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Dirichlet Average of Generalized Miller-Ross Function and Fractional Derivative

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Abstract The object of the present paper is to establish the results of single Dirichlet average of Generalized Miller-Ross Function, using Riemann-Liouville Fractional Integral. The Generalized Miller-Ross Function can be measured as a Dirichlet average and connected with fractional calculus. In this paper the solution comes in compact form of single Dirichlet average of Generalized Miller-Ross Function. The special cases of our results are same as earlier obtained by Saxena et al. [12], for single Dirichlet average of Generalized Miller-Ross Function.

Keywords: *Dirichlet average, Generalized Miller-Ross Function, fractional derivative and Fractional calculus operators*

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1. Introduction

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions like x^t, e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n, e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Recently, Gupta and Agarwal [9,10] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [7].

In the present paper the Dirichlet average of Generalized Miller-Ross Function has been obtained.

2. Definitions

Some definitions which are necessary in the preparation of this paper.

2.1. Standard Simplex in $R^n, n \geq 1$:

Denote the standard simplex in R^n, $n \geq 1$ by [[1], p. 62].

$$E = E_n = \{S(u_1, u_2, \ldots u_n):$$
$$u_1 \geq 0, \ldots u_n \geq 0, u_1 + u_2 + \ldots + u_n \leq 1\}.$$

2.2. Dirichlet Measure

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1} The complex measure μ_b is defined by E [1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \ldots - u_{k-1})$$
$$b_k^{-1} du_1 \ldots du_{k-1}$$

known as Dirichlet measure.

Here

$$B(b) = B(b1,\ldots bk) = \frac{\Gamma(b_1)\ldots\Gamma(b_k)}{\Gamma(b_1+\ldots+b_k)},$$

$$C_> = \left\{z \in z : z \neq 0, |ph\, z| < \pi/2\right\},$$

open right half plane and $C_>$ k is the k^{th} Cartesian power of $C_>$.

2.3. Dirichlet Average[[1], p. 75]

Let Ω be the convex set in $C_>$, let $z = (z_1,\ldots,z_k) \in \Omega^k, k \geq 2$ and let $u.z$ be a convex combination of $z_1,\ldots,z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z)\, d\mu_b(u) \tag{2.3}$$

F is the Dirichlet measure of f with variables $z = (z_1,\ldots,z_k)$ and parameters $b = (b_1,\ldots b_k)$.

Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \ldots - u_{k-1.}$$

If $k = 1$, define $F(b,z) = f(z)$

2.4. Generalized Miller-Ross Function

This function is introduced by the author as follows:

$$ {}_0^aN_{p,q}^{\alpha,\beta}\left(a_1\ldots a_p;,b_1\ldots b_q;x^0\right) = {}_0^aN_{p,q}^{\alpha,\beta}(x) = \sum_{n=o}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{a^n x^{n+\beta}}{\Gamma(\alpha n+\beta+1)}. \tag{2.4}$$

Here, p upper parameters $a_{1,}a_{2,}\ldots a_p$ and q lower parameters $b_{1,}b_{2,}\ldots b_q, \alpha, \beta \epsilon C, R(\alpha) > 0, R(\beta) > 0$ and $(a_j)_k(b_j)_k$ are pochammer symbols. The function (3.6.1.1) is defined when none of the denominator parameters $b_j s, j = 1,2,\ldots q$ is a negative integer or zero. If any parameter a_j is negative then the function (2.4) terminates into a polynomial in x. By using ratio test, it is evident that function (2.4) is convergent for all x, when $q \geq p$, it is convergent for $|x| < 1$ when $p = q+1$, divergent when $p > q+1$. In some cases the series is convergent for $x = 1, x = -1$. Let us consider take,

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$$

when $p = q+1$, the series is absolutely convergent for $|x| = 1$ if $R(\beta) < 0$, convergent for $x = -1$ if $0 \leq R(\beta) < 1$ and divergent for $|x| = 1$ if $1 \leq R(\beta)$ which is a special case of Wright function.

