Turkish Journal of Analysis and Number Theory, 2022, Vol. 10, No. 1, 21-26
Available online at http://pubs.sciepub.com/tjant/10/1/5
Published by Science and Education Publishing
DOI:10.12691/tjant-10-1-5

Some Spectral Characteristic Numbers of Direct Sum of Operators

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Received November 14, 2022; Revised December 23, 2022; Accepted December 30, 2022

Abstract In the present study, firstly, some algebraic inequalities are proved, which will be used later. By making use of these relations, some evaluations are found related the gaps between norm and numerical radius, spectral radius and Crawford number for diagonal block operator matrices on the infinite direct sum of Hilbert spaces. Later on, the gaps between some spectral characteristic numbers (operator norm, lower and upper bounds of spectrum set and numerical radius) of the infinite direct sum of Hilbert space operators relatively to the same spectral characteristics of the coordinate operators are investigated. Then, the obtained results are supported by applications. The open problem posed by Demuth in 2015 and the works of Kittaneh and his researcher group in this area had an important effect on the formation of the subject discussed in this study.

Keywords: *operator norm, spectral radius, numerical radius, CRAWFORD number*

Cite This Article: Zameddin I. Ismailov, and Pembe Ipek Al, "Some Spectral Characteristic Numbers of Direct Sum of Operators." *Turkish Journal of Analysis and Number Theory*, vol. 10, no. 1 (2022): 21-26. doi: 10.12691/tjant-10-1-5.

1. Introduction

As is known in the mathematical literature, obtaining the spectrum set, numerical range set of a given operator and and calculating spectral radii, numerical radii and Crawford number is one of the fundamental questions of the spectral theory of linear operators. Generally, finding the spectrum set and numerical range of non-selfadjoint linear bounded operators is theoretically and technically quite difficult. For the calculation of the spectral radius $r(A)$ of the linear bounded operator in any Banach space there is one formula $r(A) = \lim_{n\to\infty} \| A^n \|^{\frac{1}{n}}$ (see [10]). On the other hand, for the spectral and the numerical radius, the following inequalities hold $r(A) \le w(A) \le \| A\|$ and $\frac{1}{2}\| A\| \le w(A) \le \| A\|$ for $A \in L(H)$.

In addition, for the linear normal bounded operator A in Hilbert space we have the following relations $r(A) = w(A) = \| A\|$.

It is beneficial to recall that for the spectrum set $\sigma(A)$ and numerical range $W(A)$ of any linear bounded operator A the following spectral inclusion holds $\sigma(A) \subset \overline{W(A)}$ (see [8,10] for more information).

In [13], some spectral radius inequalities for 2×2 block operator matrix, sum, product, and commutators of two linear bounded Hilbert space operators have been examined.

In [1], some estimates for numerical and spectral radii of the Frobenius companion matrix have been obtained.

In [2], some upper and lower bounds for the numerical indices in Hilbert space operators have been found.

In [3], some estimates have been obtained for spectral and numerical radii of the product, sum, commutator, anticommutator of two Hilbert spaces operators.

In [7], several numerical radius inequalities have been proved for $n\times n$ block operator matrices in the direct sum of Hilbert spaces. In [5], the numerical radius inequalities have been given in for $n\times n$ accretive matrices.

In [4], several new norms and numerical radius inequalities have been researched for 2×2 block operator matrices.

In [14], several new $\mathbb{A}$-numerical radius inequalities have been offered for many type $n\times n$ block operator matrices in the direct sum of Hilbert spaces.

In [19], subadditivity of the spectral radius of commutative two operators in Banach spaces has been investigated. In [21], by the same author the subadditivity and submultiplicativity properties of local spectral radius of bounded positive operators have been researched in Banach spaces. In [20], the same properties of local spectral radius in partially ordered Banach spaces have been established. In [22], several inequalities for the spectral radius of a positive commutator of positive operators have been surveyed in Banach space ordered by a normal and generating core. In [9,12], the numerical range and numerical radius of some Volterra integral operator in Hilbert Lebesgue spaces at finite interval have been considered.

