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Generalized *k*-Order Fibonacci and Lucas Quaternions

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Abstract In this study, we define a new interesting generalization of quaternions called as generalized *k*-order Fibonacci and Lucas quaternions. We give some important results with specific choices. Depending on the d_i and q choices, we obtain *k*-order Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas quaternions. For *k*=2, we obtain the recurrence relations of known special numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas quaternions. By multiplying the choices we made, we can obtain the quaternion definitions for other special numbers. We give generating functions for these quaternions. Also, we identify and prove the matrix representations for generalized *k*-order Fibonacci and Lucas quaternions. In this way, we obtain the matrix representations for usual Fibonacci, Lucas, Pell and the other special numbers known with the d_i and q values we chose and give some properties about matrix representations for generalized *k*-order Fibonacci and Lucas quaternions.

Keywords: *Fibonacci Numbers, Lucas Numbers, Generalized k-Order Fibonacci and Lucas Numbers, Quaternions, Generalized k-Order Fibonacci and Lucas Quaternions, Matrix Representations*

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1. Introduction

Quaternion is a number system that expands complex figures into one real and three imaginary confines in mathematics. Irish mathematician Sir William Rowan Hamilton defined quaternion algebra in 1843 in [1] and applied it to mathematics in 3D space. Quaternions do not have the property of dicker. Although vectors and matrices have replaced quaternions in numerous operations, they are still used in theoretical and applied mathematics. Its main use is the calculation of rotational motion in 3D space. The family of quaternion computation plays an important role in mathematics and countless fields similar to algebraic systems, dispose fields or non-commutative division algebras and matrices in commutative rings and geometry. These studies are seen in [2].

Quaternion algebra is defined by H (Hamilton). They are also defined as Clifford algebra classification $Cl_{0,2}(R) = Cl_{3,0}^{0}(R)$. H algebra has a significant place in analysis. As a consequence of the Frobenius theorem, it is one of the four finite-dimensional quotient algebras containing the field of real numbers as subrings (the others being real numbers, complex numbers and octanions).

The definition of the quaternion family is given below:

$$H = \{q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_i \in \mathbb{R}, i = 0,1,2,3\}$$

Quaternions are the four-dimensional vector space over R having basis $\{e_0, e_1, e_2, e_3\}$. The multiplication table for the basis of H is in Table 1 as the following:

Table 1. The multiplication table

$\times$	e_0	e_2	e_2	e_3
e_0	1	e_1	e_2	e_3
e_1	e_1	-1	e_3	$-e_2$
e_2	e_2	$-e_3$	-1	e_1
e_3	e_3	e_2	$-e_1$	-1

Let q be a quaternion. It shows as follows:

$$q = \sum_{i=0}^{3} q_i e_i \in H.$$

Quaternions consists two parts. The first part is called as a scalar part $S_q = q_0e_0$ and the other part is vectorial part of $\overrightarrow{V_q} = \sum_{i=1}^{3} q_i e_i$. Then we can write as $q = S_q + \overrightarrow{V_q}$.

The conjugate of quaternion $\overline{q}$ is defined as $\overline{q} = S_q - \overrightarrow{V_q} = q_0e_0 - \sum_{i=1}^{3} q_i e_i$.

Let q and p be two quaternions such that;

$$q=\sum_{i=0}^{3} q_i e_i \text{ and } p=\sum_{i=0}^{3} p_i e_i .$$

The quaternion arithmetic is defined for $q_i, p_i \in \mathbb{R}$ and $i=0,1,2,3$ by the following:

- $q=p \Leftrightarrow q_i=p_i$
- $q+p=\sum_{i=0}^{3}(q_i+p_i)e_i$
- $kq=\sum_{i=0}^{3} kq_i e_i$ for $k\in\mathbb{R}$
- $qp=S_qS_p+S_q\overrightarrow{V_p}+\overrightarrow{V_q}S_p-\overrightarrow{V_q}\overrightarrow{V_p}+\overrightarrow{V_q}\times\overrightarrow{V_p}$, where

$\overrightarrow{V_q}\overrightarrow{V_p}=\sum_{i=1}^{3} q_i p_i$ and

$$\overrightarrow{V_q}\times\overrightarrow{V_p}=(q_2p_3-q_3p_2)e_1-(q_1p_3-q_3p_1)e_2 +(q_1p_2-q_2p_1)e_3$$

The norm of quaternion q is defined

$$\|q\|=N(q)=q\overline{q}=q_0^2+q_1^2+q_2^2+q_3^2$$

For results on quaternion theory, one can see in [1,2,3].

