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Error Analysis on Triangle Type Functions Concerning Extremum Values in Ordinary High School Mathematics Teaching

Guoqiang Dang*

School of Mathematical Sciences, Beijing Normal University, Beijing, P.R. China
*Corresponding author: dang@mail.bnu.edu.cn

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Abstract This research is focusing on the extremum of a given function which involves triangle functions. We collect some typical examples from ordinary high school mathematics teaching and demonstrate several typical mistakes which were usually posed by ordinary high school students usually and analyze the reasons for the errors. These results will be used to help ordinary high school students to well understand AM-GM inequalities and related applications for maximum and minimum values of triangle type functions.

Keywords: *AM-GM inequalities, error analysis, extremum values*

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1. Introduction

The starting point of this investigation is to clarify some various errors in computing the maximum or minimum values of a given function which involves triangle types functions by using the arithmetic mean and geometry mean (AM-GM) inequalities [1,2] which usually occur in high school mathematics study. Inequalities are one of the key tools to solve mathematical problems in applied sciences, and also play an important role in ordinary high school mathematics teaching and examinations. So we collect three typical related problems from high school mathematics teaching which aim to illustrate some typical mistakes happening usually by ordinary high school students and give some useful suggestions.

AM-GM inequalities read:

Let $a,b\in\mathrm{R}^+$, and denote $AM=\frac{a+b}{2}$, $GM=\sqrt{ab}$. Then $AM\geq GM$. Equalities occur if and only if $a=b$.

Maximum value or minimum value can be obtained upon the fixed value of $a+b$ by using $GM\leq AM$ or ab by using $AM\geq GM$ respectively. The counterexample $y=\ln x+\frac{4}{\ln x}$ shows that the condition $a,b\in\mathrm{R}^+$ is necessary. Another counterexample, $y=|\sin x|+\frac{4}{|\sin x|}$, although $|\sin x|+\frac{4}{|\sin x|}\geq 2\sqrt{|\sin x|\times\frac{4}{|\sin x|}}=4,$ anyway, $|\sin x|=\frac{4}{|\sin x|},|\sin x|=2$ is wrong, so $a=b$ is necessary.

The paper is organized as follows: in the second chapter we present three typical examples and give error analysis, in the third chapter, we give a further analysis of the triangle type functions concerning maximum and minimum values, and give some useful suggestions for ordinary high school students.

2. Examples

The following examples were selected from problems or examinations in ordinary high school mathematics, which vividly describe the typical error in the applications of AM-GM inequalities.

2.1. Example 1

For any $x\in(0,\pi)$, $f(x)=x^3+x$, $x\in R$, the inequality

$$f(x\sin x-1)+f(\cos x-a)\leq 0 \qquad (1)$$

is always holds, then the minimum value of the entire number a is ( ):

A. 1. B. 2. C. 3. D. 4

The error answer. Because $f(x)=x^3+x$, $x\in R$ is an odd function, hence we rewrite the inequality

$$f(x\sin x-1)+f(\cos x-a)\leq 0$$

as

$$f(x\sin x-1)\le f(a-\cos x).$$

Noting that $f(x)$ is monotone increasing on R, we know $x\sin x-1\le a-\cos x,$ so $x\sin x+\cos x-1\le a.$ Furthermore, by the knowledge of trigonometric functions, we obtain the following,

$$\begin{aligned}&x\sin x+\cos x-1\\&=\sqrt{x^2+1}\left(\frac{x}{\sqrt{x^2+1}}\sin x+\frac{1}{\sqrt{x^2+1}}\cos x\right)-1\\&=\sqrt{x^2+1}\left(\cos\theta\sin x+\sin\theta\cos x\right)-1\\&=\sqrt{x^2+1}\sin\left(x+\theta\right)-1,\end{aligned}\quad (2)$$

where $\tan\theta=\frac{1}{x}\in(\frac{1}{\pi},+\infty).$ But $-1\le\sin(x+\theta)\le 1,$ therefore, $\sqrt{1+x^2}\sin(x+\theta)\le\sqrt{1+x^2}<\sqrt{1+\pi^2}.$

So $\sqrt{1+\pi^2}-1\le a$, then the minimum value of a is 3.