The open problem posed by Demuth in 2015 and the works of Kittaneh and his researcher group in this area had an important effect on the formation of the subject discussed in this study (see, e.g. [6,13,14]).

This paper is organized as follows: The first section is devoted to introduction proving. We contrive by the necessary auxiliary theorem in Section 2. In the last section, we prove our main results. Also, the obtained results are supported by applications.

2. Some Auxiliary Important Results

In this section, we will prove certain auxiliary results from which it will be used later.

From [11] and [17], we have the following theorem.

Theorem 1 *Let* $n \in \mathbb{N}$. *For the numbers* $a_1, a_2, ..., a_n \in \mathbb{R}$ *and* $b_1, b_2, ..., b_n \in \mathbb{R}$, *the inequalities*

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \le \max_{1\le m\le n}(a_m - b_m),$$

$$\max_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \min_{1\le m\le n} b_m \le \max_{1\le m\le n}(a_m - b_m) + \sum_{\substack{k,m=1\\k<m}}^{n} |a_k - a_m| + \sum_{\substack{k,m=1\\k<m}}^{n} |b_k - b_m|$$

are true.

Proof. For all cases, we prove by mathematical induction method.

For $n = 2$, it is clear that

$$\begin{aligned}
&\max\{a_1, a_2\} - \max\{b_1, b_2\} \\
&= \frac{1}{2}[(a_1 + a_2 + |a_1 - a_2|) - (b_1 + b_2 + |b_1 - b_2|)] \\
&= \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) + (|a_1 - a_2| - |b_1 - b_2|)] \\
&= \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - (|b_1 - b_2| - |a_1 - a_2|)] \\
&\ge \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - ||b_1 - b_2| - |a_1 - a_2||] \\
&\ge \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - |(b_1 - b_2) - (a_1 - a_2)|] \\
&= \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - |(a_1 - b_1) - (a_2 - b_2)|] \\
&= \min\{a_1 - b_1, a_2 - b_2\}.
\end{aligned}$$

Now, assume that

$$\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m \ge \min_{1\le m\le n-1}(a_m - b_m)$$

for any $n \in \mathbb{N}$, $n > 2$.

Then, one can easily have

$$\begin{aligned}
&\max\{a_1, ..., a_n\} - \max\{b_1, ..., b_n\} \\
&= \max\{\max_{1\le m\le n-1} a_m, a_n\} - \max\{\max_{1\le m\le n-1} b_m, b_n\} \\
&\ge \min\{\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m, a_n - b_n\} \\
&\ge \min\{\min_{1\le m\le n-1}(a_m - b_m), a_n - b_n\} \\
&= \min_{1\le m\le n}(a_m - b_m).
\end{aligned}$$

From this and by mathematical induction method, for any $n \in \mathbb{N}$ the following inequality hold

$$\min_{1\le m\le n}(a_m - b_m) \le \max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m.$$

Similarly, for $n = 2$ by simple calculations we again have that

$$\begin{aligned}
&\max\{a_1, a_2\} - \max\{b_1, b_2\} \\
&= \frac{1}{2}[(a_1 + a_2 + |a_1 - a_2|) - (b_1 + b_2 + |b_1 - b_2|)] \\
&= \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) + (|a_1 - a_2| - |b_1 - b_2|)] \\
&\le \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - ||a_1 - a_2| - |b_1 - b_2||] \\
&\le \frac{1}{2}[((a_1 - b_1) + (a_2 - b_2)) - |(a_1 - b_1) - (a_2 - b_2)|] \\
&= \max\{a_1 - b_1, a_2 - b_2\}.
\end{aligned}$$

Now, assume that for $n \in \mathbb{N}$, $n > 2$

$$\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m \le \max_{1\le m\le n-1}(a_m - b_m).$$

From this assumption, one can have

$$\begin{aligned}
&\max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \\
&= \max\{\max_{1\le m\le n-1} a_m, a_n\} - \max\{\max_{1\le m\le n-1} b_m, b_n\} \\
&\le \max\{\max_{1\le m\le n-1} a_m - \max_{1\le m\le n-1} b_m, a_n - b_n\} \\
&\le \max\{\min_{1\le m\le n-1}(a_m - b_m), a_n - b_n\} \\
&= \max_{1\le m\le n}(a_m - b_m).
\end{aligned}$$

Consequently, by mathematical induction method it is obtained

$$\max_{1\le m\le n} a_m - \max_{1\le m\le n} b_m \le \max_{1\le m\le n}(a_m - b_m)$$

for any $n \in \mathbb{N}$.

