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On Fixed Points for Chatterjea's Maps in b-Metric Spaces

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Abstract In this paper we find sufficient conditions for the existence and uniqueness of fixed points of Chatterjea's maps in b-metric space. These conditions do not involve the b-metric constant. We establish a priori error estimate for the sequence of successive iterations. The error estimate, which we present is better that the well-known one for a wide class of Chatterjea's maps in metric spaces.

Keywords: *fixed point, Chatterjea's map, b-Metric space, a priori error estimate*

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1. Introduction

Fixed point theory has got wide applications in different branches of mathematics. Since the work of S. Banach [3] known as the Banach Contraction Principle, many mathematicians have extended and generalized the results in [3]. Some of the classical generalizations of [3] are presented in [14]. The concept of a b-metric space as a generalization of a metric space is introduced in [2] and a contraction mapping theorem is proved there. Since then results about fixed points, variational principles and applications were obtained in b-metric spaces. We will cite just a few recent results in these directions [1,5,7,8,9,10,11,12,13,16].

We recall some definitions and properties for b-metric spaces [12,13,16].

Definition 1.1. Let X be a non-empty set, $s \geq 1$. A functional $\rho : X \times X \to \mathbb{R}$ is called a b-metric if it satisfies the following conditions:

$\rho(x,y) \geq 0$ for all $x,y \in X$ and $\rho(x,y) = 0$ iff $x = y$;

$\rho(x,y) = \rho(y,x)$ for all $x,y \in X$;

$\rho(x,y) \leq s(\rho(x,z) + \rho(z,y))$ for all $x,y,z \in X$.

The ordered pair (X,ρ) is called a b-metric space (with constant s).

Any metric space is a b-metric space with $s = 1$.

An example of b-metric is the functional $\rho : l_p \times l_p \to \mathbb{R}$, $\rho_p(x,y) = \sum_{i=1}^{\infty} |x_i - y_i|^p$. It is easy to see that in this case $s = 2^{p-1}$.

Other classical example of b-metric space is $\mathbb{R}$ endowed with the b-metric function $\rho_p(x,y) = |x-y|^p$ for $p \in [1,+\infty)$. It is easy to see that in this case $s = 2^{p-1}$ and for $p = 1$ we get the metric space of the real numbers with a metric $\rho_1(x,y) = |x-y|$.

Definition 1.2. Let (X,ρ) be a b-metric space.

A sequence $\{x_n\}_{n=1}^{\infty}$ is called b-convergent if there exists $x \in X$, such that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the inequality $\rho(x, x_n) < \varepsilon$ holds true for all $n \geq N$;

A sequence $\{x_n\}_{n=1}^{\infty}$ is called b-Cauchy sequence if for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the inequality $\rho(x_m, x_n) < \varepsilon$ holds true for all $n > m \geq N$;

The b-metric space (X,ρ) is called complete b-metric space if any Cauchy sequence is convergent;

A subset $A \subseteq X$ is called b-bounded if $\sup\{\rho(x,y) : x,y \in A\} < \infty$;

If the set A is b-bounded then the number $\sup\{\rho(x,y) : x,y \in A\}$ is called its b-diameter and is denoted with $\delta_b(A)$.

A subset $A \subseteq X$ is called b-closed if for any convergent sequence $\{x_n\}_{n=1}^{\infty} \subset A$ the convergence $\lim_{n\to\infty} x_n = x$ implies $x \in A$.

A b-metric function ρ is called continuous if for any $y \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(y,\varepsilon) > 0$ such that there holds the inequality $|\rho(y,x) - \rho(y,z)| < \varepsilon$, provided that $\rho(x,z) < \delta$. It is easy to observe that if ρ is continuous and x_n is b-convergent to x then $\rho(y,x_n) \to \rho(y,x)$.

Every b-convergent sequence in b-metric space is a b-Cauchy sequence. If a sequence is a b-convergent in b-metric space then its limit is unique. In general a b-metric function is not continuous [5,10].