Remark 1. Here we have a key question whether the function $\sin(x+\theta)$ can equal 1 or not. The error is that the conditions $x=\pi$ and $\sin(x+\theta)=1$ cannot be held at the same time. Now we give the right answer as follows.

The right answer. Noting that $f(x)=x^3+x$ is an odd function, the inequality (1) becomes

$$f(x\sin x-1)\le f(a-\cos x).$$

Since $f(x)$ is monotone increasing, we have $x\sin x-1\le a-\cos x,$ then $x\sin x+\cos x-1\le a.$

Defining $g(x)=x\sin x+\cos x-1$, we aim to compute the maximum value of $g(x)$. Differentiating $g(x)$, we have $g'(x)=\sin x+x\cos x-\sin x=x\cos x.$

Let $g'(x)=0,$ we have $x=\frac{\pi}{2}.$ Noting that if $x\in(0,\frac{\pi}{2})$, $g'(x)>0,$ so $g(x)$ is monotone increasing; and if $x\in(\frac{\pi}{2},\pi)$, $g'(x)<0$, hence $g(x)$ is monotone decreasing. Thus, the maximum value of $g(x)$ is $g(\frac{\pi}{2})=\frac{\pi}{2}-1\le a,$ then the minimum value of a equals 1.

Remark 2. The right answer used the monotonicity of the function to transform the inequality and used the derivative function to obtain the maximum value, thus obtaining the value of a.

Remark 3. When $\tan\theta=\frac{1}{x}\in(\frac{1}{\pi},+\infty)$ in (2), then $\theta\in(\arctan\frac{1}{\pi},\frac{\pi}{2}).$ But if $x+\theta=\frac{\pi}{2},$ then

$$x=\frac{\pi}{2}-\theta\in(0,\frac{\pi}{2}-\arctan\frac{1}{\pi})\approx(0,1.2626),$$

so $x<1.2626$. Then

$$\begin{aligned}&x\sin x+\cos x-1=\sqrt{x^2+1}\sin\left(x+\theta\right)-1\\&\le\sqrt{1.2626^2+1}-1\approx 0.6106.\end{aligned}$$

Therefore, in this situation, we have $a=1$.

Remark 4. According to $f(x)=x^3+x$, $x\in R$, the inequality $f(x\sin x-1)+f(\cos x-a)\le 0$ is transformed into the following form:

$$(x\sin x-1)^3+(x\sin x-1)+(\cos x-a)^3+(\cos x-a)\le 0. \quad (3)$$

But the solving of inequality (3) is very complicated, and this is not recommended, so we omit it here.

2.2. Example 2

Suppose that $\theta\in[0,\frac{\pi}{2}]$, compute the maximum value of $y=\sin^2\theta\cdot\cos\theta$.

The error answer. By the AM-GM inequality, we have

$$\begin{aligned}y&=\sin^2\theta\cos\theta\\&\le\frac{1}{2}(\sin^4\theta+\cos^2\theta)\\&=\frac{1}{2}(\sin^4\theta-sin^2\theta+1)\\&=\frac{1}{2}(\sin^2\theta-\frac{1}{2})^2+\frac{3}{8}.\end{aligned}$$

Thus when $\sin^2\theta=0$ or $\sin^2\theta=1$, $y_{max}=\frac{1}{2}$.

Remark 5. The error answer is wrong because if $\sin^2\theta=0$, or $\sin^2\theta=1$, then $y=\sin^2\theta\cos\theta=0$, but it contradicts with $y_{max}=\frac{1}{2}$.

The right answer. Because

$$y=\sin^2\theta\cos\theta=(1-\cos^2\theta)\cos\theta=\cos\theta-\cos^3\theta,$$

then $y'=\sin\theta(3\cos^2\theta-1).$ Thus we have $\sin\theta=0$ or $\cos^2\theta=\frac{1}{3}.$ Therefore, when $0<\theta<\varphi,$ where $\cos\varphi=\frac{\sqrt{3}}{3},$ we have $y'>0,$ when $\varphi<\theta<\frac{\pi}{2},$ we have $y'<0,$ therefore,

$$y_{max}=\sin^2\phi\cos\phi=\frac{2}{3}\cdot\frac{\sqrt{3}}{3}=\frac{2\sqrt{3}}{9}.$$

We shall give another right answer as follows.