Now, we prove the second part of the theorem.

For $n = 2$ it is clear that

$$\begin{aligned}
&2\max\{a_1, a_2\} - 2\min\{b_1, b_2\} \\
&= (a_1 + a_2 + |a_1 - a_2|) - (b_1 + b_2 - |b_1 - b_2|) \\
&= ((a_1 - b_1) + (a_2 - b_2)) + (|a_1 - a_2| + |b_1 - b_2|) \\
&\ge ((a_1 - b_1) + (a_2 - b_2)) + |(a_1 - a_2) - (b_1 - b_2)| \\
&= ((a_1 - b_1) + (a_2 - b_2)) + |(a_1 - b_1) - (a_2 - b_2)| \\
&= 2\max\{(a_1 - b_1), (a_2 - b_2)\}.
\end{aligned}$$

Then, we have

$$\max\{a_1 - b_1, a_2 - b_2\} \le \max\{a_1, a_2\} - \min\{b_1, b_2\}.$$

On the other hand, we get

$$\begin{aligned}
&2\max\{a_1, a_2\} - 2\min\{b_1, b_2\} \\
&= (a_1 + a_2 + |a_1 - a_2|) - (b_1 + b_2 - |b_1 - b_2|) \\
&= ((a_1 - b_1) + (a_2 - b_2)) + |(a_1 - b_1) - (a_2 - b_2)| \\
&+ [|a_1 - a_2| + |b_1 - b_2| - |(a_1 - b_1) - (a_2 - b_2)|] \\
&\le [(a_1 - b_1) + (a_2 - b_2) + |(a_1 - b_1) - (a_2 - b_2)|] \\
&+ ||a_1 - a_2| + |b_1 - b_2| - |(a_1 - b_1) - (a_2 - b_2)|| \\
&= [(a_1 - b_1) + (a_2 - b_2) + |(a_1 - b_1) - (a_2 - b_2)|] \\
&+ |||a_1 - a_2| + |b_1 - b_2|| - |(a_1 - b_1) - (a_2 - b_2)||
\end{aligned}$$

$$\leq [(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|]$$
$$+||a_1-a_2|+|b_1-b_2|-(a_1-b_1)-(a_2-b_2)|$$
$$=[(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|]$$
$$+|(|a_1-a_2|+(a_1-a_2))+(|b_1-b_2|-(b_1-b_2))|$$
$$=[(a_1-b_1)+(a_2-b_2)+|(a_1-b_1)-(a_2-b_2)|]$$
$$+2|a_1-a_2|+2|b_1-b_2|.$$

Then, for $n=2$, we have

$$\max\{a_1,a_2\}-\min\{b_1,b_2\}$$
$$\leq \max\{a_1-b_1,a_2-b_2\}+|a_1-a_2|+|b_1-b_2|.$$

Now, assume that the mentioned inequalities hold for $k=n-1$. Then for $k=n$, it is clear that

