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Stability Results of the Additive and Quartic Functional Equations in Random p-Normed Spaces

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Abstract In this paper, we investigate the Hyers-Ulam stability of mixed type additive and quartic functional equations in Random p-normed spaces by direct and fixed-point method.

Keywords: *Hyers-Ulam stability, additive functional equation, quartic functional equation, random p-normed spaces, fixed point method, direct method*

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1. Introduction

In 1940, the stability problems of functional equations about homomorphism of groups was introduced by Ulam [1]. In 1941, Hyers [2] gave an affirmative answer to Ulam's question for additive groups (under the assumption that groups are Banach spaces). Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference $\|f(x+y)-f(x)-f(y)\| \leq \varepsilon\,(\|x\|^p+\|y\|^p)$ for all $\varepsilon > 0$ and $p \in [0,1)$. Following the same approach as Rassias, Gajda [5] gave an affirmative solution of this problem for $p > 1$ and also proved that it is possible to solve the Rassias-type $p = 1$. Also, in 1994, Rassias generalization theorem was delivered by Gavruta [6] who replaced $\varepsilon(\|x\|^p+\|y\|^p)$ by a control function $\phi(x,y)$. The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. J.M. Rassias [4] followed the modern approach of the Th.M. Rassias [7] theorem in which he replaced the factor product of norms instead of sum of norms.

The functional equations

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

are known as additive functional equation. Each additive solution of a functional equation must be an additive mapping. For Stability of additive, quadratic, cubic and quartic functional equations in random normed spaces, we may refer [8,9,10,11,12].

Some notions and conventions of the theory of random and random p-normed spaces are taken in our paper as in [11,12,13,14,15].

Throughout the paper Δ^+ is the distribution functions space, that is, the space of all mappings V: R ∪ {−∞, ∞} → [0,1], such that F is left continuous and increasing on R, V(0) = 0 and V(+∞) = 1. $D^+ \subset \Delta^+$ consisting of all functions $V \in \Delta^+$ for which $l^-V(+\infty) = 1$, where $l^-\phi(s)$ denotes $l^-\phi(s) = \lim_{t\to s^-} \phi(s)$. The space Δ^+ is partially ordered by the usual point wise ordering of functions, i.e., $V \leq W \Leftrightarrow V(t) \leq W(t)\ \forall t \in R$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0 \ if\ t \leq 0, \\ 1\ if\ t > 0. \end{cases}$$

In 2012, SA Mohiuddine *et al.* [16] are to present a relationship between three various disciplines the theory of functional equation

$$2f\left(\frac{x+y+z}{2}\right) = f(x)+f(y)+f(z) \tag{1.2}$$

Recently, in 2019, Senthil Kumar *et al.* [17] proved the general solution of Hyers-Ulam stability of mixed type additive and quartic functional equations of the form

$$\begin{aligned} &f(2x+y)+f(2x-y)+f(x+2y)+f(x-2y) \\ &=8\left[f(x+y)+f(x-y)\right]+f(2x)-5f(x) \\ &\quad +7f(-x)+2f(2y)-5f(-y)-9f(y), \end{aligned} \tag{1.3}$$

We focus on the ensuing mixed type functional equation derived from additive and quartic mappings. We validate the Hyers-Ulam stability of equation (1.3) in p-normed spaces. It is not hard for one for one that the mixed type function $f(x) = ax + bx^4$ is a solution of the equation (1.3).

In this section, we determine the generalized Hyers-Ulam stability of mixed type additive quartic functional equations in random p-normed space, by using direct method and fixed-point method. Following definitions and notions will be usedto prove our main results:

Definition 1.1 [11] (t-norm) $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly t - norm) if T satisfies the following conditions:

i) T is commutative and associative;

ii) T is a continuous;

iii) $T(a,1)=a$ for all $a\in[0,1]$;

iv) $T(a,b)\le T(c,d)$, whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Examples of continuous t-norm are $T(x,y)=xy$, $T(x,y)=\max\{x+y-1,0\}$ and $T(x,y)=min(x,y)$.

Recall that, if T is a t-norm and $\{x_n\}$ are given numbers in [0,1], then, $T_{i=1}^{n}x_i$ is defined by recursively by

$$T_{i=1}^{n}x_i=\begin{cases}x_1 & if\ n=1,\\ \mathrm{T}(\mathrm{T}_{i=1}^{n-1}x_i,x_n) & if\ n\ge 2,\end{cases}$$

$T_{i=n}^{\infty}x_i$ is defined as $T_{i=n}^{\infty}x_{n+i}$.

Definition 1.2. [11]. A Random Normed space (briefly RN-space) is a triple (X,Θ,T), where X is a vector space, T is a continuous t-norm and $\Theta:\mathrm{X}\to D^{+}$(for all $x\in X,\Theta(x)$ is denoted by Θ_x), satisfying the following conditions:

i) $\Theta_x(t)=\varepsilon_0(t)$, for all $t>0$ if and only if $x=0$;

ii) $\Theta_{\alpha x}(t)=\Theta_x\left(\frac{t}{|\alpha|}\right)$, for all $x\in X,t\ge 0$ and $\alpha\in\mathcal{R}$ with $\alpha\ne 0$;

iii) $\Theta_{x+y}(t+u)\ge T\left(\Theta_x(t),\Theta_y(u)\right)$ for all $x,y\in X$ and $t,u\ge 0$.

Definition 1.3. [11] Let (X,Θ,T) be a RN-space.

RN1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x\in X$ if $\lim_{n\to\infty}\Theta_{x_n-x}(t)=1$, $t>0$.

RN2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty}\Theta_{x_n-x_l}(t)=1$, $t>0$.

RN3) A RN-space (X,Θ,T) is said to be complete if every Cauchy sequence in X is convergent.

Definition 1.4. [13] Let X be a real linear space $p\in R^{+}$ with $0<p\le 1$ and T be a continuous t-norm. The triple (X,Θ,T) is called a random p-normed space if a mapping $\Theta:X\to D^{+}$(for all $x\in X,\Theta(x)$ is denoted by Θ_x), satisfying the following conditions:

i) $\Theta_x(t)=\varepsilon_0(t)$, for all $t>0$ if and only if $x=0$;

ii) $\Theta_{\alpha x}(t)=\Theta_x\left(\frac{t}{|\alpha|^p}\right)$, for all $x\in X,t\ge 0$ and $\alpha\ne 0$;

iii) $\Theta_{x+y}(t+u)\ge T\left(\Theta_x(t),\Theta_y(u)\right)$, for all $x,y\in X$ and $t,u\ge 0$.

Note that every p-normed space $(X,\|.\|)$ defines a random p-normed space (X,Θ,T_M) where $\Theta_x(t)=\frac{t}{t+\|x\|}$, , for all $t>0$ and T_M is the minimum t-norm. This space is called the induced random p-normed space.

Definition 1.5. [13] Let (X,Θ,T) is called a random p-normed space.

1. A sequence $\{x_n\}$ in X is said to be convergent to a point $x\in X$ if for all $t>0$ and $\lambda>0$, there exists a positive integer N such that $\Theta_{x_n-x}(t)>1-\lambda$ whenever $n\ge N$.

2. A sequence $\{x_n\}$ in X is called a Cauchy convergent if for all $t>0$ and $\lambda>0$, there exists a positive integer N such that $\Theta_{x_n-x_m}(t)>1-\lambda$ whenever $n\ge m\ge N$.