Fibonacci and Lucas quaternions were defined by Horadam in 1963 in [4] and introduced the recurrence relations of quaternions in [5] in 1993. Iyer studied some properties about Fibonacci and Lucas quaternions in [6]. Halici gave Binet's formula, generating function of Fibonacci quaternion and some properties in [7]. With a description similar to Fibonacci quaternions, Cimen and Ipek defined Pell and Pell-Lucas quaternions in [8]. Liana and Wloch defined a new kind of quaternion called Jacobsthal and Jacobsthal-Lucas in [9] and they provided interesting properties of these quaternions. Also, Polatli, Kizilates and Kesim defined a new type of quaternion on split *k*-Fibonacci and *k*-Lucas numbers in [10]. Cerda-Morales defined the Tribonacci quaternion by generalizing the Fibonacci quaternions one step in [11]. Tasci and Yalcin generalized the Fibonacci quaternions as Fibonacci *p*-quaternions in 2015 in [12]. Also, Tasci defined the Padovan and Pell-Padovan quaternions in [13]. Asci and Aydinyuz generalized all of these studies and defined *k*-order Fibonacci quaternions in [14]. They obtained Fibonacci, Tribonacci, Tetranacci and similar quaternions for special cases.

In this paper, we define a new generalization and move the Fibonacci and Lucas quaternions to order *k*. With this generalization, we can obtain quaternions of special numbers such as Fibonacci, Lucas, Pell and Jacobsthal depending on special cases and give the recurrence relations, generating functions and some properties of these quaternions. In the last part of this study, we employ the quaternion family to matrix theory by giving the matrix representations of the quaternions we have defined.

2. Generalized *k*-Order Fibonacci and Lucas Numbers

In this section, we recall the generalized *k*-order Fibonacci and Lucas numbers defined by Asci and Aydinyuz in [15] in 2021.

Definition 1. Asci and Aydinyuz defined the generalized $k-$order Fibonacci and Lucas numbers in [15] by the recurrence relation

$$V_n^{(k)}=d_1V_{n-1}^{(k)}+d_2V_{n-2}^{(k)}+d_3V_{n-3}^{(k)}+\ldots+d_kV_{n-k}^{(k)}$$

with the initial conditions for, $n>k\geq 2$,

$$V_1^{(k)}=V_2^{(k)}=V_3^{(k)}=\ldots=V_{k-2}^{(k)}=0, V_{k-1}^{(k)}=q, V_k^{(k)}=d_1.$$

Let's look at what strings of numbers the relation $V_n^{(k)}$ turns into for some special choices.

- If we choose $k=2$, we obtain in Table 2 as follows:

Table 2. Some special cases of $V_n^{(k)}$

Special Cases	Sequences
$d_1=d_2=1, q=1$	Fibonacci Sequence
$d_1=d_2=1, q=2$	Lucas Sequence
$d_1=2, d_2=1, q=1$	Pell Sequence
$d_1=2, d_2=1, q=2$	Pell-Lucas Sequence
$d_1=1, d_2=2, q=1$	Jacobsthal Sequence
$d_1=1, d_2=2, q=2$	Jacobsthal-Lucas Sequence

By increasing these special selections, we can obtain other special numbers.

- For $k=3$ and $d_1=d_2=d_3=1,\ q=1;$ the Tribonacci sequence is obtained.
- For $d_1=d_2=d_3=\ldots=d_k=1$ and $q=1;$ the $k-$order Fibonacci sequence is obtained.
- For $d_1=d_2=d_3=\ldots=d_k=1$ and $q=2$; the $k-$order Lucas sequence is obtained.
- For $d_1=2,\ d_2=d_3=\ldots=d_k=1$ and $q=1$; the $k-$order Pell sequence is obtained.
- For $d_1=2,\ d_2=d_3=\ldots=d_k=1$ and $q=2$; the $k-$order Pell-Lucas sequence is obtained.
- For $d_1=1,\ d_2=2,\ d_3=\ldots=d_k=1$ and $q=1;$ the *k*-order Jacobsthal sequence is obtained.
- For $d_1=1,\ d_2=2,\ d_3=\ldots=d_k=1$ and $q=2$; the $k-$order Jacobsthal-Luas sequence is obtained.