2.5. Fractional Derivative [[8], p. 181]

The theory of fractional derivative with respect to an arbitrary function has been used by Erdelyi [8]. The general definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.5}$$

where $\mathrm{Re}(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$

3. Equivalence

In this section we shall prove the equivalence of single Dirichlet average of Generalized Miller-Ross Function $(k = 2)$ with the fractional derivative i.e.

$$S\left(\beta,\beta',x,y\right) = \frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta}(x-y)^{1-\beta-\beta'} D_{x-y}^{-\beta'}\, {}_0^aN_{p,q}^{\alpha,\beta}(x)(x-y)^{\beta-1}. \tag{3.1}$$

Proof:

$$S\left(\beta,\beta',x,y\right) = \sum_{n=o}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{a^n}{\Gamma(\alpha n+\beta+1)} R_n(\beta,\beta',x,y)$$

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{a^n}{\Gamma(\alpha n+\beta+1)} \frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta\Gamma\beta'} \int_0^1 [ux+(1-u)y]^{n+\beta} u^{\beta-1}(1-u)^{\beta'-1} du$$

Putting $u(x-y) = t$, we have

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{a^n}{\Gamma(\alpha n+\beta+1)} \frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta\Gamma\beta'} \int_0^{x-y} [t+y]^{n+\beta} \left(\frac{t}{x-y}\right)^{\beta-1} \left(1-\frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}.$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta\Gamma\beta'} \int_0^{x-y} \sum_{n=o}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n} \frac{a^n}{\Gamma(\alpha n+\beta+1)} [t+y]^{n+\beta} (t)^{\beta-1} (x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta\Gamma\beta'}$$

$$\int_0^{x-y} {}_0^a N_{p,q}^{\alpha,\beta}\left(y+t\right)\left(t\right)^{\beta-1}\left(x-y-t\right)^{\beta'-1} dt .$$

Hence by the definition of fractional derivative, we get

$$S\left(\beta,\beta',x,y\right) = (x-y)^{1-\beta-\beta'}$$

$$\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\beta} D_{x-y}^{-\beta'} {}_0^a N_{p,q}^{\alpha,\beta}\left(x\right)(x-y)^{\beta-1}.$$

This completes the analysis.

4. Particular Cases

If $\beta' = \gamma-\beta, \beta = 0, y = 0, \alpha = 1, a = 1$ and no upper and lower parameter in (3.1) then

$$S\left(0,\gamma-\beta,x,0\right) = (x)^{1-\gamma} \frac{\Gamma\gamma}{\Gamma\beta} D_x^{-\gamma} e^x (x)^{\beta-1} \quad (4.1)$$

$$= {}_1F_1\left(\beta;\gamma;x\right).$$

This confluent hyper geometric function [11]

$$S(\beta,\gamma-\beta,x,0) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma s \frac{\Gamma\left(\beta-s\right)}{\Gamma\left(\gamma-s\right)}\left(-x\right)^{-s} ds .$$

Then

$$S(\beta,\gamma-\beta,x,0) = \frac{\Gamma\left(\gamma\right)}{\Gamma(\beta)}\left[H_{1,2}^{1,1}\left(-x\middle|\begin{matrix}(1-\beta,1)\\ (0,1),(1-\gamma,1)\end{matrix}\right)\right]$$

(ii) If $\beta' = \xi-\beta$ and from (4.1), then

$$S\left(\beta,\xi-\beta;x,0\right) = \frac{\Gamma(\xi)}{\Gamma(\beta)} x^{1-\xi} D_x^{\beta-\xi} e^x x^{\beta-1}$$

$$S\left(\beta,\xi-\beta;x,0\right) = {}_1F_1\left(\beta;\xi;x\right) = \Gamma\left(\xi\right) E_{1,\xi}^{\beta}(x)$$

where $E_{1,\xi}^{\beta}(x)$ be the generalization of Mittag-Leffler function [12].

5. Applications

Dirichlet average is average given by Dirichlet. The Dirichlet average of elementary function like power function, exponential function etc is given by many notable mathematician, Actually, We have convert the elementary function into the summation form after that taking Dirichlet average of those function, using fractional integral and get new results. These results will be used in future by mathematician and scientist. Thus we have find a connection Dirichlet average of a function and fractional integral.

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