3. The random p-normed space (X,Θ,T) is said to be complete if every Cauchy sequence is convergent to a point in X.

Theorem 1.6. [14]. If (X,Θ,T) is a random normed space and $\{x_n\}$ is a sequence of X such that $x_n\to x$, then $\lim_{n\to\infty}\Theta_{x_n}(t)=\Theta_x(t)$.

2. Stability Results for Additive Functional Equation by Using Direct Method

In this section, we investigate the generalized Hyers-Ulam stability problem of the functional equation (1.2), in random p-normed spaces under the minimum t-norm T_M by using direct method.

In this paper, let X be a linear space, (X,Θ',T) be a random p-normed space and (X,Θ,T_M) be a complete random p-normed space. We determine the stability of the additive functional equation defined by

$$D_{f(x,y,z)}=2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),\quad(2.1)$$

in random p-normed spaces by using Direct method.

Theorem 2.1. Let $\phi:X^3\to Z$ be amapping such that for some $0<\alpha<2$.

$$\Theta'_{\phi(2x,2y,2z)}(t)\ge\Theta'_{\alpha(x,y,z)}(t)\qquad(2.2)$$

and $\lim_{n\to\infty}\Theta'_{\phi(2^nx,2^ny,2^nz)}(2^{np}t)=1$, for all $x,y,z\in X$ and all $t>0$. If $f:X\to Y$ is an odd mapping with $f(0)=0$ such that

$$\Theta_{Df(x,y,z)}(t)\ge\Theta'_{\phi(x,y,z)}(t)\qquad(2.3)$$

for all $x,y,z\in X$ and all $t>0$, then there exists a unique additive mapping $Q:X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge\Theta'_{\phi(x,2x,x)}(2^p-\alpha^p)(t),\qquad(2.4)$$

for all $x\in X$ and all $t>0$.

Proof. Replacing y by $2x$ and z by x in (2.3), we obtain

$$\Theta_{f(x)-\frac{f(2x)}{2}}(t)\ge\Theta'_{\phi(x,2x,x)}\left(2^pt\right),\qquad(2.5)$$

for all $x\in X$ and all $t>0$.

Replacing x by 2^nx in (2.5), we obtain

$$\Theta_{\frac{f(2^nx)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}}}(t)\ge\Theta'_{\phi\left(2^nx,2^{n+1}x,2^nx\right)}\left(2^p\left(2^{np}\right)t\right)\ge\Theta'_{\phi((x,2x,x))}\left(\left(\frac{2}{\alpha}\right)^{np}(2^p)t\right),\qquad(2.6)$$

for all $x\in X$ and all $t>0$.

Since

$$f(x)-\frac{f(2^nx)}{2^n}=\sum_{j=0}^{n-1}\left(\frac{f(2^jx)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right),\qquad(2.7)$$

for all $x \in X$ and all $t > 0$.

$$\Theta_{f(x)-\frac{f(2^n x)}{2^n}}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right)$$

$$= \Theta_{\sum_{j=0}^{n-1}\left(\frac{f(2^j x)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right)}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right)$$

$$\geq T_{M\, j=0}^{\;n-1}\left(\Theta_{\left(\frac{f(2^j x)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right)}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right) \quad (2.8)$$

$$= T_M\left(\Theta'_{\phi(x,2x,x)},\Theta'_{\phi(x,2x,x)},\ldots,\Theta'_{\phi(x,2x,x)}(t)\right)$$

$$= \Theta'_{\phi(x,2x,x)}(t),$$

for all $x \in X$ and all $t > 0$.

Replacing x by $2^m x$ in (2.8), we get

$$\Theta_{\frac{f(2^m x)}{2^m}-\frac{f(2^{m+n}x)}{2^{m+n}}}(t) \geq \Theta'_{\phi(x,2x,x)}\left(\frac{(2)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right) \quad (2.9)$$

for all $x \in X$ and all $m, n \in Z$ with $n > m \geq 0$. It follows from

$$\lim_{n,m\to\infty}\Theta'_{\phi(x,2x,x)}\left(\frac{(2)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right) = 1,$$

that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy in (X, Θ, T_M), and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}\left\{\frac{f(2^n x)}{2^n}\right\}$, for all $x \in X$ and all $t > 0$. Fix $x \in X$ and put $m = 0$ in (2.9). Then, we obtain

$$\Theta_{f(x)-\frac{f(2^n x)}{2^n}}(t) \geq \Theta'_{\phi((x,2x,x))}\left(\frac{(2)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right),$$

for all $x \in X$ and all $t > 0$. For every $s > 0$,

$$\Theta_{f(x)-Q(x)}(s+t)$$

$$\geq T_M\left(\Theta_{f(x)-\frac{f(2^n x)}{2^n}}, \Theta_{\frac{f(2^n x)}{2^n}-Q(x)}(s)\right) \quad (2.10)$$

$$\geq T_M\left(\Theta'_{\phi((x,2x,x))}\left(\frac{(2)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right), \Theta_{\frac{f(2^n x)}{2^n}-Q(x)}(s)\right),$$

for all $x \in X$ and all $t > 0$. Taking the limit $n \to \infty$ in (2.10), we obtain

$$\Theta_{f(x)-Q(x)}(s+t) \geq \Theta'_{\phi((x,2x,x))}\left((2^p - \alpha^p)t\right), \quad (2.11)$$

Thus, since s is arbitrary and taking $s \to 0$ in (2.11), we have

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi((x,2x,x))}\left((2^p - \alpha^p)t\right),$$

for all $x \in X$ and all $t > 0$. Thus, the condition (2.4) holds for all $x \in X$ and all $t > 0$. If we replace (x, y, z) by $(2^n x, 2^n y, 2^n z)$ in (2.3), then

$$\Theta_{\frac{Df(2^n x, 2^n y, 2^n z)}{2^n}}(t) \geq \Theta'_{\phi(2x,2y,2z)}\left(2^{np} t\right), \quad (2.12)$$

for all $x, y, z \in X$ and all $t > 0$. Letting $n \to \infty$ in (2.12), we find that $\Theta_{DQ(x,y,z)} = 1$, for all $t > 0$, which implies $DQ(x, y, z) = 0$, for all $x, y, z \in X$. Therefore, the mapping Q is additive. Now, we prove that the additive mapping Q is unique. Let us assume that there exists another mapping $R : X \to Y$ which satisfies (2.4). For fixed $x \in X$, $Q(2^n x) = 2^n Q(x)$ and $R(2^n x) = 2^n R(x)$, all $n \in Z^+$. It follows from (2.4) that

$$\Theta_{Q(x)-R(x)}(t) = \Theta_{\frac{Q(2^n x)}{2^n}-\frac{R(2^n x)}{2^n}}(t)$$

$$\geq T_M\left(\Theta_{\frac{Q(2^n x)}{2^n}-\frac{f(2^n x)}{2^n}}\left(\frac{t}{2}\right), \Theta_{\frac{f(2^n x)}{2^n}-\frac{R(2^n x)}{2^n}}\left(\frac{t}{2}\right)\right)$$

$$\geq \Theta'_{\phi(x,2x,x)}\left(\left(2^p - \alpha^p\right)\left(\frac{2}{\alpha}\right)^{np} t\right)$$

as $\lim_{n\to\infty}(2^p - \alpha^p)\left(\frac{2}{\alpha}\right)^{np}(t) = \infty$, we have $\Theta_{Q(x)-R(x)}(t) = 1$ for all $t > 0$. Thus, $Q(x) = R(x)$, for all $x \in X$. Hence, it is completed.

