5.2) we have

$$\mathcal{M}(Jx_{n-1},x_n)$$
$$\le[\mathcal{M}(x_{n-1},x_n)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}$$
$$.[\mathcal{M}(x_{n-1},Jx_{n-1}).\mathcal{M}(x_n,Jx_n)]^{\beta_0}$$
$$\times[\mathcal{M}(Jx_{n-1},Jx_n).\mathcal{M}(x_n,Jx_n)]^{\frac{\gamma_0}{3\mathbb{K}}}$$
$$.[\mathcal{M}(x_{n-1},x_{n-1}).\mathcal{M}(x_n,x_n)]^{\frac{\delta_0}{4\mathbb{K}}},$$

which implies

$$\mathcal{M}(x_n,x_{n+1})$$
$$\le[\mathcal{M}(x_{n-1},x_n)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}$$
$$.[\mathcal{M}(x_{n-1},x_n).\mathcal{M}(x_n,x_{n+1})]^{\beta_0}\quad(5.4)$$
$$\times[\mathcal{M}(x_{n-1},x_{n+1}).\mathcal{M}(x_n,x_n)]^{\frac{\gamma_0}{3\mathbb{K}}}$$
$$.[\mathcal{M}(x_{n-1},x_{n-1}).\mathcal{M}(x_n,x_n)]^{\frac{\delta_0}{4\mathbb{K}}}.$$

From ($\mathcal{M}3$) we have

$$\mathcal{M}(x_{n-1},x_{n+1}) \le [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{\mathbb{K}},$$

and

$$\mathcal{M}(x_n,x_n) \le [\mathcal{M}(x_n,x_{n+1})]^{2\mathbb{K}} \le [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{2\mathbb{K}},$$

and so

$$[\mathcal{M}(x_{n-1},x_{n+1}).\mathcal{M}(x_n,x_n)]^{\frac{1}{3\mathbb{K}}} \le [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]. \quad (5.5)$$

Also,

$$\mathcal{M}(x_{n-1},x_{n-1}) \le [\mathcal{M}(x_n,x_{n-1})]^{2\mathbb{K}} \le [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]^{2\mathbb{K}}.$$

Then

$$[\mathcal{M}(x_{n-1},x_{n-1}).\mathcal{M}(x_n,x_n)]^{\frac{1}{4\mathbb{K}}} \le [\mathcal{M}(x_n,x_{n-1}).\mathcal{M}(x_n,x_{n+1})]. \quad (5.6)$$

Hence, by (5.4), (5.5) and (5.6) we get

$$\mathcal{M}(x_n,x_{n+1}) \le [\mathcal{M}(x_{n-1},x_n)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}.[\mathcal{M}(x_n,x_{n+1})]^{\beta_0+\gamma_0+\delta_0},$$

and then

$$\mathcal{M}(x_n,x_{n+1}) \le [\mathcal{M}(x_{n-1},x_n)]^{c},$$

where

$$c=\frac{\left[\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}+\beta_0+\gamma_0+\delta_0\right]}{1-(\beta_0+\gamma_0+\delta_0)}.$$

Now since $(\mathbb{K}+1)(\alpha_0+\beta_0+\gamma_0+\delta_0)<1$, then

$$\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}+\beta_0+\gamma_0+\delta_0+\frac{\beta_0+\gamma_0+\delta_0}{\mathbb{K}}<\frac{1}{\mathbb{K}},$$

which implies

$$\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}+\beta_0+\gamma_0+\delta_0<\frac{1}{\mathbb{K}}[1-(\beta_0+\gamma_0+\delta_0)].$$

Then by lemma 2.14 we have $\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1$. Now, since $\lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1$ exists (and finite), so $\{x_n\}$ is a cauchy sequence. Since $(\mathcal{X},\mathcal{M},\mathbb{K})$ is complete multiplicative b-metric-like space, the sequence $\{x_n\}$ in $\mathcal{X}$ converges to $z\in\mathcal{X}$ so that

$$\lim_{m,n\to\infty}\mathcal{M}(x_n,z)=\mathcal{M}(z,z) = \lim_{m,n\to\infty}\mathcal{M}(x_n,x_m)=1.$$