As far as we will consider only b-metrics we will omit the letter b in the above definitions.

Definition 1.3. ([14]) Let (X,ρ) be a metric space. A map $T:X\to X$ is a Hardy Rogers map is there exist nonnegative constants a_i, $i=1,2,3,4,5$ satisfying $\sum_{i=1}^{5} a_i < 1$ such that for each $x,y\in X$ the inequality

$$\rho(Tx,Ty)\le a_1\rho(x,y)+a_2\rho(x,Tx)+a_3\rho(y,Ty)+a_4\rho(x,Ty)+a_5\rho(y,Tx)$$

holds for all $x,y\in X$.

As pointed in [15] from the symmetry of the function ρ it follows that $a_2=a_3$ and $a_4=a_5$. Therefore if T is a Hardy-Rogers contraction then there exist $k_1, k_2, k_3\ge 0$, such that $k_1+2k_2+2k_3<1$ and there holds the inequality

$$\rho(Tx,Ty)\le k_1\rho(x,y)+k_2(\rho(x,Tx)+\rho(y,Ty))+k_3(\rho(x,Ty)+\rho(y,Tx)).$$

Generalizations of Hardy Rogers map in b-metric space are investigated in [8,13].

If $k_1=k_2=0$ and $k_3\in[0,1/2)$ in the above inequality we get a generalization of Chatterjea's map [6] in b-metric space.

Definition 1.4. Let (X,ρ) be a b-metric space. A map $T:X\to X$ is called Chatterjea's map if there exists $k\in[0,1/2)$ such that the inequality

$$\rho(Tx,Ty)\le k\left(\rho(Tx,y)+\rho(Ty,x)\right)$$

holds for all $x,y\in X$.

We will denote for the rest of the article $\alpha=\frac{k}{1-k}$, where k is the constant from the definition of Chatterjea's map. From $k\in[0,1/2)$ it follows that $\alpha\in[0,1)$.

2. Fixed Points for Chatterjea's Maps in b-Metric Spaces

Theorem 2.1. Let (X,ρ) be a complete b-metric space, ρ be a continuous function, $T:X\to X$ be a Chatterjea's map, such that the inequality $\sup_{n\in\mathbb{N}}\left\{\rho\left(T^n x,x\right)\right\}<\infty$ holds for any $x\in X$. Then

(i) there exists a unique fixed point say ξ of T;

(ii) for any $x_0\in A$ the sequence $\{x_n\}_{n=1}^{\infty}$ converges to ξ, where $x_{n+1}=Tx_n$, $n=0,1,2,\ldots$;

(iii) there holds the a priori error estimate

$$\rho\left(\xi,T^m x\right)\le\alpha^m\sup_{j\in\mathbb{N}}\rho\left(T^j x,x\right). \tag{2.1}$$

Lemma 2.2. Let (X,ρ) be a b-metric space and let $T:X\to X$ be a Chatterjea's map. Then for any $x\in X$ there holds the inequality

$$\rho\left(T^n x,T^m x\right)\le\left(\frac{k}{1-k}\right)^m\sup_{2\le j\le n}\left\{\rho\left(T^j x,x\right)\right\} \tag{2.2}$$

for any $n>m\ge 1$.

Proof. Let us denote $r_n(x)=\rho\left(T^n x,x\right)$ and $x_{m,n}=\rho\left(T^n x,T^m x\right)$. We consider the sequence

$$x_{2,1},x_{3,1},x_{3,2},\ldots x_{n-1,n-2},x_{n,1},x_{n,2},\ldots,x_{n,n-1},x_{n+1,1},\ldots \tag{2.3}$$

We will prove inequality (2.2) by induction on the sequence (2.3). Let us denote by i the sum of the indices of the sequence in (2.3).

Let $i=3$, i.e. $n=2$ and $m=1$. Then $x_{2,1}\le kr_2(x)\le\frac{k}{1-k}\rho\left(T^2x,x\right)$.