Theorem 2.2. Let $\phi : X^3 \to Z$ be a mapping such that for some $2 < \alpha$,

$$\Theta'_{\phi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)}(t) \geq \Theta'_{\phi(x,y,z)}(\alpha^p t) \quad (2.13)$$

and $\lim_{n\to\infty}\Theta'_{2^n\phi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)}(t) = 1$, for all $x, y, z \in X$ and all $t > 0$. If $f : X \to Y$be an odd mapping with $f(0) = 0$, which satisfies (2.3), then there exists a unique additive mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},x,\frac{x}{2}\right)}\left((2^p - \alpha^p)t\right), \quad (2.14)$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x = z = \frac{x}{2}$ and $y = x$ in (2.3), we obtain

$$\Theta_{f(x)-2f\left(\frac{x}{2}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},x,\frac{x}{2}\right)}(\alpha^p t), \quad (2.15)$$

for all $x, y \in X$ and all $t > 0$.

Replacing x by $\frac{x}{2^n}$ in (2.15), we obtain

$$\Theta_{2^n f\left(\frac{x}{2^n}\right)-2^{n+1} f\left(\frac{x}{2^{n+1}}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},x,\frac{x}{2}\right)}\left(\left(\frac{\alpha}{2}\right)^{np}(2^p)t\right), \quad (2.16)$$

for all $x \in X$ and all $t > 0$.

Since

$$f(x) - 2^n f\left(\frac{x}{2^n}\right) = \sum_{j=0}^{n-1}\left(2^j f\left(\frac{x}{2^j}\right) - 2^{(j+1)} f\left(\frac{x}{2^{j+1}}\right)\right), \quad (2.17)$$

for all $x \in X$ and all $t > 0$.

From inequality (2.16) and (2.17), we get

$$\Theta_{f(x)-2^n f\left(\frac{x}{2^n}\right)}\left(\sum_{j=0}^{n-1}\left(\frac{\left(\frac{2}{\alpha}\right)^{jp}}{(2)^p}t\right)\right) \geq \Theta'_{\phi\left(\frac{x}{2},x,\frac{x}{2}\right)}(t), \quad (2.18)$$

for all $x \in X$ and all $t > 0$.

Replacing x by $\frac{x}{2^m}$ in (2.18), we get

$$\Theta_{2^m f\left(\frac{x}{2^m}\right)-2^{(m+n)} f\left(\frac{x}{2^{m+n}}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},x,\frac{x}{2}\right)}\left(\frac{(2)^p}{\sum_{j=m}^{n+m-1}\left(\frac{2}{\alpha}\right)^{jp}}t\right), \quad (2.19)$$

for all $x \in X$ and all $m, n \in Z$ with $n > m \geq 0$. Then, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy in (X, Θ, T_M), and so it converges to some point $Q(x) \in Y$. Now, we can define a mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$, for all $x \in X$ and all $t > 0$. The remaining part goes through in a similar method to the corresponding Theorem 2.1.

Corollary 2.3. Let X be a linear space, (Z, Θ', T_M) be a random p-normed space and (Y, Θ, T_M) be a complete random p-normed space. Assume δ is a positive real number and $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies

$$\Theta_{Df(x,y,z)}(t) \geq \Theta'_{\delta z_0}(t), \quad (2.20)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\delta z_0}\left(\left(2^p - 1\right)t\right), \quad (2.21)$$

for all $x, y, z \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^3 \to Z$ be defined by $\phi(x, y, z) = \delta z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. This completes the proof.

Corollary 2.4. Let X be a linear space, (Z, Θ', T_M) be a random p-normed space and (Y, Θ, T_M) be a complete random p-normed space. Assume r is a positive real number with $r \neq 3$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies

$$\phi_{Df(x,y,z)}(t) \geq \phi'_{(\|x\|^r+\|y\|^r+\|y\|^r)z_0}(t), \quad (2.22)$$

for all $x, y, z \in X$ and all $t > 0$, then there exists a unique mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{2^r\|x\|^r z_0}(2^p - 2^{pr})(t), \quad (2.23)$$

for all $x \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^3 \to Z$ be defined by $\phi(x, y) = (\|x\|^r + \|y\|^r + \|z\|^r)z_0$. Then, the proof follows from Theorem 2.1and Theorem 2.2 by $\alpha = 2^r$. This completes the proof.

3. Stability Results for Mixed Type Functional Equations

In this section, we prove the generalized Hyers-Ulam stability of mixed type Additive Quartic functional equations in random p-normed space, by using direct method. For any mapping $f : X \to Y$ defined by

$$\begin{aligned} D_{f(x,y)} &= f(2x+y) + f(2x-y) + f(x+2y) + f(x-2y) \\ &-8[f(x+y) + f(x-y)] - f(2x) + 5f(x) \\ &-7f(-x) - 2f(2y) + 5f(-y) + 9f(y) \end{aligned} \quad (3.1)$$

Theorem 3.1. Let $\phi : X^2 \to Z$ be a mapping such that for some $0 < \alpha < 2$.

$$\Theta'_{\phi(2x,2y)}(t) \geq \Theta'_{\alpha\phi(x,y)}(t), \quad (3.2)$$

and $\lim_{n\to\infty} \Theta'_{\phi(2^n x, 2^n y)}(2^{2np} t) = 1$ for all $x, y \in X$ and all $t > 0$. If an even mapping $f : X \to Y$ with $f(0) = 0$ satisfying

$$\Theta_{Df(x,y)}(t) \geq \Theta'_{\phi(x,y)}(t), \quad (3.3)$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi(x,0)}(2^{3p}(2^p - \alpha^p)t), \quad (3.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing y by 0, in (3.3), we obtain

$$\Theta_{f(x)-\frac{f(2x)}{2^4}}(t) \geq \Theta'_{\phi(x,0)}(2^{4p} t), \quad (3.5)$$

for all $x \in X$ and all $t > 0$.

Replacing x by $2^n x$ in (3.5) we obtain

$$\begin{aligned} &\Theta_{\frac{f(2^n x)}{2^{4n}} - \frac{f(2^{n+1}x)}{2^{4(n+1)}}}(t) \\ &\geq \Theta'_{\phi(2^n x, 2^n x)}\left(2^{4p}\left(2^{4np}\right)t\right) \qquad (3.6) \\ &\geq \Theta'_{\phi(x,0)}\left(\left(\frac{2}{\alpha}\right)^{np}(2^{4p})t\right), \end{aligned}$$

for all $x \in X$ and all $t > 0$.