3. Generalized *k*-Order Fibonacci and Lucas Quaternions

In this section, we firstly define the generalized *k*-order Fibonacci and Lucas quaternions. Also, we give the generating functions of these quaternions and obtain some interesting properties. Finally, we identify and prove the matrix representations.

Definition 2. The *nth* generalized *k*-order Fibonacci and Lucas quaternions $QV_n^{(k)}$ are defined as

$$QV_n^{(k)}=V_n^{(k)}e_0+V_{n+1}^{(k)}e_1+V_{n+2}^{(k)}e_2+V_{n+3}^{(k)}e_3 \quad (3.1)$$

where $V_n^{(k)}$ is *nth* generalized *k*-order Fibonacci and Lucas numbers.

Corollary 1. For, $k=2$, we obtain these quaternions in Table 3 as follows:

Table 3. Some special cases of $QV_n^{(k)}$

Special Cases	Quaternions
$d_1=d_2=1, q=1$	Fibonacci Quaternion QF_n in [7]
$d_1=d_2=1, q=2$	Lucas Quaternion QL_n in [6]
$d_1=2,\ d_2=1,\ q=1$	Pell Quaternion QP_n in [8]
$d_1=2,\ d_2=1,\ q=2$	Pell-Lucas Quaternion QQ_n in [8]
$d_1=1,\ d_2=2,\ q=1$	Jacobsthal Quaternion QJ_n in [9]
$d_1=1,\ d_2=2,\ q=2$	Jacobsthal-Lucas Quaternion Qj_n in [9]

Corollary 2. If we choose $k=3$, $d_1=d_2=d_3=1$ and $q=1$, the Tribonacci quaternion QT_n is obtained in [11].

Corollary 3. If we choose $d_1=d_2=d_3=...=d_k=1$ and $q=1$, the k- order Fibonacci quaternions $QF_n^{(k)}$ are obtained in [14].

Quaternion definitions of the other special numbers can be reached by making similar choices.

Definition 3. The conjugate of generalized k-order Fibonacci and Lucas quaternions $QV_n^{(k)}$ is defined by

$$\overline{QV_n^{(k)}}=V_n^{(k)}e_0-V_{n+1}^{(k)}e_1-V_{n+2}^{(k)}e_2-V_{n+3}^{(k)}e_3.$$

Definition 4. The norm of generalized k-order Fibonacci and Lucas quaternions $QV_n^{(k)}$ is defined by

$$\left\|QV_n^{(k)}\right\|=N_{QV_n^{(k)}}=\left(V_n^{(k)}\right)^2+\left(V_{n+1}^{(k)}\right)^2+\left(V_{n+2}^{(k)}\right)^2+\left(V_{n+3}^{(k)}\right)^2..$$

Proposition 1. For $n>0$ and $k\geq 2$, the following properties are obtained:

1. $QV_n^{(k)}+\overline{QV_n^{(k)}}=2V_n^{(k)}$
2. $\left(QV_n^{(k)}\right)^2+QV_n^{(k)}\overline{QV_n^{(k)}}=2V_n^{(k)}QV_n^{(k)}$
3. $QV_n^{(k)}\overline{QV_n^{(k)}}=\left(V_n^{(k)}\right)^2+\left(V_{n+1}^{(k)}\right)^2+\left(V_{n+2}^{(k)}\right)^2+\left(V_{n+3}^{(k)}\right)^2$
4. $QV_{n+1}^{(k)}-QV_n^{(k)}=QV_n^{(k)}-QV_{n-k}^{(k)}$.