Let $i=4$, i.e. $n=3$ and $m=1$. Then

$$x_{3,1}\le k\left(r_3(x)+x_{2,1}\right)\le k\left(1+\frac{k}{1-k}\right)\sup_{2\le j\le 3} r_j(x)=\frac{k}{1-k}\sup_{2\le j\le 3}\rho\left(T^j x,x\right).$$

Let inequality (2.2) holds for $i=p$.

We will prove that (2.2) holds true for $i=p+1$. Let $n+m=p$. There are two cases: If $m<n$ then we consider $x_{n,m+1}$, if $m=n-1$ then we consider $x_{n+1,1}$.

Case I) There are two subcases: $m<n-2$ and $m=n-2$. Let first $m<n-2$. Then

$$\begin{aligned}x_{n,m+1}&\le k\left(x_{n,m}+x_{n-1,m+1}\right)\\&\le k\left(\left(\frac{k}{1-k}\right)^m\sup_{2\le j\le n} r_j(x)+\left(\frac{k}{1-k}\right)^{m+1}\sup_{2\le j\le n-1} r_j(x)\right)\\&=k\left(\frac{k}{1-k}\right)^m\left(1+\frac{k}{1-k}\right)\sup_{2\le j\le n} r_j(x)\\&=\left(\frac{k}{1-k}\right)^{m+1}\sup_{2\le j\le n}\rho\left(T^j x,x\right).\end{aligned}$$

Let now $m=n-2$. Then

$$\begin{aligned}x_{n,m+1}&\le k\left(x_{n,m}+x_{n-1,m+1}\right)=kx_{n,m}\\&\le k\left(\frac{k}{1-k}\right)^m\sup_{2\le j\le n} r_j(x)\\&=\left(\frac{k}{1-k}\right)^{m+1}\sup_{2\le j\le n}\rho\left(T^j x,x\right).\end{aligned}$$

Case II)

$$\begin{aligned} x_{n+1,1} &\le k\left(r_{n+1}(x)+x_{n,1}\right)\\ &\le k\left(\sup_{2\le j\le n+1} r_j(x)+\left(\frac{k}{1-k}\right)\sup_{2\le j\le n} r_j(x)\right)\\ &= k\left(1+\frac{k}{1-k}\right)\sup_{2\le j\le n+1} r_j(x)\\ &= \frac{k}{1-k}\sup_{2\le j\le n+1}\rho\left(T^j x, x\right). \end{aligned}$$

Proof. of Theorem 2.1 (i) Let $x\in X$ be arbitrary.

Let us put $M=\sup_{j\ge 2}\rho\left(T^j x,x\right)$. From Lemma 2.2 we have that the inequality

$$\rho\left(T^n x, T^m x\right)\le \alpha^m \sup_{2\le j\le n}\rho\left(T^j x,x\right)\le \alpha^m M$$

holds for every $n>m\ge 1$. Consequently the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ is a Cauchy sequence. From the assumption that X is complete b-metric space it follows that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ is b-convergent. Therefore it follows that there exists $\xi=\lim_{n\to\infty}T^n x\in X$. Let us fix $n\in\mathbb{N}$. After taking a limit on $m\to\infty$ from the assumption that the b-metric is continuous and using that T is Chatterjea's map we get the inequality

$$\begin{aligned} \rho(T\xi,\xi) &= \lim_{m\to\infty}\rho\left(T\xi, T^m x\right)\\ &\le \lim_{m\to\infty}\left(k\left(\rho\left(T\xi,T^{m-1}x\right)+\rho\left(\xi,T^m x\right)\right)\right)\\ &= k\left(\rho(T\xi,\xi)+\rho(\xi,\xi)\right)=k\rho(T\xi,\xi) \end{aligned}$$

and therefore $\rho(T\xi,\xi)=0$ i.e. ξ is a fixed point for T. Let suppose that there are two fixed points $\xi\ne\eta$. Then from the inequality

$$\begin{aligned} \rho(\xi,\eta) &= \rho(T\xi,T\eta)\le k\left(\rho(T\xi,\eta)+\rho(T\eta,\xi)\right)\\ &= 2k\rho(\xi,\eta) \end{aligned}$$

and the assumption that $k\in[0,1/2)$ it follows that $\xi=\eta$.