Since

$$f(x) - \frac{f(2^n x)}{2^{4n}} = \sum_{j=0}^{n-1}\left(\frac{f(2^j x)}{2^{4j}} - \frac{f(2^{j+1}x)}{2^{4(j+1)}}\right),$$

So,

$$\begin{aligned} &\Theta_{f(x) - \frac{f(2^n x)}{2^{4n}}}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2^4)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right) \\ &= \Theta_{\sum_{j=0}^{n-1}\left(\frac{f(2^j x)}{2^{4j}} - \frac{f(2^{j+1}x)}{2^{4(j+1)}}\right)}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2^4)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right) \\ &\geq T_{M\,j=0}^{\;n-1}\left(\Theta_{\left(\frac{f(2^j x)}{2^{4j}} - \frac{f(2^{j+1}x)}{2^{4(j+1)}}\right)}\left(\frac{1}{(2^4)^p}\left(\frac{\alpha}{2}\right)^{jp} t\right)\right) \qquad (3.7) \\ &= T_M\left(\Theta'_{\phi(x,0)}, \Theta'_{\phi(x,0)}, \ldots, \Theta'_{\phi(x,0)}(t)\right) \\ &= \Theta'_{\phi(x,0)}(t), \end{aligned}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $2^m x$ in (3.7), we get

$$\Theta_{\frac{f(2^m x)}{2^{4m}} - \frac{f(2^{m+1}x)}{2^{4(m+1)}}}(t) \geq \Theta'_{\phi(x,0)}\left(\frac{(2^4)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right) \qquad (3.8)$$

for all $x \in X$ and all $m, n \in Z$ with $n > m \geq 0$. It follows from

$$\lim_{n,m\to\infty} \Theta'_{\phi(x,0)}\left(\frac{(2^4)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right) = 1$$

that the sequence $\left\{\frac{f(2^n x)}{2^{4n}}\right\}$ is a Cauchy in (X, Θ, T_M) and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}\left\{\frac{f(2^n x)}{2^{4n}}\right\}$, for all $x \in X$ and all $t > 0$. Fix $x \in X$ and put $m = 0$ in (3.8). Then, we obtain

$$\Theta_{f(x) - \frac{f(2^n x)}{2^{4n}}}(t) \geq \Theta'_{\phi(x,0)}\left(\frac{(2^4)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right),$$

for all $x \in X$ and all $t > 0$. For every $s > 0$,

$$\begin{aligned} &\Theta_{f(x)-Q(x)}(s+t) \\ &\geq T_M\left(\Theta_{f(x) - \frac{f(2^n x)}{2^{4n}}}, \Theta_{\frac{f(2^n x)}{2^{4n}} - Q(x)}(s)\right) \qquad (3.9) \\ &\geq T_M\left(\Theta'_{\phi(x,0)}\left(\frac{(2^4)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}} t\right), \Theta_{\frac{f(2^n x)}{2^{4n}} - Q(x)}(s)\right), \end{aligned}$$

for all $x \in X$ and all $t > 0$. Taking the limit $n \to \infty$ in (3.9), we obtain

$$\Theta_{f(x)-Q(x)}(s+t) \geq \Theta'_{\phi(x,0)}\left(2^{3p}\left(2^p - \alpha^p\right)t\right) \qquad (3.10)$$

Thus, since s is arbitrary and taking limit $s \to 0$ in (3.10), we have

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi(x,0)}(2^{3p}(2^p - \alpha^p)t),$$

for all $x \in X$ and all $t > 0$. Thus, the condition (3.4), holds for all $x \in X$ and all $t > 0$. If we replace (x, y) by $(2^n x, 2^n y)$ in (3.3), then

$$\Theta_{\frac{Df(2^n x, 2^n y)}{2^{4n}}}(t) \geq \Theta'_{\Theta(2x,2y)}\left(2^{np} t\right), \qquad (3.11)$$

for all $x, y \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.11), we find that $\Theta_{DQ(x,y)} = 1$ for all $t > 0$, which implies $DQ(x, y) = 0$, for all $x, y \in X$. Therefore, the mapping Q is quartic. Now, we prove that the quartic mapping Q is unique. Let us assume that there exists another mapping $R : X \to Y$ which satisfies (3.3). For fixed $x \in X$, $Q(2^n x) = 2^{4n} Q(x)$ and $R(2^n x) = 2^{4n} R(x)$, all $n \in Z^+$. It follows from (3.3) that

$$\begin{aligned} &\Theta_{Q(x)-R(x)}(t) = \Theta_{\frac{Q(2^n x)}{2^{4n}} - \frac{R(2^n x)}{2^{4n}}}(t) \\ &\geq T_M\left(\Theta_{\frac{Q(2^n x)}{2^{4n}} - \frac{f(2^n x)}{2^{4n}}}\left(\frac{t}{2}\right), \Theta_{\frac{f(2^n x)}{2^{4n}} - \frac{R(2^n x)}{2^{4n}}}\left(\frac{t}{2}\right)\right) \\ &\geq \Theta'_{\phi(x,0)}\left(2^{3p}\left(2^p - \alpha^p\right)\left(\frac{2}{\alpha}\right)^{np} t\right). \end{aligned}$$

as, $\lim_{n\to\infty} 2^{3p}(2^p - \alpha^p)\left(\frac{2}{\alpha}\right)^{np}(t) = \infty$, we have $\Theta_{Q(x)-R(x)}(t) = 1$ for all $t > 0$. Thus, $Q(x) = R(x)$, for all $x \in X$. Therefore, the proof is completed.

Theorem 3.2. Let $\phi : X^2 \to Z$ be a mapping such that for some $2 < \alpha$,

$$\Theta'_{\phi\left(\frac{x}{2},\frac{y}{2}\right)}(t) \geq \Theta'_{\phi(x,0)}(\alpha^{4p} t), \tag{3.12}$$

and $\lim_{n\to\infty} \Theta'_{2^{4n}\phi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}(t) = 1$ for all $x, y \in X$ and all $t > 0$. If an even mapping $f : X \to Y$ with $f(0) = 0$ which satisfies (3.3), then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi(x,0)}(2^{3p}(2^p - \alpha^p)t), \tag{3.13}$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing x by $\frac{x}{2}$ and y by 0 in (3.3), we obtain

$$\Theta_{f(x)-2^4 f\left(\frac{x}{2}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},0\right)}(2\alpha^p t), \tag{3.14}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $\frac{x}{2^n}$ in (3.14), we obtain

$$\Theta_{2^{4n} f\left(\frac{x}{2^n}\right)-2^{4(n+1)} f\left(\frac{x}{2^n}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},0\right)}\left(\left(\frac{\alpha}{2}\right)^{np}(2^{4p})t\right), \tag{3.15}$$

for all $x \in X$ and all $t > 0$.

Since

$$f(x) - 2^{4n} f\left(\frac{x}{2^n}\right) = \sum_{j=0}^{n-1}\left(2^{4j} f\left(\frac{x}{2^j}\right) - 2^{4(j+1)} f\left(\frac{x}{2^{j+1}}\right)\right), \tag{3.16}$$

for all $x \in X$ and all $t > 0$.

From inequality (3.15) and (3.16), we get

$$\Theta_{f(x)-2^{4n} f\left(\frac{x}{2^n}\right)}\left(\sum_{j=0}^{n-1}\frac{\left(\frac{2}{\alpha}\right)^{jp}}{\left(2^4\right)^p}t\right) \geq \Theta'_{\phi\left(\frac{x}{2},0\right)}(t), \tag{3.17}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $\frac{x}{2^m}$ in (3.17), we get

$$\Theta_{2^{4m} f\left(\frac{x}{2^m}\right)-2^{4(m+n)} f\left(\frac{x}{2^{m+n}}\right)}(t) \geq \Theta'_{\phi\left(\frac{x}{2},0\right)}\left(\frac{\left(2^4\right)^p}{\sum_{j=m}^{n+m-1}\left(\frac{2}{\alpha}\right)^{jp}}t\right), \tag{3.18}$$

for all $x \in X$ and all $m, n \in Z$ with $n > m \geq 0$. Then, the sequence $\left\{2^{4n} f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy in (X, Θ, T_M), and so it converges to some point $Q(x) \in Y$. Now, we can define a quartic mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}\left\{2^{4n} f\left(\frac{x}{2}\right)\right\}$, for all $x \in X$ and all $t > 0$. The remaining part goes through in a similar method to the corresponding Theorem 3.1.