$$\max_{1\leq m\leq n} a_m-\min_{1\leq m\leq n} b_m$$
$$=\max\{\max_{1\leq m\leq n-1} a_m, a_n\}-\min\{\min_{1\leq m\leq n-1} b_m, b_n\}$$
$$\leq \max\{\max_{1\leq m\leq n-1} a_m-\min_{1\leq m\leq n-1} b_m, a_n-b_n\}$$
$$+|\max_{1\leq m\leq n-1} a_m-a_n|+|\min_{1\leq m\leq n-1} b_m-b_n|$$
$$\leq \max\{\max_{1\leq m\leq n-1}(a_m-b_m)+\sum_{\substack{k,m=1\\k<m}}^{n-1}|a_k-a_m|$$
$$+\sum_{\substack{k,m=1\\k<m}}^{n-1}|b_k-b_m|,|a_n-b_n|\}+|\max_{1\leq m\leq n-1} a_m-a_n|$$
$$+|\min_{1\leq m\leq n-1} b_m-b_n|$$
$$\leq \max\{\max_{1\leq m\leq n-1}(a_m-b_m),|a_n-b_n|\}+\sum_{\substack{k,m=1\\k<m}}^{n-1}|a_k-a_m|$$
$$+\sum_{\substack{k,m=1\\k<m}}^{n-1}|b_k-b_m|+\sum_{k=1}^{n-1}|a_k-a_n|+\sum_{k=1}^{n-1}|b_k-b_n|$$
$$=\max_{1\leq m\leq n}(a_m-b_m)+\sum_{\substack{k,m=1\\k<m}}^{n}|a_k-a_m|+\sum_{\substack{k,m=1\\k<m}}^{n}|b_k-b_m|.$$

On the other hand, we have

$$\max_{1\leq m\leq n} a_m-\min_{1\leq m\leq n} b_m$$
$$=\max\{\max_{1\leq m\leq n-1} a_m, a_n\}-\min\{\min_{1\leq m\leq n-1} b_m, b_n\}$$
$$\geq \max\{\max_{1\leq m\leq n-1} a_m-\min_{1\leq m\leq n-1} b_m, a_n-b_n\}$$
$$\geq \max\{\max_{1\leq m\leq n-1}(a_m-b_m), a_n-b_n\}$$
$$=\max_{1\leq m\leq n}(a_m-b_m).$$

Theorem 2 *For the sequences of real numbers* (a_n) *and* (b_n) *the following inequalities hold*

$$\inf_{n\geq 1}(a_n-b_n)\leq \sup_{n\geq 1}(a_n)-\sup_{n\geq 1}(b_n)\leq \sup_{n\geq 1}(a_n-b_n)$$

and

$$\sup_{n\geq 1}(a_n-b_n)\leq \sup_{n\geq 1}(a_n)-\inf_{n\geq 1}(b_n)$$
$$\leq \sup_{n\geq 1}(a_n-b_n)+\sum_{\substack{k,m=1\\k<m}}^{\infty}|a_k-a_m|+\sum_{\substack{k,m=1\\k<m}}^{\infty}|b_k-b_m|.$$

Proof. For the prove this claims, it will be used of Theorem 1.

For any $n\geq 1$, we have

$$\max_{n\geq 1} a_n-\max_{n\geq 1} b_n\leq \max_{n\geq 1} a_n-\max_{1\leq m\leq n} b_m$$
$$=\left(\max_{1\leq m\leq n} a_m-\max_{1\leq m\leq n} b_m\right)$$
$$+\left(\max_{n\geq 1} a_n-\max_{1\leq m\leq n} a_m\right)$$
$$\leq \max_{1\leq m\leq n}(a_m-b_m)+\left(\max_{n\geq 1} a_n-\max_{1\leq m\leq n} a_m\right)$$
$$\leq \max_{n\geq 1}(a_n-b_n)+\left(\max_{n\geq 1} a_n-\max_{1\leq m\leq n} a_m\right).$$

Then, from the last equation we have

$$\max_{n\geq 1} a_n-\max_{n\geq 1} b_n\leq \max_{n\geq 1}(a_n-b_n).$$

For any $n\geq 1$, it is true

$$\max_{n\geq 1} a_n-\max_{n\geq 1} b_n$$
$$\geq \left(\max_{1\leq m\leq n} a_m-\max_{1\leq m\leq n} b_m\right)$$
$$+\left(\max_{1\leq m\leq n} b_m-\max_{n\geq 1} b_n\right)$$
$$\geq \min_{1\leq m\leq n}(a_m-b_m)+\left(\max_{1\leq m\leq n} b_m-\max_{n\geq 1} b_n\right)$$
$$\geq \inf_{1\leq m\leq n}(a_m-b_m)+\left(\max_{1\leq m\leq n} b_m-\max_{n\geq 1} b_n\right).$$