As $z\in\bigcap_{i=1}^m E_i$. Since $x_0\in E_1$, subsequence $\{x_{m(n-1)}\}_{n=1}^{\infty}\in E_1$, the subsequence $\{x_{m(n-1)+1}\}_{n=1}^{\infty}\in E_2$ and similarly the subsequence $\{x_{mn-1}\}_{n=1}^{\infty}\in E_m$. All the m subsequences are convergent in the closed sets E_j, and all have the same limit $z\in\bigcap_{i=1}^m E_i$. Assume that there exists $n_0\in N$ such that the following inequalities Satisfied:

$$[\mathcal{M}(x_{n_0},x_{n_0+1})]^{\frac{1}{2\mathbb{K}}}>\mathcal{M}(x_{n_0},z)$$

$$\textit{and } [\mathcal{M}(x_{n_0+1},x_{n_0+2})]^{\frac{1}{2\mathbb{K}}}>\mathcal{M}(x_{n_0+1},z).$$

Then

$$\begin{aligned}&\mathcal{M}(x_{n_0},x_{n_0+1})\\ &\le[\mathcal{M}(x_{n_0},z).\mathcal{M}(Jx_{n_0},z)]^{\mathbb{K}},\\ &<[\mathcal{M}(x_{n_0},x_{n_0+1})]^{\frac{1}{2}}.[\mathcal{M}(x_{n_0+1},x_{n_0+2})]^{\frac{1}{2}},\\ &<[\mathcal{M}(x_{n_0},x_{n_0+1})]^{\frac{1}{2}}.[\mathcal{M}(x_{n_0},x_{n_0+1})]^{\frac{1}{2}},\\ &=\mathcal{M}(x_{n_0},x_{n_0+1}),\end{aligned}$$

which is contradiction. Hence, for every $n\in N$, we have

$$[\mathcal{M}(x_{n_0},x_{n+1})]^{\frac{1}{2\mathbb{K}}}>\mathcal{M}(x_{n_0},z)$$

$$\textit{and } [\mathcal{M}(x_{n+1},x_{n+2})]^{\frac{1}{2\mathbb{K}}}>\mathcal{M}(x_{n+1},z).$$

and so by (5.2) we have

$$\begin{aligned}&\mathcal{M}(x_{n+1},Jz)\\ &\le[\mathcal{M}(x_n,z)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}\\ &.[\mathcal{M}(x_n,x_{n+1}).\mathcal{M}(z,Jz)]^{\beta_0}\\ &\times[\mathcal{M}(x_n,Jz).\mathcal{M}(z,x_{n+1})]^{\frac{\gamma_0}{3\mathbb{K}}}\\ &.[\mathcal{M}(x_n,x_n).\mathcal{M}(z,Jz)]^{\frac{\delta_0}{4\mathbb{K}}},\end{aligned} \quad (5.7)$$

or

$$\begin{aligned}&\mathcal{M}(x_{n+2},Jz)\\ &\le[\mathcal{M}(x_{n+1},z)]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}}\\ &.[\mathcal{M}(x_{n+1},x_{n+2}).\mathcal{M}(z,Jz)]^{\beta_0}\\ &\times[\mathcal{M}(x_{n+1},Jz).\mathcal{M}(z,x_{n+2})]^{\frac{\gamma_0}{3\mathbb{K}}}\\ &.[\mathcal{M}(x_{n+1},x_{n+1}).\mathcal{M}(z,Jz)]^{\frac{\delta_0}{4\mathbb{K}}}.\end{aligned} \quad (5.8)$$