Theorem 1. The recurrence relation of the generalized k-order Fibonacci and Lucas quaternions is given by

$$QV_n^{(k)}=\sum_{j=1}^{k}d_jQV_{n-j}^{(k)}\ \ for\ n\geq 2.$$

Proof: From the definition of generalized k-order Fibonacci and Lucas numbers and quaternions, we obtain as

$$\begin{aligned}\sum_{j=1}^{k}d_jQV_{n-j}^{(k)}&=e_0\left(d_1V_{n-1}^{(k)}+d_2V_{n-2}^{(k)}+...+d_kV_{n-k}^{(k)}\right)\\&+e_1\left(d_1V_n^{(k)}+d_2V_{n-1}^{(k)}+...+d_kV_{n-k+1}^{(k)}\right)\\&+e_2\left(d_1V_{n+1}^{(k)}+d_2V_n^{(k)}+...+d_kV_{n-k+2}^{(k)}\right)\\&+e_3\left(d_1V_{n+2}^{(k)}+d_2V_{n+1}^{(k)}+...+d_kV_{n-k+3}^{(k)}\right)\\&=e_0V_n^{(k)}+e_1V_{n+1}^{(k)}+e_2V_{n+2}^{(k)}+e_3V_{n+3}^{(k)}\\&=QV_n^{(k)}.\end{aligned}$$

Theorem 2. The generating function for the generalized k-order Fibonacci and Lucas quaternions is defined as

$$g(t)=\sum_{n=0}^{\infty}QV_n^{(k)}t^n=\frac{\begin{bmatrix}QV_0^{(k)}+t\left(QV_1^{(k)}-d_1QV_0^{(k)}\right)\\+t^2\left(QV_2^{(k)}-d_1QV_1^{(k)}-d_2QV_0^{(k)}\right)\end{bmatrix}}{1-\sum_{i=1}^{k}d_it^i}. \quad (3.2)$$

Proof: We can give the following proof, with $g(t)$ being the generating function of the generalized k-order Fibonacci and Lucas quaternions $\left\{QV_n^{(k)}\right\}$.

$$\begin{aligned}&g(t)-d_1tg(t)-...-d_kt^kg(t)\\&=QV_0^{(k)}+t\left(QV_1^{(k)}-d_1QV_0^{(k)}\right)\\&\quad+t^2\left(QV_2^{(k)}-d_1QV_1^{(k)}-d_2QV_0^{(k)}\right)\\&\quad+t^3\left(QV_3^{(k)}-d_1QV_2^{(k)}-d_2QV_1^{(k)}-d_3QV_0^{(k)}\right)\\&\quad+\sum_{n=4}^{\infty}t^n\left(QV_n^{(k)}-\sum_{j=1}^{n-1}d_jQV_{n-j}^{(k)}\right)\end{aligned}$$

By doing the necessary operations, we can obtain the generating function as follows:

$$g(t)=\frac{\begin{bmatrix}QV_0^{(k)}+t\left(QV_1^{(k)}-d_1QV_0^{(k)}\right)\\+t^2\left(QV_2^{(k)}-d_1QV_1^{(k)}-d_2QV_0^{(k)}\right)\end{bmatrix}}{1-\sum_{i=1}^{k}d_it^i}.$$

Corollary 4. Let's consider the generating functions with special choices given in (3.2) as follows:

1. For $d_1=d_2=...=d_k=1$ and $q=1$, the generating function of k-order Fibonacci quaternions is obtained in [14] as

$$g(t)=\frac{\begin{bmatrix}QF_0^{(k)}+t\left(QF_1^{(k)}-QF_0^{(k)}\right)\\+t^2\left(QF_2^{(k)}-QF_1^{(k)}-QF_0^{(k)}\right)\end{bmatrix}}{1-\sum_{i=1}^{k}t^i}.$$

2. For $d_1=1$, $d_2=...=d_k=1$ and $q=1$, the generating function of $k-$order Pell quaternions is obtained as

$$g(t)=\frac{\begin{bmatrix}QP_0^{(k)}+t\left(QP_1^{(k)}-2QP_0^{(k)}\right)\\+t^2\left(QP_2^{(k)}-2QP_1^{(k)}-QP_0^{(k)}\right)\end{bmatrix}}{1-2t-\sum_{i=2}^{k}t^i}.$$

3. For $d_1 = 1,\ d_2 = 2,\ d_3 = ... = d_k = 1$ and $q = 1$, the generating function of $k-$order Jacobsthal quaternions is obtained as

$$g(t) = \frac{\left[QJ_0^{(k)} + t\left(QJ_1^{(k)} - QJ_0^{(k)}\right) + t^2\left(QJ_2^{(k)} - QJ_1^{(k)} - 2QJ_0^{(k)}\right)\right]}{1 - t - 2t^2 - \sum_{i=3}^{k} t^i}.$$

By changing our choices for d_i and q, we can get the generating functions of other special numbers.