From the last relation, it is obtained

$$\inf_{n\geq 1}(a_n-b_n)\leq \max_{n\geq 1} a_n-\max_{n\geq 1} b_n.$$

On the other hand, from the following simple calculations we have

$$\sup_{n\geq 1} a_n-\inf_{n\geq 1} b_n\leq \sup_{n\geq 1} a_n-\inf_{1\leq m\leq n} b_m$$
$$=\left(\sup_{m\geq 1} a_m-\inf_{1\leq m\leq n} b_m\right)+\left(\sup_{n\geq 1} a_n-\sup_{1\leq m\leq n} a_m\right)$$
$$=\sup_{1\leq m\leq n}(a_m-b_m)+\sum_{\substack{k,m=1\\k<m}}^{n}|a_k-a_m|+\sum_{\substack{k,m=1\\k<m}}^{n}|b_k-b_m|$$
$$+\left(\sup_{n\geq 1} a_n-\sup_{1\leq m\leq n} a_m\right).$$

Then, from this inequality, we have

$$\sup_{n\geq 1} a_n-\inf_{n\geq 1} b_n$$
$$\leq \sup_{n\geq 1}(a_n-b_n)+\sum_{\substack{k,m=1\\k<m}}^{\infty}|a_k-a_m|+\sum_{\substack{k,m=1\\k<m}}^{\infty}|b_k-b_m|.$$

Later on, from the Theorem 1 it is obtained that for any $n\geq 1$

$$\sup_{n\geq 1} a_n-\inf_{n\geq 1} b_n\geq \sup_{1\leq m\leq n} a_m-\inf_{n\geq 1} b_n$$
$$=\left(\sup_{1\leq m\leq n} a_m-\inf_{1\leq m\leq n} b_m\right)+\left(\inf_{1\leq m\leq n} b_m-\inf_{n\geq 1} b_n\right)$$
$$\geq \sup_{1\leq m\leq n}(a_m-b_m)+\left(\inf_{1\leq m\leq n} b_m-\inf_{n\geq 1} b_n\right)$$
$$\geq \sup_{1\leq m\leq n}(a_m-b_m).$$

Hence,

$$\sup_{n\geq 1}(a_n - b_n) \leq \sup_{n\geq 1} a_n - \inf_{n\geq 1} b_n.$$

3. Some Relations between Spectral Characteristic Numbers of Infinite Direct Sum of Hilbert Space Operators and Their Coordinate Operators

Let $\sigma(A)$ and $W(A)$ be the spectrum and numerical range sets of the linear bounded operator A in any Hilbert space H, respectively (see [8]). Also, assume that

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

$$\tau(A) = \inf\{|\lambda| : \lambda \in \sigma(A)\}.$$

and

$$gap(A;n,w) = \|A\| - w(A),$$

$$gap(A;n,r) = \|A\| - r(A),$$

$$gap(A;n,\tau) = \|A\| - \tau(A).$$

It is well known that $\sigma(A) \subset \overline{W(A)}$ for any $A \in L(H)$ (for more information see [8,10]).

In addition, let H_n is a Hilbert space, $A_n \in L(H_n)$, for $n \geq 1$, and $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n$.

Remember that some connections between some spectral characteristic numbers of the direct sum of Hilbert space operators with same numbers of coordinate operators have been investigated in [16,18].

Using Theorem 2, the following results can be proved.