Assume that (5.7) holds. Then, by taking limit as $n \to \infty$ in (5.7), we get

$$\lim_{n\to\infty} \mathcal{M}(x_{n+1}, Jz) \le \left[\mathcal{M}(z, Jz)\right]^{\beta_0} . \left[\lim_{n\to\infty} \mathcal{M}(x_n, Jz)\right]^{\frac{\gamma_0}{3\mathbb{K}}},$$

and hence by using proposition 2.10(2) we obtain

$$\left[\mathcal{M}(z, Jz)\right]^{\frac{1}{\mathbb{K}}} \le \left[\mathcal{M}(z, Jz)\right]^{\beta_0} . \left[\mathcal{M}(z, Jz)\right]^{\frac{\gamma_0}{3}}.$$

Therefore,

$$\left[\mathcal{M}(z, Jz)\right]^{\left(\frac{1}{\mathbb{K}} - \beta_0 - \frac{J}{3}\right)} \le 1.$$

If we take $\alpha_0, \beta_0, \gamma_0, \delta_0 \ge 0$ and $\alpha_0 + \beta_0 + \gamma_0 + \delta_0 < \frac{1}{\mathbb{K}+1} < \frac{1}{\mathbb{K}}$, then $\beta_0 + \frac{\gamma_0}{3} \le \beta_0 + \gamma_0 < \frac{1}{\mathbb{K}}$. That is, $\frac{1}{\mathbb{K}} - \beta_0 - \frac{\gamma_0}{3} > 1$. Hence, $\mathcal{M}(z, Jz) = 1$, i.e., $z = Jz$. If (5.8) holds, then by a similar method, we can deduce that $z = Jz$

If we take $E_i = \mathcal{X}$ in the above theorem for all m, then we have the following corollary.

Corollary 5.2. *Let $(\mathcal{X}, \mathcal{M}, \mathbb{K})$ be a complete multiplicative b-metric-like space, and let J be a self-mapping on $\mathcal{X}$. Suppose that*

$$\left[\mathcal{M}(x, Jx)\right]^{\frac{1}{2\mathbb{K}}} \le \mathcal{M}(x, y).$$

This implies

$$\mathcal{M}(Jx, Jy) \le \left[\mathcal{M}(x, y)\right]^{\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}}} . \left[\mathcal{M}(x, Jx) . \mathcal{M}(y, Jy)\right]^{\beta_0} \times \left[\mathcal{M}(x, Jy) . \mathcal{M}(y, Jx)\right]^{\frac{\gamma_0}{3\mathbb{K}}} . \left[\mathcal{M}(x, x) . \mathcal{M}(y, y)\right]^{\frac{\delta_0}{4\mathbb{K}}},$$

$\forall x, y \in \mathcal{X}$, *where* $\alpha_0, \beta_0, \gamma_0, \delta_0 \ge 0$ *and* $\alpha_0 + \beta_0 + \gamma_0 + \delta_0 < \frac{1}{\mathbb{K}+1}$. *Then J has a fixed point.*

If in theorem (5.1) we take $\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}} = \beta_0 = \frac{\gamma_0}{3\mathbb{K}} = \frac{\delta_0}{4\mathbb{K}} = S$, then we deduce the following corollary.

Corollary 5.3. *Let $(\mathcal{X}, \mathcal{M}, \mathbb{K})$ be a multiplicative b-metric-like space, and let $\{E_i\}_{i=1}^m$ be a family of non-empty closed subsets of $\mathcal{X}$ with $\mathcal{Y} = \bigcup_{i=1}^m E_i$. Let $J : \mathcal{Y} \to \mathcal{Y}$ be a map satisfying*

$$J(E_i) \subseteq E_{i+1}, i = 1, 2, ..., m \text{ where } E_{m+1} = E_1.$$

Assume that

$$\left[\mathcal{M}(x, Jx)\right]^{\frac{1}{2\mathbb{K}}} \le \mathcal{M}(x, y).$$

This implies

$$\left[\mathcal{M}(Jx, Jy)\right] \le \mathcal{M}(x, y) . \mathcal{M}(x, Jx) . \mathcal{M}(y, Jy) . \mathcal{M}(x, Jy) . \mathcal{M}(y, Jx) . \left[\mathcal{M}(x, x) . \mathcal{M}(y, y)\right]^S,$$

for all $x \in E_i$ and $y \in E_{i+1}$, where

$$0 \le S \le \frac{1}{(\mathbb{K}+1)(7\mathbb{K}+1) + \mathbb{K}}.$$

Then J has a fixed point in $\bigcap_{i=1}^m E_i$.