Corollary 5. Some special cases of generating functions are obtained given in (3.2) for $k = 2$ in Table 4 as follows:

Table 4. Generating Functions

Special Cases	Quaternions	Generating Functions
-	Horadam quaternion in [5]	$\frac{QH_0^{(k)} + t\left(QH_1^{(k)} - d_1 QH_0^{(k)}\right)}{1 - d_1 t - d_2 t^2}$
$d_1 = d_2 = 1,\ q = 1$	Fibonacci quaternion in [7]	$\frac{(0,1,1,2) + t(1,0,1,1)}{1 - t - t^2}$
$d_1 = d_2 = 1,\ q = 2$	Lucas quaternion in [6]	$\frac{(2,1,3,4) + t(-1,2,1,3)}{1 - t - t^2}$
$d_1 = 2, d_2 = 1,\ q = 1$	Pell quaternion in [8]	$\frac{(0,1,2,5) + t(1,0,1,2)}{1 - 2t - t^2}$
$d_1 = 2, d_2 = 1,\ q = 2$	Pell-Lucas quaternion in [8]	$\frac{(2,1,4,9) + t(-3,2,1,4)}{1 - 2t - t^2}$
$d_1 = 1, d_2 = 2,\ q = 1$	Jacobsthal quaternion	$\frac{(0,1,1,3) + t(1,0,2,2)}{1 - t - 2t^2}$
$d_1 = 1, d_2 = 2,\ q = 2$	Jacobsthal-Lucas quaternion	$\frac{(2,1,5,7) + t(-1,4,2,10)}{1 - t - 2t^2}$

Now let's identify the matrix of the generalized k-order Fibonacci and Lucas quaternions. We introduce the matrices Q_k, V_k and $E_{k,n}$. Let Q_k, V_k and $E_{k,n}$ determine as

$$Q_k = \begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_{k-1} & d_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, V_k = \begin{bmatrix} QV_{k-1}^{(k)} & QV_{k-2}^{(k)} & QV_{k-3}^{(k)} & \cdots & QV_1^{(k)} & QV_0^{(k)} \\ QV_{k-2}^{(k)} & QV_{k-3}^{(k)} & QV_{k-4}^{(k)} & \cdots & QV_0^{(k)} & QV_{-1}^{(k)} \\ QV_{k-3}^{(k)} & QV_{k-4}^{(k)} & QV_{k-5}^{(k)} & \cdots & QV_{-1}^{(k)} & QV_{-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ QV_1^{(k)} & QV_0^{(k)} & QV_{-1}^{(k)} & \cdots & QV_{3-k}^{(k)} & QV_{2-k}^{(k)} \\ QV_0^{(k)} & QV_{-1}^{(k)} & QV_{-2}^{(k)} & \cdots & QV_{2-k}^{(k)} & QV_{1-k}^{(k)} \end{bmatrix}$$

and

$$E_{k,n} = \begin{bmatrix} QV_{n+k-1}^{(k)} & QV_{n+k-2}^{(k)} & QV_{n+k-3}^{(k)} & \cdots & QV_{n+1}^{(k)} & QV_n^{(k)} \\ QV_{n+k-2}^{(k)} & QV_{n+k-3}^{(k)} & QV_{n+k-4}^{(k)} & \cdots & QV_n^{(k)} & QV_{n-1}^{(k)} \\ QV_{n+k-3}^{(k)} & QV_{n+k-4}^{(k)} & QV_{n+k-5}^{(k)} & \cdots & QV_{n-1}^{(k)} & QV_{n-2}^{(k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ QV_{n+1}^{(k)} & QV_n^{(k)} & QV_{n-1}^{(k)} & \cdots & QV_{n+3-k}^{(k)} & QV_{n+2-k}^{(k)} \\ QV_n^{(k)} & QV_{n-1}^{(k)} & QV_{n-2}^{(k)} & \cdots & QV_{n+2-k}^{(k)} & QV_{n+1-k}^{(k)} \end{bmatrix}.$$