By the AM-GM inequality, we have

$$\begin{aligned}y^2&=\sin^4\theta\cos^2\theta=\sin^4\theta(1-\sin^2\theta)\\&=\frac{1}{2}\sin^2\theta\sin^2\theta(2-2\sin^2\theta)\\&\le\frac{1}{2}[\frac{\sin^2\theta+\sin^2\theta+(2-2\sin^2\theta)}{3}]^3=\frac{4}{27}.\end{aligned}$$

Hence $y_{max}^2=\frac{4}{27},$ thus $y_{max}=\frac{2\sqrt{3}}{9},$ where

$$\sin^2\theta=2-2\sin^2\theta, \textit{and } \sin^2\theta=\frac{2}{3}.$$

2.3. Example 3

Compute the maximum value of the function

$$f(x)=\cos^3 x+\sin^2 x-\cos x.$$

The wrong answer.

$$\begin{aligned} f(x)&=\cos^3 x+\sin^2 x-\cos x\\ &=(\cos x-1)\left(\cos^2 x-1\right)\\ &=(1-\cos x)^2(\cos x+1)\\ &\le\left(\frac{1-\cos x+1-\cos+\cos x+1}{3}\right)^3\\ &=\frac{(3-\cos x)^3}{27}\le\frac{64}{27}. \end{aligned} \tag{4}$$

Remark 6. In the wrong answer above, the necessary condition of AM-GM inequalities has been missed. Here should be $1-\cos x=1+\cos x$, it follows that $\cos x=0$. Therefore the maximum value of $f(x)=0$, but it is wrong.

The right answer. By AM-GM inequalities, we have the following.

$$\begin{aligned} f(x)&=\cos^3 x+\sin^2 x-\cos x\\ &=(1-\cos x)^2(\cos x+1)\\ &=4(\cos x+1)\cdot\frac{1-\cos x}{2}\cdot\frac{1-\cos x}{2}\\ &\le 4\cdot\left(\frac{\cos x+1+\frac{1-\cos x}{2}+\frac{1-\cos x}{2}}{3}\right)^3\\ &=\frac{32}{27}. \end{aligned} \tag{5}$$

Remark 7. In (4), while applying the AM-GM inequalities, we should convert the product of

$$4(\cos x+1)\cdot\frac{1-\cos x}{2}\cdot\frac{1-\cos x}{2}$$

to a fixed constant number, such as

$$\cos x+1+\frac{1-\cos x}{2}+\frac{1-\cos x}{2}=2,$$

$$1-\cos x+1-\cos x+2+2\cos x=4,$$

otherwise yields the wrong results, for instance, see (4), $1-\cos x+1-\cos+\cos x+1=3-\cos x$ which is not a constant number. Another solution is as follows:

$$\begin{aligned} f(x)&=\cos^3 x+\sin^2 x-\cos x\\ &=(1-\cos x)^2(\cos x+1)\\ &=\frac{1}{2}\cdot(1-\cos x)(1-\cos x)\cdot 2\cdot(1+\cos x)\\ &\le\frac{1}{2}\cdot\left(\frac{1-\cos x+1-\cos x+2+2\cos x}{3}\right)^3\\ &=\frac{1}{2}\left(\frac{4}{3}\right)^3=\frac{32}{27}. \end{aligned}$$

3. Conclusions

We discuss three typical examples aiming to find the extremum of a giving function that involves triangle-type functions in high school mathematics teaching and give some useful suggestions for the high school students who are hard to understand the problems of AM-GM inequalities and their applications. Students should pay much attention to the domain of triangle type functions in the problems and the necessary and sufficient conditions of AM-GM inequalities. Given a function $y=f(x)$ on a given interval, to find the maximum or minimum, the general method is to use the algebraic and geometric properties of the function, or use derivatives to obtain the monotonicity of the function on the intervals. Furthermore, it should be noted that if the function is composed of two functions, e. x.,

$$f(x)=s(x)\pm t(x),$$
$$f(x)=s(x)t(x),$$
$$f(x)=s(x)/t(x),$$

$$f(x)=s(t(x)).$$

But the extremum of function $f(x)$ is not necessarily equal to the sum, the difference, the product, the quotient, or the composition of the maximum or minimum values of the function $s(x)$ and $t(x)$.

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