If in Corollary(5.2) we take $\frac{\alpha_0(\mathbb{K}+1)}{\mathbb{K}} = \beta_0 = \frac{\gamma_0}{3\mathbb{K}} = \frac{\delta_0}{4\mathbb{K}} = S$, then we deduce the following corollary.

Corollary 5.4. *Let $(\mathcal{X}, \mathcal{M}, \mathbb{K})$ be a complete multiplicative b-metric-like space, and let J be a self-mapping on $\mathcal{X}$. Assume that*

$$\left[\mathcal{M}(x, Jx)\right]^{\frac{1}{3\mathbb{K}}} \le \mathcal{M}(x, y).$$

This implies

$$\mathcal{M}(Jx, Jy) \le \mathcal{M}(x, y) . \mathcal{M}(x, Jx) . \mathcal{M}(y, Jy) . \mathcal{M}(x, Jy) . \mathcal{M}(y, Jx) . \left[\mathcal{M}(x, x) . \mathcal{M}(y, y)\right]^S,$$

for all $x \in E_i$ and $y \in E_{i+1}$, where

$$0 \le S \le \frac{1}{(\mathbb{K}+1)(7\mathbb{K}+1) + \mathbb{K}}.$$

Then J has a fixed point in $\bigcap_{i=1}^m E_i$.

Corollary 5.5. *Let $(\mathcal{X}, \Psi)$ be a complete multiplicative metric-like space, $m \in N$, let $E_1, E_2, ..., E_m$ be non-empty closed subsets of $\mathcal{X}$ and $\mathcal{Y} = \bigcup_{i=1}^m E_i$. Suppose that $J : \mathcal{Y} \to \mathcal{Y}$ is an operator such that*

(1) $\mathcal{Y} = \bigcup_{i=1}^m E_i$ is a cyclic representation of $\mathcal{X}$ with respect to J;

(2) Suppose there exists $0 \le S < \frac{1}{17}$ such that

$$\left[\int_0^{\Psi(x,Jx)} \rho(t)^{dt}\right]^{\frac{1}{2}} \le \int_0^{\Psi(x,y)} \rho(t)^{dt} \Rightarrow \int_0^{\Psi(Jx,Jy)} \rho(t)^{dt} \le \left[\int_0^{\mathcal{G}(x,y)} \rho(t)^{dt}\right]^S,$$

where

$$\mathcal{G}(x, y) = \Psi(x, y) . \Psi(x, Jx) . \Psi(x, Jy) . \Psi(y, Jy) . \Psi(y, Jx) . \Psi(y, y),$$

for $x \in E_i$, $i = 1, 2, ..., m$, where $E_{m+1} = E_1$, and $\rho : [0, \infty) \to [0, \infty)$ is Lebesgue-integrable mapping satisfying $\int_0^{\varepsilon} \rho(t)^{dt} > 0$ for $\varepsilon > 0$. Then J has a fixed point.