Lemma 1. Let $n \geq 1$. Then, we get as

$$E_{k,n+1} = Q_k E_{k,n}.$$

Theorem 3. For $n \geq 1$, we get as

$$E_{k,n} = Q_k^n V_k \tag{3.3}$$

Proof: The proof is provided by the induction method on n. If $n=1$, we get as

$$E_{k,1} = Q_k V_k$$

Suppose that it is true for n

$$E_{k,n} = Q_k^n V_k$$

Then for, $n+1$, we get the proof of the theorem as

$$\begin{aligned} Q_k^{n+1} V_k &= Q_k Q_k^n V_k \\ &= Q_k E_{k,n} \\ &= E_{k,n+1}. \end{aligned}$$

Corollary 6. For $k = 2$;

1. The matrix representation of the Horadam Quaternions is obtained in [5] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} d_1 & d_2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QV_1 & QV_0 \\ QV_0 & QV_{-1} \end{bmatrix} \\ &= \begin{bmatrix} QV_{n+1} & QV_n \\ QV_n & QV_{n-1} \end{bmatrix} \\ &= E_{2,n} \end{aligned}$$

2. For $d_1 = d_2 = 1$ and $q = 1$, the matrix representation of the Fibonacci quaternions is obtained in [7] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QF_1 & QF_0 \\ QF_0 & QF_{-1} \end{bmatrix} \\ &= \begin{bmatrix} QF_{n+1} & QF_n \\ QF_n & QF_{n-1} \end{bmatrix} \end{aligned}$$

3. For $d_1 = d_2 = 1$ and $q = 2$, the matrix representation of the Lucas quaternions is obtained in [6] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QL_1 & QL_0 \\ QL_0 & QL_{-1} \end{bmatrix} \\ &= \begin{bmatrix} QL_{n+1} & QL_n \\ QL_n & QL_{n-1} \end{bmatrix} \end{aligned}$$

4. For $d_1 = 2$, $d_2 = 1$ and $q = 1$, the matrix representation of the Pell quaternions is obtained in [8] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QP_1 & QP_0 \\ QP_0 & QP_{-1} \end{bmatrix} \\ &= \begin{bmatrix} QP_{n+1} & QP_n \\ QP_n & QP_{n-1} \end{bmatrix} \end{aligned}$$

5. For $d_1 = 2$, $d_2 = 1$ and $q = 2$, the matrix representation of the Pell-Lucas quaternions is obtained in [8] as follows:

$$Q_2^n V_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QQ_1 & QQ_0 \\ QQ_0 & QQ_{-1} \end{bmatrix} = \begin{bmatrix} QQ_{n+1} & QQ_n \\ QQ_n & QQ_{n-1} \end{bmatrix}$$

6. For $d_1 = 1$, $d_2 = 2$ and $q = 1$, the matrix representation of the Jacobsthal quaternions is obtained in [9] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} QJ_1 & QJ_0 \\ QJ_0 & QJ_{-1} \end{bmatrix} \\ &= \begin{bmatrix} QJ_{n+1} & QJ_n \\ QJ_n & QJ_{n-1} \end{bmatrix} \end{aligned}$$

7. For $d_1 = 1$, $d_2 = 2$ and $q = 2$, the matrix representation of the Jacobsthal-Lucas quaternions is obtained in [9] as follows:

$$\begin{aligned} Q_2^n V_2 &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} Qj_1 & Qj_0 \\ Qj_0 & Qj_{-1} \end{bmatrix} \\ &= \begin{bmatrix} Qj_{n+1} & Qj_n \\ Qj_n & Qj_{n-1} \end{bmatrix} \end{aligned}$$

Corollary 7. For $k = 3$, $d_1 = d_2 = d_3 = 1$ and $q = 1$, the matrix representation of the Tribonacci quaternions is obtained in [11] as follows:

$$\begin{aligned} Q_3^n V_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} QT_2 & QT_1 & QT_0 \\ QT_1 & QT_0 & QT_{-1} \\ QT_0 & QT_{-1} & QT_{-2} \end{bmatrix} \\ &= \begin{bmatrix} QT_{n+2} & QT_{n+1} & QT_n \\ QT_{n+1} & QT_n & QT_{n-1} \\ QT_n & QT_{n-1} & QT_{n-2} \end{bmatrix}. \end{aligned}$$

Corollary 8. For $d_1 = d_2 = \ldots = d_k = 1$ and $q = 1$; the matrix representation of the k-order Fibonacci quaternions are shown in [14].