(ii) The proof follows from (i), because any sequence $\left\{T^n x_0\right\}_{n=1}^{\infty}$ is convergent to the fixed point of T, which is unique.

(iii) Let $x\in X$ be arbitrary. From Lemma 2.2 we have the inequality

$$\rho\left(T^n x,T^m x\right)\le\alpha^m\sup_{j\in\mathbb{N}}\rho\left(T^j x,x\right)$$

holds for every $n>m\ge 1$ and every $x\in X$. From (ii) it follows that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ converges to the unique fixed point ξ. Therefore using the continuity of ρ and Lemma 2.2 we get

$$\rho\left(\xi,T^m x\right)=\lim_{n\to\infty}\rho\left(T^n x,T^m x\right)\le\alpha^m\sup_{j\in\mathbb{N}}\rho\left(T^j x,x\right).$$

As far as any metric space is a b-metric space, then Theorem 2.1 holds true for arbitrary metric space. If (X,d) is a complete metric space and T be Chatterjea's map then the a priori error estimate is well known [4]

$$d\left(\xi,T^m x\right)\le\frac{\alpha^m}{1-\alpha}d(Tx,x). \quad (2.4)$$

If we assume that $\sup_{j\in\mathbb{N}}\rho\left(T^j x,x\right)\le\rho(Tx,x)$ then we will get from Theorem 2.1 the a priori estimate

$$\rho\left(\xi,T^m x\right)\le\alpha^m\rho(Tx,x). \quad (2.5)$$

Let us mention that in this case the a priori estimate (2.5) is better, than (2.4).

Let $\varepsilon\in\left(0,\rho(Tx,x)\right)$, $m_\alpha\in\mathbb{N}$ be the smallest number, that satisfies (2.5) and $n_\alpha\in\mathbb{N}$ be the smallest number, that satisfies (2.4). Then

$$\begin{aligned} n_\alpha-m_\alpha &\ge \left|\frac{\log\frac{\varepsilon(1-\alpha)}{\rho(Tx,x)}}{\log\alpha}\right|-\left(\left|\frac{\log\frac{\varepsilon}{\rho(Tx,x)}}{\log\alpha}\right|+1\right)\\ &= \left|\frac{\log(1-\alpha)}{\log\alpha}\right|-1. \end{aligned}$$

If k gets close to $1/2$ then α gets closer to 1 and therefore $n_\alpha-m_\alpha$ gets closer to infinity.

We would like to point out that if the space is a metric space than using the triangle inequality we can obtain (2.5) from (2.1).

Example 2.3. Let us consider the b-metric space $\left(\mathbb{R},\rho_p\right)$ for $p\ge 1$. Let $0<\alpha<\beta$ be two arbitrary positive real numbers. Let us define the map $T_\alpha^\beta:[0,+\infty)\to[0,+\infty)$, by $T_\alpha^\beta x=\begin{cases}\alpha, & x\in[\beta,+\infty)\\ 0, & x\in[0,\beta)\end{cases}$ (Figure 1), which is a variation of the classical examples from [14]. It is well known that $T_{1/2}^2$ is Chatterjea's map and $T_{1/2}^1$ is not Chatterjea's map in the metric space $(\mathbb{R},\rho_1)$ [14]. It is easy to observe that the Picard iteration sequence $x_n=T_\alpha^\beta x_{n-1}$ converges to the fixed point $x=0$ for any initial point $x_1\in[0,+\infty)$.