Corollary 5.6. *Let* $(\mathcal{X},\Psi)$ *be a complete multiplicative metric-like space, and let* $J:\mathcal{X}\to\mathcal{X}$ *such that for any* $x,y\in\mathcal{X}$ *there exists* $0\le S<\frac{1}{17}$ *such that*

$$\left[\int_0^{\Psi(x,Jx)}\rho(t)^{dt}\right]^{\frac{1}{2}}\le\int_0^{\Psi(x,y)}\rho(t)^{dt}$$
$$\Rightarrow\int_0^{\Psi(Jx,Jy)}\rho(t)^{dt}\le\left[\int_0^{\mathcal{G}(x,y)}\rho(t)^{dt}\right]^{S},$$

where

$$\mathcal{G}(x,y)=\Psi(x,y).\Psi(x,Jx).\Psi(x,Jy)$$
$$.\Psi(y,Jy).\Psi(y,Jx).\Psi(y,y),$$

and $\rho:[0,\infty)\to[0,\infty)$ *is Lebesgue-integrable mapping satisfying* $\int_0^{\varepsilon}\rho(t)^{dt}>0$ *for* $\varepsilon>0$. *Then J has a fixed point.*

6. Application to Integral Equations

Integral equation method is very useful for solving problems in applied fields. Many papers consist on the problem of existence of solutions of nonlinear integral equations and the results are established by using different fixed point techniques, see e.g., [1,8,13,14]. Inspired by the work. Consider the following integral equation

$$p(u)=\int_1^U\left(H(u,s)\,f(s,p(s))\right)^{ds}\ \textit{for all } u\in[1,U],\ (6.1)$$

where $U>1$, $f:[1,U]\times\mathbb{R}^+\to\mathbb{R}^+$ and

$$H:[1,U]\times[1,U]\to[0,\infty)$$

are continuous functions. Let $\mathcal{X}=C([1,U])$ be a set of real continuous functions on $[1,U]$. We endow $\mathcal{X}$ with the complete multiplicative b-metric-like

$$\mathcal{M}_\infty(p,q)=\sup_{u\in[1,U]}\left(|p(u)|.|q(u)|\right)^2\ \textit{for all } p,q\in\mathcal{X}.$$

Clearly, $(\mathcal{X},\mathcal{M}_\infty,1)$ is a complete multiplicative b-metric-like space.

Let $(\alpha_0,\beta_0)\in\mathcal{X}^2$, $(\alpha_1,\beta_1)\in\mathbb{R}^2$ be such that

$$\alpha_1\le\alpha_0(u)\le\beta_0(u)\le\beta_1\ \textit{for all } u\in[1,U].\qquad(6.2)$$

Assume that for all $u\in[1,U]$, we have

$$\alpha_0(u)\le\int_1^U\left(H(u,s)\,f(s,\beta_0(s))\right)^{ds},\qquad(6.3)$$

and

$$\beta_0(u)\le\int_1^U\left(H(u,s)\,f(s,\alpha_0(s))\right)^{ds}.\qquad(6.4)$$

Let, for all $s\in[1,U]$, $f(s,\cdot)$ be a decreasing function, that is,

$$x,y\in\mathbb{R},x\ge y\Rightarrow f(s,x)\le f(s,y).\qquad(6.5)$$

Assume that

$$\sup_{u\in[1,U]}\int_1^U H(u,s)^{ds}\le 1.\qquad(6.6)$$

Also suppose that for all $s\in[1,U]$ for all $x,y\in\mathbb{R}$ with ($x\le\beta_1$ and $y\ge\alpha_1$) or ($y\le\beta_1$ and $x\ge\alpha_1$).

$$[f(s,x).f(s,y)]$$
$$\le\left(\begin{array}{l}(x,y)^{2\alpha_0}.\left((x,Jx)^2.(y,Jy)^2\right)^{\beta_0}\\ .\left((x,Jy)^2.(y,Jx)^2\right)^{\frac{\gamma_0}{3}}.\left(x^4.y^4\right)^{\frac{\delta_0}{4}}\end{array}\right)^{\frac{1}{2}}\qquad(6.7)$$

where $\alpha_0,\beta_0,\gamma_0,\delta_0\ge 0$ and $\alpha_0+\beta_0+\gamma_0+\delta_0<\frac{1}{3}$.