Theorem 3 *For the direct sum of operators* $A = \bigoplus_{m=1}^{\infty} A_m$ *in* $H = \bigoplus_{m=1}^{\infty} H_m$, *the following inequalities hold*

$$\inf_{m\geq 1} gap(A_m;n_m,w_m) \leq gap(A;n,w) \leq \sup_{m\geq 1} gap(A_m;n_m,w_m),$$

$$\inf_{m\geq 1} gap(A_m;n_m,r_m) \leq gap(A;n,r) \leq \sup_{m\geq 1} gap(A_m;n_m,r_m),$$

and

$$\inf_{m\geq 1} gap(A_m;n_m,\tau_m) \leq gap(A;n,\tau)$$

$$\leq \sup_{m\geq 1} gap(A_m;n_m,\tau_m) + \sum_{\substack{k,m=1\\ k<m}}^{n} |n_k - n_m| + \sum_{\substack{k,m=1\\ k<m}}^{\infty} |\tau_k - \tau_m|,$$

where, $n = \|A\|$, $w = w(A)$, $r = r(A)$, $\tau = \tau(A)$ *and* $n_m = n(A_m) = \|A_m\|$, $w_m = w(A_m)$, $r_m = r(A_m)$, $\tau_m = \tau(A_m)$ *for* $m \geq 1$.

Proof. From the [15], we know

$$\|A\| = \sup_{n\geq 1}\|A_n\|.$$

Also, from the [16], we know

$$w(A) = \sup_{n\geq 1} w(A_n),$$

$$r(A) = \sup_{n\geq 1} r(A_n),$$

and

$$\tau(A) = \inf_{n\geq 1} \tau(A_n).$$

Consequently,

$$gap(A;\|A\|,w(A)) = \|A\| - w(A) = \sup_{m\geq 1}\|A_m\| - \sup_{m\geq 1} w(A_m),$$

$$gap(A;n,r) = \|A\| - r(A) = \sup_{m\geq 1}\|A_m\| - \sup_{m\geq 1} r(A_m),$$

$$gap(A;n,\tau) = \|A\| - \tau(A) = \sup_{m\geq 1}\|A_m\| - \inf_{m\geq 1} \tau(A_m).$$

Now assumed that, firstly

$$a_m = \|A_m\|,\ b_m = w(A_m),\ m \geq 1,$$

secondly

$$a_m = \|A_m\|,\ b_m = r(A_m),\ m \geq 1,$$

and thirdly

$$a_m = \|A_m\|,\ b_m = \tau(A_m),\ m \geq 1.$$

In this case, apply the Theorem 2 in according places it will be obtained validity of the claims of theorem, respectively.

Now, it will be proved the following result.

Theorem 4 *Let* $A = \bigoplus_{n=1}^{\infty} A_n$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A \in L(H)$, $A \neq 0$, $r(A) \neq 0$. *If* $m_n, m \in (0,1]$, $n = 1,2,\ldots$ *are smallest numbers satisfy the following conditions*

$$r(A_n) \leq m_n \|A_n\|$$

and

$$r(A) \leq m\|A\|$$

respectively, then

$$\inf_{n\geq 1} m_n \leq m \leq \sup_{n\geq 1} m_n.$$

Proof. It is clear that

$$\sup_{n\geq 1} r(A_n) \leq \left(\sup_{n\geq 1} m_n\right)\left(\sup_{n\geq 1}\|A_n\|\right).$$

Then, from [15] and [16]

$$r(A) \leq \left(\sup_{n\geq 1} m_n\right)\|A\|.$$

From this and since m is the smallest, it is obtained that

$$m \leq \sup_{n\geq 1} m_n.$$

On the contrary, since $m \in (0,1]$ is a smallest number satisfying the condition

$$r(A) \le m\|A\|,$$

using the technique in proof of Theorem 2 the following inequalities hold

$$0 \le m\|A\| - r(A) \le \sup_{n\ge1}\{m\|A_n\| - r(A_n)\}.$$

Then, at least one of $n_0 \ge 1$

$$r(A_{n_0}) \le m\|A_{n_0}\|.$$

Consequently, it is implies that

$$m_{n_0} \le m.$$

Hence,

$$\inf_{n\ge1} m_n \le m.$$

Example Let $\mathcal{H}_1 = \mathbb{C}, \mathcal{H}_2 = \mathbb{C}^2,$

$$A_1 = a,\ a > 0,\ A_1 : \mathbb{C} \to \mathbb{C},$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, A_2 : \mathbb{C}^2 \to \mathbb{C}^2,$$