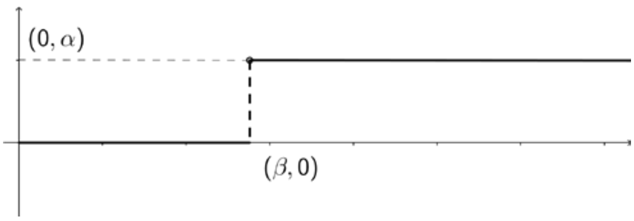

Figure 1

If $x,y\in[0,\beta)$ or $x,y\in[\beta,+\infty)$, then T_α^β satisfies the condition in Definition 1.4 for any $k\in\left[0,\frac{1}{2}\right)$, because

$\rho_p(Tx,Ty) = |Tx - Ty|^p = 0$. If $y \in [0,\beta)$ and $x \in [\beta,+\infty)$, then we get $\rho_p(Tx,y) + \rho_p(Ty,x) = |\alpha - y|^p + \beta^p$ and $\rho_p(Tx,Ty) = \alpha^p$. Using the inequality

$$\inf\left\{|\alpha - y|^p + x^p : y \in [0,\beta),\ x \in [\beta,+\infty)\right\} = \beta^p$$

we get that there holds

$$\rho_p(Tx,Ty) = \alpha^p \le k\beta^p \le k\left(\rho_p(Tx,y) + \rho_p(Ty,x)\right) \quad (2.6)$$

for any $k \ge \left(\frac{\alpha}{\beta}\right)^p$. Therefore if $2\alpha \ge \beta$ then T_α^β is not a Chatterjea's map in $(\mathbb{R},\rho_1)$. For any arbitrary $0 < \alpha < \beta$ we can choose $p \in [1,+\infty)$, such that $\left(\frac{\alpha}{\beta}\right)^p \in \left[0,\frac{1}{2}\right)$. Consequently for any map T_α^β we can endow $(\mathbb{R},\rho_1)$ with a suitable b-metric $\rho_p(x-y) = |x-y|^p$ so that T_α^β to satisfy the condition in Definition 1.4 in $\left(\mathbb{R},\rho_p\right)$.

Let us consider the particular case $2\alpha \ge \beta$ and $p > 1$. If we choose in this case $k \ge \left(\frac{\alpha}{\beta}\right)^p \ge \left(\frac{1}{2}\right)^p \in \left[0,\frac{1}{2}\right)$, provided that we have considered the b-metric space $(\mathbb{R},\rho_p)$, $p > 1$, then $k.s \ge \frac{1}{2}$, because $s = 2^{p-1}$ in $(\mathbb{R},\rho_p)$. Consequently T_α^β does not satisfy the conditions in ([16] Theorem 3) for any $p \in (1,+\infty)$ in $(\mathbb{R},\rho_p)$ and thus Theorem 2.1 extends ([12] Theorem 3) in the case when $\sup_{n\in\mathbb{N}} \rho\left(T^n x, x\right) < \infty$.

In the particular case $T_{1/2}^1$ we get that $k.s = \frac{1}{2}$, provided that k is chosen so that inequality (2.6) to hold in $\left(\mathbb{R},\rho_p\right)$ and therefore ([12] Theorem 3) could not be applied.

When applying fixed point theorems for approximating of a solution of the equation $Tx = x$ we usually find an initial starting point x_0, which belongs to a neighborhood U of the solution ξ, such that $T: U \to U$ and U is bounded and closed. Thus the next Corollary can be applied in a wide class of problems.

Corollary 2.3. Let $\left(X,\rho\right)$ be a complete b-metric space, ρ be a continuous function, $A \subseteq X$ be a b-bounded and b-closed set, $T: A \to A$ be Chatterjea's map. Then

there exists a unique fixed point say ξ of T;

for any $x_0 \in A$ the sequence $\{x_n\}_{n=1}^\infty$ converges to ξ, where $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$;

there holds the a priori error estimate $\rho\left(\xi, T^n x\right) \le \alpha^m \delta_b\left(A\right)$.

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