Corollary 3.3. Let X be a linear space, (Z, Θ', T_M) be a random p-normed space and (Y, Θ, T_M) be complete a random p-normed space. Assume δ is a positive real number and $z_0 \in Z$. If an even mapping $f : X \to Y$ with $f(0) = 0$ which satisfies

$$\Theta_{Df(x,y)}(t) \geq \Theta'_{\delta z_0}(t), \tag{3.19}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\delta z_0}(2^{3p}(2^p - 1)t), \tag{3.20}$$

for all $x \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^2 \to Z$ be defined by $\phi(x, y) = \delta z_0$. Then, the proof follows from Theorem 3.1 by $\alpha = 1$. This completes the proof.

Corollary 3.4. Let X be a linear space, (Z, Θ', T_M) be a random p-normed space and (Y, Θ, T_M) be a complete random p-normed space. Assume r is a positive real number with $r \neq 2$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping $f(0) = 0$ which satisfies

$$\Theta_{Df(x,y)}(t) \geq \Theta'_{\left(\|x\|^r+\|y\|^r\right)z_0}(t), \tag{3.21}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{2^r x^r z_0}\left(2^{3p}\left(2^p - 2^{pr}\right)t\right), \tag{3.22}$$

for all $x \in X$ and all $t > 0$.

Proof. Let a mapping $\phi : X^2 \to Z$ be defined by $\phi(x, y) = (\|x\|^r + \|y\|^r)z_0$. Then, the proof follows from Theorem 3.1 and Theorem 3.2 by $\alpha = 2^r$. This completes the proof.

Theorem 3.5. Let $\phi : X^2 \to Z$ be a mapping such that for some $0 < \alpha < 2$.

$$\Theta'_{\phi(2x,2y)}(t) \geq \Theta'_{\alpha(x,y)}(t) \tag{3.23}$$

and $\lim_{n\to\infty} \Theta'_{\phi(2^n x, 2^n y)}(2^{np} t) = 1$ for all $x, y \in X$ and all $t > 0$. If $f : X \to Y$ is an odd mapping with $f(0) = 0$ such that

$$\Theta_{Df(x,y)}(t) \geq \Theta'_{\phi(x,y)}(t), \tag{3.24}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique additive mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \Theta'_{\phi(x,0)}((2^p - \alpha^p)t), \tag{3.25}$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing y by 0 in, (3.24), we obtain

$$\Theta_{f(x)-\frac{f(2x)}{2}}(t) \geq \Theta'_{\phi(x,0)}(2^p t), \tag{3.26}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $2^n x$ in (3.26), we obtain

$$\Theta_{\frac{f(2^n x)}{2^n}-\frac{f(2^{n+1}x)}{2^{n+1}}}(t) \ge \Theta'_{\phi(2^n x,0)}\left(2^p\left(2^{np}\right)t\right) \ge \Theta'_{\phi((x,0))}\left(\left(\frac{2}{\alpha}\right)^{np}(2^p)t\right), \tag{3.27}$$

for all $x \in X$ and all $t > 0$.

Since

$$f(x)-\frac{f(2^n x)}{2^n} = \sum_{j=0}^{n-1}\left(\frac{f(2^j x)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right), \tag{3.28}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned}
&\Theta_{f(x)-\frac{f(2^n x)}{2^n}}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp}t\right)\right)\\
&=\Theta_{\sum_{j=0}^{n-1}\left(\frac{f(2^j x)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right)}\left(\sum_{j=0}^{n-1}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp}t\right)\right)\\
&\ge T_{M\,j=0}^{\;n-1}\left(\Theta_{\left(\frac{f(2^j x)}{2^j}-\frac{f(2^{j+1}x)}{2^{j+1}}\right)}\left(\frac{1}{(2)^p}\left(\frac{\alpha}{2}\right)^{jp}t\right)\right)\\
&=T_M\left(\Theta'_{\phi(x,0)},\Theta'_{\phi(x,0)},\ldots,\Theta'_{\phi(x,0)}(t)\right)\\
&=\Theta'_{\phi(x,0)}(t),
\end{aligned}$$

for all $x \in X$ and all $t > 0$.

Replacing x by $2^m x$ in (3.29), we get

$$\Theta_{\frac{f(2^m x)}{2^m}-\frac{f(2^{m+n}x)}{2^{m+n}}}(t) \ge \Theta'_{\phi(x,0)}\left(\frac{(2)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}}t\right), \tag{3.30}$$

for all $x \in X$ and all $m, n \in Z$ with $n > m \ge 0$. It follows from

$$\lim_{n,m\to\infty}\Theta'_{\phi(x,0)}\left(\frac{(2)^p}{\sum_{j=m}^{m+n-1}\left(\frac{\alpha}{2}\right)^{jp}}t\right)=1,$$

that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is Cauchy in (X, Θ, T_M), and so it converges to some point $Q(x) \in Y$. We can define a mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty}\left\{\frac{f(2^n x)}{2^n}\right\}$, for all $x \in X$ and all $t > 0$. Fix $x \in X$ and put $m = 0$ in (3.30), Then, we obtain

$$\Theta_{f(x)-\frac{f(2^n x)}{2^n}}(t) \ge \Theta'_{\phi((x,0))}\left(\frac{(2)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}}t\right),$$

for all $x \in X$ and all $t > 0$. For every $s > 0$,

$$\begin{aligned}
&\Theta_{f(x)-Q(x)}(s+t)\\
&\ge T_M\left(\Theta_{f(x)-\frac{f(2^n x)}{2^n}},\Theta_{\frac{f(2^n x)}{2^n}-Q(x)}(s)\right)\\
&\ge T_M\left(\Theta'_{\phi((x,0))}\left(\frac{(2)^p}{\sum_{j=0}^{n-1}\left(\frac{\alpha}{2}\right)^{jp}}t\right),\Theta_{\frac{f(2^n x)}{2^n}-Q(x)}(s)\right),
\end{aligned} \tag{3.31}$$

for all $x \in X$ and all $t > 0$. Taking the limit $n \to \infty$ in (3.31), we obtain

$$\Theta_{f(x)-Q(x)}(s+t) \ge \Theta'_{\phi((x,0))}\left(2^p-\alpha^p\right)(t), \tag{3.32}$$

Thus, since s is arbitrary and taking $s \to 0$ in (3.32), we have

$$\Theta_{f(x)-Q(x)}(t) \ge \Theta'_{\phi((x,0))}\left((2^p-\alpha^p)t\right),$$

for all $x \in X$ and all $t > 0$. Thus, the condition (3.25), holds for all $x \in X$ and all $t > 0$. If we replace (x, y, z) by $(2^n x, 2^n y)$ in (3.24), then

$$\Theta_{\frac{Df(2^n x, 2^n y)}{2^n}}(t) \ge \Theta'_{\phi(2x,2y)}\left(2^{np}t\right), \tag{3.33}$$

for all $x, y \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.33), we find that $\Theta_{DQ(x,y)} = 1$, for all $t > 0$, which implies $DQ(x, y) = 0$, for all $x, y \in X$. Therefore, the mapping Q is additive. Now, we prove that the additive mapping Q is unique. Let us assume that there exists another mapping $R : X \to Y$ which satisfies (3.25). For fixed $x \in X$, $Q(2^n x) = 2^n Q(x)$ and $R(2^n x) = 2^n R(x)$, all $n \in Z^+$. It follows from (3.25) that

$$\begin{aligned}
&\Theta_{Q(x)-R(x)}(t)=\Theta_{\frac{Q(2^n x)}{2^n}-\frac{R(2^n x)}{2^n}}(t)\\
&\ge T_M\left(\Theta_{\frac{Q(2^n x)}{2^n}-\frac{f(2^n x)}{2^n}}\left(\frac{t}{2}\right),\Theta_{\frac{f(2^n x)}{2^n}-\frac{R(2^n x)}{2^n}}\left(\frac{t}{2}\right)\right)\\
&\ge \Theta'_{\phi(x,0)}\left(\left(2^p-\alpha^p\right)\left(\frac{2}{\alpha}\right)^{np}t\right).
\end{aligned}$$

as $\lim_{n\to\infty}(2^p-\alpha^p)\left(\frac{2}{\alpha}\right)^{np}(t)=\infty$, we have $\Theta_{Q(x)-R(x)}(t)=1$ for all $t>0$. Thus, $Q(x)=R(x)$, for all $x\in X$. Hence, the proof is complete.