Theorem 6.1. *Under assumptions (6.2)-(6.7), integral equation (6.1) has a solution in*

$$\{p\in C[1,U]:\alpha_0\le p(u)\le\beta_0\ \textit{for all } u\in[1,U]\}.$$

Proof. Define the closed subsets of $\mathcal{X}$, E_1 and E_2 by

$$E_1=\{p\in\mathcal{X}:p\le\beta_0\},$$

and

$$E_2=\{p\in\mathcal{X}:p\le\alpha_0\}.$$

Also define the mapping $J:\mathcal{X}\to\mathcal{X}$ by

$$Jp(u)=\int_1^U\left(H(u,s)\,f(s,p(s))\right)^{ds}\ \textit{for all } u\in[1,U].$$

Let us prove that

$$J(E_1)\subseteq E_2\ \textit{and}\ J(E_2)\subseteq E_1.\qquad(6.8)$$

Suppose that $p\in E_1$, that is,

$$p(s)\le\beta_0(s)\ \textit{for all } s\in[1,U].$$

Applying condition (6.5), since $H(u,s)\ge 0$ for all $u,s\in[1,U]$, we obtain that

$$H(u,s)\,f(s,p(s))\ge H(u,s)\,f(s,\beta_0(s))$$
$$\textit{for all } u,s\in[1,U].$$

The above inequality with condition (6.3) imply that

$$\int_1^U\left(H(u,s)\,f(s,p(s))\right)^{ds}$$
$$\ge\int_1^U\left(H(u,s)\,f(s,\beta(s))\right)^{ds}\ge\alpha_0(u),$$

for all $u,s\in[1,U]$. Then we have $Jp\in E_2$.

Similarly, let $p\in E_2$, that is,

$$p(s) \le \alpha_0(s) \ \textit{for all } s \in [1,U].$$

Using condition (6.5), since $H(u,s) \ge 0$ for all $u,s \in [1,U]$, we obtain that

$$H(u,s)\, f(s,p(s)) \le H(u,s)\, f(s,\alpha_0(s))$$
$$\textit{for all } u,s \in [1,U].$$

This implies the above inequality with condition (6.4)

$$\int_1^U \left(H(u,s)\, f(s,p(s))\right)^{ds} \le \int_1^U \left(H(u,s)\, f(s,\alpha_0(s))\right)^{ds} \le \beta_0(u),$$

for all $u,s \in [1,U]$. Then we have $Jp \in E_1$. Also, we deduce that (6.8) holds.

Now, let $(p,q) \in E_1 \times E_2$, that is, for all $u \in [1,U]$,

$$p(u) \le \beta_0(u), q(u) \le \alpha_0(u).$$

From condition(6.2) that for all $u \in [1,U]$,

$$p(u) \le \beta_1, q(u) \le \alpha_1.$$

Now, by condition (6.6) and (6.7), we have, for all $u \in [1,U]$,

$$(|Jx|.|Jy|)^2 = \left[\begin{matrix} \left|\int_1^U \left(H(u,s)\, f(s,x(s))\right)^{ds}\right| \\ \cdot\left|\int_1^U \left(H(u,s)\, f(s,y(s))\right)^{ds}\right| \end{matrix}\right]^2,$$

$$\le \left[\begin{matrix} \int_1^U \left(H(u,s)\left|f(s,x(s))\right|\right)^{ds} \\ \cdot\int_1^U \left(H(u,s)\left|f(s,y(s))\right|\right)^{ds} \end{matrix}\right]^2,$$

$$= \left[\int_1^U \left(H(u,s)\left|f(s,x(s))\right|\left|f(s,y(s))\right|\right)^{ds}\right]^2$$

$$= \left[\int_1^U \left(H(u,s)\left(\begin{matrix} (x.y)^{2\times 2\alpha_0}\, .\left((x.Jx)^2 .(y.Jy)^2\right)^{\beta_0} \\ .\left((x.Jy)^2 .(y.Jx)^2\right)^{\frac{\gamma_0}{3}} .\left(x^4 .y^4\right)^{\frac{\delta_0}{4}} \end{matrix}\right)^{\frac{1}{2}}\right)^{ds}\right]^2,$$