We can obtain matrix representations of other special quaternions such as k-order Pell, Jacobsthal by making similar choices.

Theorem 4. Let $n \geq 1$ be integer. Then, we get as

$$\begin{bmatrix} d_1 & d_2 & d_3 & \cdots & d_{k-1} & d_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{k\times k}^n \begin{bmatrix} QV_{k-1}^{(k)} \\ QV_{k-2}^{(k)} \\ QV_{k-3}^{(k)} \\ \vdots \\ QV_1^{(k)} \\ QV_0^{(k)} \end{bmatrix}_{k\times 1} = \begin{bmatrix} QV_{n+k-1}^{(k)} \\ QV_{n+k-2}^{(k)} \\ QV_{n+k-3}^{(k)} \\ \vdots \\ QV_{n+1}^{(k)} \\ QV_n^{(k)} \end{bmatrix}_{k\times 1}.$$

Theorem 5. Let m and n be an integer. Then, we get as

$$QV_{n+m}^{(k)} = V_{n+1}^{(k)} QV_m^{(k)} + \sum_{j=0}^{k-2} \left(Q_{m-(k-j-1)}^{(k)} \sum_{p=0}^{j} V_{n-p}^{(k)} \right)$$

where $V_n^{(k)}$ is the *nth* generalized k-order Fibonacci and Lucas numbers.

Proof: For $k \geq 2$, the Q_k^n matrix is defined in [15] as

$$Q_k^n = \begin{bmatrix} V_{n+1}^{(k)} & \cdots & V_n^{(k)} + V_{n-1}^{(k)} + V_{n-2}^{(k)} & V_n^{(k)} + V_{n-1}^{(k)} & V_n^{(k)} \\ V_n^{(k)} & \cdots & V_{n-1}^{(k)} + V_{n-2}^{(k)} + V_{n-3}^{(k)} & V_{n+1}^{(k)} + V_n^{(k)} & V_{n-1}^{(k)} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ V_{n-k+3}^{(k)} & \cdots & V_{n-k+2}^{(k)} + V_{n-k+1}^{(k)} + V_{n-k}^{(k)} & V_{n-k+2}^{(k)} + V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} \\ V_{n-k+2}^{(k)} & \cdots & V_{n-k+1}^{(k)} + V_{n-k}^{(k)} + V_{n-k-1}^{(k)} & V_{n-k+3}^{(k)} + V_{n-k+2}^{(k)} & V_{n-k+1}^{(k)} \end{bmatrix}$$

If we use (3.3), we get as follows

$$Q_k^{n+m} = Q_k^n Q_k^m$$

and

$$Q_k^n V_k = E_{k,n}$$

Then, we have

$$E_{k,n+m} = Q_k^{n+m} V_k = Q_k^n Q_k^m V_k = Q_k^n E_{k,m}.$$

If the equality of matrices is used, we get for

$$\begin{aligned} Q_{n+m}^{(k)} &= V_n^{(k)} QV_{m+1-k}^{(k)} + \left(V_n^{(k)} + V_{n-1}^{(k)}\right) QV_{m+2-k}^{(k)} \\ &+ \left(V_n^{(k)} + V_{n-1}^{(k)} + V_{n-2}^{(k)}\right) QV_{m+3-k}^{(k)} + \dots + \\ &+ \left(V_n^{(k)} + V_{n-1}^{(k)} + \dots + V_{n+2-k}^{(k)}\right) QV_{m-1}^{(k)} + V_{n+1}^{(k)} QV_m^{(k)} \end{aligned}$$

4. Conclusion

In this study, we defined the generalized k-order Fibonacci and Lucas quaternion family by making a new generalization. Depending on the d_i and q choices, we gained k-order Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas quaternions. For, k=2, we obtained the recurrence relations of known special numbers as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas quaternions. By multiplying the choices we have made, we obtained the quaternion definitions for other special numbers. We gave the generating functions for these quaternions and we obtained the generating functions of special numbers. Also, we identified and proved the matrix representations for generalized k-order Fibonacci and Lucas quaternions. In this way, we obtained the matrix representations for usual Fibonacci, Lucas, Pell and the other special numbers known with the d_i and q values we have chosen and gave some properties about matrix representations for generalized k-order Fibonacci and Lucas quaternions.

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