Theorem 3.6. Let $\phi: X^2\to Z$ be a mapping such that for some $2<\alpha$,

$$\Theta'_{\phi\left(\frac{x}{2},\frac{y}{2}\right)}(t)\ge \Theta'_{\phi(x,y)}(\alpha^p t) \tag{3.34}$$

and $\lim_{n\to\infty}\Theta'_{2^n\phi\left(\frac{x}{2^n},\frac{y}{2^n}\right)}(t)=1$, for all $x,y\in X$ and all $t>0$. If $f: X\to Y$ is an odd mapping with $f(0)=0$, which satisfies (3.24), then there exists a unique additive mapping $Q: X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge \Theta'_{\phi\left(\frac{x}{2},0\right)}\left((2^p-\alpha^p)t\right), \tag{3.35}$$

for all $x,y\in X$ and all $t>0$.

Proof. Replacing $x$by $\frac{x}{2}$ and y by 0 in (3.24), we obtain

$$\Theta_{f(x)-2f\left(\frac{x}{2}\right)}(t)\ge \Theta'_{\phi\left(\frac{x}{2},0\right)}(\alpha^p t), \tag{3.36}$$

for all $x,y\in X$ and all $t>0$.

Replacing x by $\frac{x}{2^n}$ in (3.36), we obtain

$$\Theta_{2^n f\left(\frac{x}{2^n}\right)-2^{n+1}f\left(\frac{x}{2^n}\right)}(t) \ge \Theta'_{\phi\left(\frac{x}{2},0\right)}\left(\left(\frac{\alpha}{2}\right)^{np}(2^p)t\right), \tag{3.37}$$

for all $x\in X$ and all $t>0$.

Since

$$f(x)-2^n f\left(\frac{x}{2^n}\right) = \sum_{j=0}^{n-1}\left(2^j f\left(\frac{x}{2^j}\right)-2^{(j+1)}f\left(\frac{x}{2^{j+1}}\right)\right), \tag{3.38}$$

for all $x\in X$ and all $t>0$.

From inequality (3.37) and (3.38), we get

$$\Theta_{f(x)-2^n f\left(\frac{x}{2^n}\right)}\left(\sum_{j=0}^{n-1}\frac{\left(\frac{2}{\alpha}\right)^{jp}}{(2)^p}t\right)\ge \Theta'_{\phi\left(\frac{x}{2},0\right)}(t), \tag{3.39}$$

for all $x\in X$ and all $t>0$.

Replacing x by $\frac{x}{2^m}$ in (3.39), we get

$$\Theta_{2^m f\left(\frac{x}{2^m}\right)-2^{(m+n)}f\left(\frac{x}{2^{m+n}}\right)}(t) \ge \Theta'_{\phi\left(\frac{x}{2},0\right)}\left(\frac{(2)^p}{\sum_{j=m}^{n+m-1}\left(\frac{2}{\alpha}\right)^{jp}}t\right), \tag{3.40}$$

for all $x\in X$ and all $m,n\in Z$ with $n>m\ge 0$. Then, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy in (X,Θ,T_M), and so it converges to some point $Q(x)\in Y$. Now, we can define a mapping $Q: X\to Y$ by $Q(x)=\lim_{n\to\infty}\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$, for all $x\in X$ and all $t>0$. The remaining part goes through in a similar method to the corresponding Theorem 3.5.

Corollary 3.7. Let X be a linear space, (Z,Θ',T_M) be a random p-normed space and (Y,Θ,T_M) be a completerandom p-normed space. Assume δ is a positive real number and z_0 in Z. If $f: X\to Y$ is a mapping with $f(0)=0$ which satisfies

$$\Theta_{Df(x,y)}(t)\ge \Theta'_{\delta z_0}(t), \tag{3.41}$$

for all $x,y\in X$ and all $t>0$, then there exists a unique mapping $Q: X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge \Theta'_{\delta z_0}\left((2^p-1)t\right), \tag{3.42}$$

for all $x,y\in X$ and all $t>0$.

Proof. Let a mapping $\phi: X^2\to Z$ be defined by $\phi(x,y)=\delta z_0$. Then, the proof follows from Theorem 3.5 by $\alpha=1$. This completes the proof.

Corollary 3.8. Let X be a linear space, (Z,Θ',T_M) be a random p-normed space and (Y,Θ,T_M) be complete a random p-normed space. Assume r is a positive real number with $r\neq 3$ and $z_0\in Z$. If $f: X\to Y$ is an odd mapping with $f(0)=0$ which satisfies

$$\phi_{Df(x,y)}(t)\ge \phi'_{(\|x\|^r+\|y\|^r)z_0}(t), \tag{3.43}$$

for all $x,y\in X$ and all $t>0$, then there exists a unique additive mapping $Q: X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge \Theta'_{2^r\|x\|^r z_0}\left((2^p-2^{pr})t\right), \tag{3.44}$$

for all $x\in X$ and all $t>0$.

Proof. Let a mapping $\phi: X^2\to Z$ be defined by $\phi(x,y)=(\|x\|^r+\|y\|^r)z_0$. Then, the proof follows from Theorem 3.5 and Theorem 3.6 by $\alpha=2^r$. This completes the proof.

4. Stability Results by Fixed Point Method

In this section, we give the generalized Hyers-Ulam stability of mixed type additive quartic functional equations in random p-normed spaces. Let us recall that a mapping $d: X^2\to[0,\infty)$ is a called a metric on a non-empty set X if

i) $d(x,y)=0$ if and only if$x=y$,
ii) $d(x,y)=d(y,x)$,
iii) $d(x,y)\le d(x,y)+d(y,z)$,

for all $x,y,z\in X$. Before proceeding to the main results in this section, we give the fixed-point theorem which plays an important role in proving our theorems.

Theorem 4.1. [18]. (Alternative fixed-point theorem) Let (E,d) be a generalized complete metric space and $\Gamma: E\to E$ be a strictly contractive function with Lipschitz constant $L<1$. Then, for each $x\in E$, either $d(\Gamma^{n+1}x,\Gamma^n x)=\infty$ for all non-negative integer $n\ge 0$ or there exists a natural number n_0 such that

i) $d(\Gamma^{n+1}x, \Gamma^n x) < \infty$, for all $n \geq n_0$;

ii) the sequence $\{\Gamma^n x\}_{n=1}^{\infty}$ converges to a fixed-point $y \in E$ of Γ;

iii) y is the unique fixed point of Γ in the set $\mathcal{F} = \{q \in E: d(\Gamma^{n_0}a, q) < \infty\}$;

iv) $d(q, y) \leq \frac{1}{1-L} d(q, \Gamma q), q \in \mathcal{F}$.