$$\le \left[\int_1^U \left(H(u,s)\left(\begin{matrix} (|x|.|y|)^{2\times 2\alpha_0}\, .\left(\begin{matrix}(|x|.|Jx|)^2 \\ .(|y|.|Jy|)^2\end{matrix}\right)^{\beta_0} \\ .\left(\begin{matrix}(|x|.|Jy|)^2 \\ .(|y|.|Jx|)^2\end{matrix}\right)^{\frac{\gamma_0}{3}} .\left(|x|^4 .|y|^4\right)^{\frac{\delta_0}{4}} \end{matrix}\right)^{\frac{1}{2}}\right)^{ds}\right]^2,$$

$$\le \left[\int_1^U \left(H(u,s)^2\left(\begin{matrix} \mathcal{M}_\infty(x,y)^{2\alpha_0}\, .\left(\begin{matrix}\mathcal{M}_\infty(x,Jx) \\ .\mathcal{M}_\infty(y,Jy)\end{matrix}\right)^{\beta_0} \\ .\left(\mathcal{M}_\infty(x,Jy).\mathcal{M}_\infty(y,Jx)\right)^{\frac{\gamma_0}{3}} \\ .\left(\mathcal{M}_\infty(x,x).\mathcal{M}_\infty(y,y)\right)^{\frac{\delta_0}{4}} \end{matrix}\right)^{\frac{1}{2}}\right)^{ds}\right]^2,$$

$$= \mathcal{M}_\infty(x,y)^{2\alpha_0} .\left(\mathcal{M}_\infty(x,Jx).\mathcal{M}_\infty(y,Jy)\right)^{\beta_0}$$
$$\times\left(\mathcal{M}_\infty(x,Jy).\mathcal{M}_\infty(y,Jx)\right)^{\frac{\gamma_0}{3}}$$
$$\times\left(\mathcal{M}_\infty(x,x).\mathcal{M}_\infty(y,y)\right)^{\frac{\delta_0}{4}} .\left(\int_1^U H(u,s)^{ds}\right)^2,$$

$$= \mathcal{M}_\infty(x,y)^{2\alpha_0} .\left(\mathcal{M}_\infty(x,Jx).\mathcal{M}_\infty(y,Jy)\right)^{\beta_0}$$
$$\times\left(\mathcal{M}_\infty(x,Jy).\mathcal{M}_\infty(y,Jx)\right)^{\frac{\gamma_0}{3}}$$
$$\times\left(\mathcal{M}_\infty(x,x).\mathcal{M}_\infty(y,y)\right)^{\frac{\delta_0}{4}},$$

which implies

$$\mathcal{M}_\infty(Jx,Jy)$$
$$\le \mathcal{M}_\infty(x,y)^{2\alpha_0} .\left(\mathcal{M}_\infty(x,Jx).\mathcal{M}_\infty(y,Jy)\right)^{\beta_0}$$
$$.\left(\mathcal{M}_\infty(x,Jy).\mathcal{M}_\infty(y,Jx)\right)^{\frac{\gamma_0}{3}} .\left(\mathcal{M}_\infty(x,x).\mathcal{M}_\infty(y,y)\right)^{\frac{\delta_0}{4}}.$$

Similarly, the above inequality hold if $(p,q) \in E_1 \times E_2$.

So, Conditions of Theorem (5.1) satisfied and J has a fixed point $\hat{w}^*$ in

$$E_1 \cap E_2 = \{p \in C([1,U]) : \alpha_0 \le p(u) \le \beta_0 \ \textit{for all } u \in [1,U]\}.$$

That is, $\hat{w}^* \in E_1 \cap E_2$ is the solution to,

$$p(u) = \int_1^U \left(H(u,s)\, f(s,p(s))\right)^{ds} \ \textit{for all } u \in [1,U],$$

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