$$A = A_1 \oplus A_2,\ A : \mathbb{C}^3 \to \mathbb{C}^3.$$

In this case, we have

$$\sigma(A_1) = \{a\},\ r(A_1) = a,\ \|A_1\| = a,\ m(A_1) = 1,$$

$$\sigma(A_2) = \{1,0\},\ r(A_2) = 1,\ \|A_2\| = \sqrt{2},\ m(A_2) = \frac{1}{\sqrt{2}},$$

and

$$\sigma(A) = \{a,1,0\},\ r(A) = \max\{1,a\},\ \|A\| = \max\{a,\sqrt{2}\}.$$

If $a > \sqrt{2}$, we have $r(A) = a$ and $\|A\| = a$. Hence, $m(A) = 1 = \sup\{m(A_1), m(A_2)\}$. If $a < 1$, we have $r(A) = 1$ and $\|A\| = \sqrt{2}$. Hence, $m(A) = \frac{1}{\sqrt{2}} = \inf\{m(A_1), m(A_2)\}$.

Let $\mathcal{H}_1 = L^2(0,1), \mathcal{H}_n = \mathbb{C}, n \ge 2,$

$$A_1 : L^2(0,1) \to L^2(0,1),\ A_1 f(t) = \pi \int_0^t f(s)ds,\ f \in L^2(0,1),$$

$$A_n : \mathbb{C} \to \mathbb{C},\ A_n x_n = \frac{1}{n-1} x_n,\ x_n \in \mathbb{C},\ n \ge 2,$$

and

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n,\ A = \bigoplus_{n=1}^{\infty} A_n,\ A : \mathcal{H} \to \mathcal{H}.$$

We have,

$$\sigma(A_1) = \{0\},\ \|A_1\| = 2,\ gap(A_1) = 2,$$

$$\sigma(A_n) = \left\{\frac{1}{n-1}\right\},\ \|A_n\| = \frac{1}{n-1},\ gap(A_n) = 0,\ n \ge 2,$$

and

$$\sigma(A) = \left\{0,1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n-1},\ldots\right\},\ \|A\| = 2,\ r(A) = 1.$$

Hence,

$$0 = \inf_{n\ge1} gap(A_n) < gap(A) = 1 < \sup_{n\ge1} gap(A_n) = 2.$$

In the direct sum $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ of Banach spaces $\mathcal{H}_n$, $n \ge 1$, consider the following operator in the form

$$T = \begin{pmatrix} A_1 & B_1 & & & & & \\ & A_2 & B_2 & & & & \\ & & A_3 & B_3 & & 0 & \\ & & & \ddots & \ddots & & \\ & 0 & & & A_n & B_n & \\ & & & & & \ddots & \ddots \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

where, $A_n \in L(\mathcal{H}_n), n \ge 1$ and $B_n \in L(\mathcal{H}_{n+1}, \mathcal{H}_n), n \ge 1$.

It can be verified that

$$\|T\| \le \sup_{n\ge1}\|A_n\| + \sup_{n\ge1}\|B_n\|$$

and

$$\bigcup_{n=1}^{\infty} \sigma(A_n) \subset \sigma(T).$$

Then, we have

$$\begin{aligned} &\|T\| - r(T) \\ &\le \sup_{n\ge1}\|A_n\| - dist\left\{\bigcup_{n=1}^{\infty}\sigma(A_n), \{0\}\right\} + \sup_{n\ge1}\|B_n\| \\ &= \sup_{n\ge1}\|A_n\| - \sup_{n\ge1} r(A_n) + \max_{n\ge1}\|B_n\| \\ &= gap(A;n,r) + \|B\|, \end{aligned}$$

where,

$$A = \begin{pmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & A_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & A_n & \\ & & & & & \ddots \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 & & & & & \\ & B_2 & & & & \\ & & B_3 & & 0 & \\ & & & \ddots & & \\ & 0 & & & B_n & \\ & & & & & \ddots \end{pmatrix}$$

$$A, B : \mathcal{H} \to \mathcal{H}.$$

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