Theorem 4.2. Let $\phi : X^3 \to D^+$ ($\phi(x,y,z)$ is denoted by $\phi_{x,y,z}$) be a mapping such that, for some $0 < \alpha < 2$.

$$\Theta'_{\phi(2x,2y,2z)}(t) \geq \Theta'_{(x,y,z)}(t), \tag{4.1}$$

for all $x, y, z \in X$ and all $t > 0$. If $f : X \to Y$ is an odd mapping with $f(0) = 0$ such that

$$\Theta_{Df(x,y,z)}(t) \geq \phi_{x,y,z}(t), \tag{4.2}$$

for all $x, y, z \in X$ and all $t > 0$, then there exists a unique mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \phi_{x,2x,x}(2-\alpha^p)(t), \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Replacing y by x in (4.2), we obtain

$$\Theta_{f(x)-\frac{f(2x)}{2}}(t) \geq \phi_{x,2x,x}(2^p t), \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

Consider a general metric d on Λ, here Λ be a set of all mappings from X into Y and introduce a generalized metric on Λ as follows:

$$d(g,h) = \inf\left\{c \in (0,\infty) \text{ s. t. } \Theta_{g(x)-h(x)}(ct) \geq \phi_{x,2x,x}(t), x \in X, t > 0\right\}$$

whereas $\inf \phi = +\infty$ It is easy to show that (Λ, d) is a complete metric space [10]. Now, let us consider a mapping $J : \Lambda \to \Lambda$ defined by

$$J_{g(x)} = \frac{1}{2} g(2x),$$

for all $g \in \Lambda$ and for all $x \in$ X. Let g, h in Λ and $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) < c$. Then, we have

$$\Theta_{g(x)-h(x)}(ct) \geq \phi_{x,2x,x}(t),$$

for all $x \in X$ and all $t > 0$, hence

$$\begin{aligned} &\Theta_{Jg(x)-Jh(x)}\left(\left(\frac{\alpha}{2}\right)^p ct\right) \\ &\geq \Theta_{g(2x)-h(2x)}\left(\alpha^p ct\right) \\ &\geq \phi_{2x,4x,2x}\left(\alpha^p t\right), \phi_{x,x,x}(t), \end{aligned} \tag{4.5}$$

for all $x \in X$ and all $t > 0$, and so, if $d(g, h) < c$, then

$$d(Jg, Jh) < \frac{\alpha^p}{2^p} d(g,h),$$

for all $g, h \in \Lambda$. Then J is a strictly contractive self-mapping on Λ with Lipschitz constant $L = \frac{\alpha^p}{2^p} < 1$.

Also, it follows from (4.2) that

$$\begin{aligned} &\Theta_{f(x)-\frac{f(2x)}{2}}\left(\frac{t}{2^p}\right) \\ &\geq \Theta_{f(x)-Jf(x)}\left(\frac{t}{2^p}\right) \geq \phi_{x,2x,x}(t), \end{aligned} \tag{4.6}$$

for all $x \in X$ and all $t > 0$, which implies that

$$d(Jg, Jh) < \frac{\alpha^p}{2^p}.$$

Using Theorem 4.1, there exists a mapping $Q : X \to Y$, which is a unique fixed point of J in the set $\Lambda_1 = \{g \in \Lambda, : d(g, h) < \infty\}$ such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

for all $x \in X$, since $\lim_{n\to\infty} d(J^n f, Q) = 0$. Again, it follows from Theorem 4.1 that

$$d(f,Q) \leq \frac{1}{1-L} d(f, JQ) \leq \frac{1}{(1-L)} = \frac{1}{\left(2-\alpha^p\right)},$$

which implies

$$\Theta_{f(x)-Q(x)}(t) \geq \phi_{x,2x,x}\big((2-\alpha^p)t\big),$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $2^n x$ and $2^n y$ in (4.2), respectively,

$$\begin{aligned} \Theta_{Df(x,y,z)}(t) &= \lim_{n\to\infty} \Theta_{Df\left(2^n x, 2^n y, 2^n z\right)}\left(2^{np} t\right) \\ &\geq \lim_{n\to\infty} \phi_{2^n x, 2^n y, 2^n z}\left(2^{2np} t\right), \end{aligned}$$

for all $x \in X$ and all $t > 0$. It follows from $\lim_{n\to\infty}\left(\left(\frac{2}{\alpha}\right)^{np} t\right) = 1$ that $DQ(x, y) = 0$. Hence, the mapping Q is additive. Now, we show that mapping Q is unique. To prove this, we assume that there exists an additive mapping $R : X \to Y$, which satisfies (4.3). Then, R is a fixed point of J in Λ_1. However, it follows from Theorem 4.1 that J has only one fixed point in Λ_1. Hence, we deduce that $Q = R$.

Theorem 4.3. Let $\phi : X^3 \to D^+$ be a mapping such that, for some $2 < \alpha$.

$$\phi_{x,2x,x}(t) \geq \phi_{2x,4x,2x}(\alpha^p t), \tag{4.7}$$

for all $x \in X$ and all $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (4.2), then there exists a unique mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \phi_{x,2x,x}\big((2-\alpha^p)t\big), \tag{4.8}$$

for all $x, y \in X$ and all $t > 0$.

Proof. Let Λ and d be as in the proof of Theorem 4.2. Then (Λ, d) becomes a complete metric space and the mapping $J : \Lambda \to \Lambda$ defined by

$$J_{g(x)} = 2g\left(\frac{x}{2}\right),$$

for all $x \in X$ and $g \in \Lambda$. Then,

$$d(Jg,Jh)<\frac{2^p}{\alpha^p}d(g,h),$$

for all $g,h\in\Lambda$. Then, J is a strictly contractive self-mapping on Λ with Lipschitz constant $L=\frac{2^p}{\alpha^p}<1$. It follows from (4.3) that $d(f,Jf)<\frac{1}{(\alpha)^p}$, we get

$$d(f,Q)\le\frac{1}{1-L}d(f,Jf)\le\frac{1}{\left(\alpha^p-2^p\right)},$$

which implies the inequality (4.2) holds for all $x\in X$ and all $t>0$. The remaining assertion goes through in a similar method to the corresponding part of Theorem 4.3. This completes the proof.

Corollary 4.4. Let X be a real p-Banach spaces, and define $\Theta_x(t)=\frac{t}{t+\|x\|}$, for all $x\in X$ and all $t>0$. Then, (X,Θ,T_M) is a complete random p-normed space. Define

$$\phi_{x,y,z}(t)=\frac{t}{t+\left(\|x\|^r+\|y\|^r+\|z\|^r\right)}$$

for all $x,y,z\in X$ and all $t>0$ in which $0<r<\alpha$. Assume that $f:X\to Y$ is a mapping with $f(0)=0$ which satisfies (4.2), then there exists a unique mapping $Q: X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge\frac{2^p\left(2^p-2^{pr}\right)}{2^p\left(2^p-2^{pr}\right)+\|x\|^r},\tag{4.9}$$

for all $x\in X$ and all $t>0$, where $\alpha=2^r$. Hence, we have

$$\|f(x)-Q(x)\|\le\frac{\|x\|^r}{2^p\left(2^p-2^{pr}\right)},\tag{4.10}$$

for all $x\in X$.

Theorem 4.5. Let $\phi: X^2\to D^+(\phi(x,y)$ is denoted by $\phi_{x,y}$) be a mappingsuch that, for some$0<\alpha<2$.

$$\Theta'_{\phi(x,y)}\left(\alpha^p t\right)\ge\Theta'_{\phi(x,y)}(t),\tag{4.11}$$

for all $x,y\in X$ and all $t>0$. If an even mapping $f: X\to Y$ with $f(0)=0$ such that

$$\Theta_{Df(x,y)}(t)\ge\phi_{x,y}(t),\tag{4.12}$$

for all $x,y\in X$ and all $t>0$, then there exists a unique quartic mapping $Q: X\to Y$ such that

$$\Theta_{f(x)-Q(x)}(t)\ge\phi_{x,0}(2^{3p}(2^p-\alpha^p)t),\tag{4.13}$$

for all $x\in X$ and all $t>0$.

Proof. Replacing y by 0 in (4.12), we obtain

$$\Theta_{f(x)-\frac{f(2x)}{2^4}}(t)\ge\phi_{x,0}\left(2^{4p}t\right),\tag{4.14}$$

for all $x\in X$ and all $t>0$.

Consider a general metric d on Λ, here Λ be a set of all mappings from X into Y and introduce a generalized metric on Λ as follows:

$$d(g,h)$$
$$=\inf\left\{c\in(0,\infty)\text{s. t. }\Theta_{g(x)-h(x)}(ct)\ge\phi_{x,0}(t),x\in X,t>0\right\},$$

whereas $\inf\phi=+\infty$ It is easy to show that (Λ,d) is a complete metric space [10]. Now, let us consider a mapping $J:\Lambda\to\Lambda$ defined by

$$J_{g(x)}=\frac{1}{2^4}g(2x),$$

for all $g\in\Lambda$ and for all $x\in X$. Let g,h in Λ and $c\in(0,\infty)$ be an arbitrary constant with $d(g,h)<c$. Then, we have

$$\Theta_{g(x)-h(x)}(ct)\ge\phi_{x,0}(t),$$

for all $x\in X$ and all $t>0$, hence

$$\begin{aligned}&\Theta_{Jg(x)-Jh(x)}\left(\left(\frac{\alpha}{2}\right)^p ct\right)\\&\ge\Theta_{g(2x)-h(2x)}\left(\alpha^p ct\right)\\&\ge\phi_{2x,0}\left(\alpha^p t\right),\phi_{x,0}(t),\end{aligned}\tag{4.15}$$

for all $x\in X$ and all $t>0$, and so, if $d(g,h)<c$, then

$$d(Jg,Jh)<\frac{\alpha^p}{2^p}d(g,h),$$

for all $g,h\in\Lambda$. Then J is a strictly contractive self-mapping on Λ with Lipschitz constant $L=\frac{\alpha^p}{2^p}<1$. Also, it follows from (4.12) that

$$\Theta_{f(x)-\frac{f(2x)}{2^4}}\left(\frac{t}{2^p}\right)\ge\Theta_{f(x)-Jf(x)}\left(\frac{t}{2^p}\right)\ge\phi_{x,0}(t),\tag{4.16}$$

for all $x\in X$ and all $t>0$, which implies that

$$d(Jg,Jh)<\frac{\alpha^p}{2^p}.$$

Using Theorem 4.1, there exists a mapping $Q: X\to Y$, which is a unique fixed point of J in the set $\Lambda_1=\{g\in\Lambda,:d(g,h)<\infty\}$ such that

$$Q(x)=\lim_{n\to\infty}\frac{f\left(2^n x\right)}{2^{4n}},$$

for all $x\in X$, since $\lim_{n\to\infty}d(J^n f,Q)=0$. Again, it follows from Theorem 4.1 that

$$\begin{aligned}d(f,Q)&\le\frac{1}{1-L}d(f,Jf)\\&\le\frac{1}{\left(2^{4p}\right)(1-L)}=\frac{1}{2^{3p}\left(2^p-\alpha^p\right)},\end{aligned}$$

which implies

$$\Theta_{f(x)-Q(x)}(t)\ge\phi_{x,0}(2^{3p}(2^p-\alpha^p)t),$$

for all $x \in X$ and all $t > 0$. Replacing x and y by $2^{4n}x$ and $2^{4n}y$ in (4.12), respectively,

$$\Theta_{Df(x,y)}(t) = \lim_{n\to\infty} \Theta_{Df\left(2^{4n}x, 2^{4n}y\right)}\left(2^{4np}t\right) \geq \lim_{n\to\infty} \phi_{2^{4n}x, 2^{4n}y}\left(2^{4np}t\right),$$

for all $x, y \in X$ and all $t > 0$. It follows from $\lim_{n\to\infty}\left(\left(\frac{2}{\alpha}\right)^{np} t\right) = 1$ that $DQ(x,y) = 0$. Hence, the mapping Q is quartic. Now, we show that mapping Q is unique. To prove this, we assume that there exists a quartic mapping $R : X \to Y$, which satisfies (4.13). Then, R is a fixed point of J in Λ_1. However, it follows from Theorem 4.1 that J has only one fixed point in Λ_1. Hence, we deduce that $Q = R$.

Theorem 4.6. Let $\phi : X^2 \to D^+$ be a mapping such that, for some $2 < \alpha$.

$$\phi_{x,0}(t) \geq \phi_{2x,0}(\alpha^p t), \tag{4.17}$$

for all $x \in X$ and all $t > 0$. If $f : X \to Y$ is a mapping with $f(0) = 0$ which satisfies (4.12), then there exists a unique mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \phi_{x,0}\left(2^{3p}\left(\alpha^p - 2^p\right)t\right), \tag{4.18}$$

for all $x \in X$ and all $t > 0$.

Proof. Let Λ and d be as in the proof of Theorem 4.5. Then (Λ, d) becomes a complete metric space and the mapping $J : \Lambda \to \Lambda$ defined by

$$J_{g(x)} = 2^4 g\left(\frac{x}{2}\right),$$

for all $x \in X$ and $g \in \Lambda$. Then,

$$d(Jg, Jh) < \frac{2^p}{\alpha^p} d(g,h),$$

for all $g, h \in \Lambda$.

Then, J is a strictly contractive self-mapping on Λ with Lipschitz constant $L = \frac{2^p}{\alpha^p} < 1$.

It follows from (4.12) that $d(f, Jf) < \frac{1}{(2\alpha)^p}$, we get

$$d(f,Q) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{2^{3p}\left(\alpha^p - 2^p\right)},$$

which implies the inequality (4.12) holds for all $x, y \in X$ and all $t > 0$. The remaining assertion goes through in a similar method to the corresponding part of Theorem 4.5.

Corollary 4.7. Let X be a real p-Banach spaces, and define $\Theta_x(t) = \frac{t}{t + \|x\|}$, for all $x, y \in X$ and all $t > 0$. Then, (X, Θ, T_M) is a complete random p-normed space. Define

$$\phi_{x,y}(t) = \frac{t}{t + \left(\|x\|^r + \|y\|^r\right)},$$

for all $x, y \in X$ and all $t > 0$ in which $0 < r < \alpha$. Assume that $f : X \to Y$ is a mapping $f(0) = 0$ which satisfies (4.12), then there exists a unique mapping $Q : X \to Y$ such that

$$\Theta_{f(x)-Q(x)}(t) \geq \frac{2^{3p}\left(2^p - 2^{pr}\right)}{2^{3p}\left(2^p - 2^{pr}\right) + \|x\|^r}, \tag{4.19}$$

for all $x, y \in X$ and all $t > 0$, where $\alpha = 2^r$. Hence, we have

$$\left\| f(x) - Q(x) \right\| \leq \frac{\|x\|^r}{2^{3p}\left(2^p - 2^{pr}\right)}, \tag{4.20}$$

for all $x, y \in X$.

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