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Some Fixed Point Results in S-Metric Spaces

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Abstract In this paper, we prove some fixed point results on complete S-metric spaces. Our results extend and improve some recent results in the references.

Keywords: *Metric space, S-metric space, fixed point*

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1. Introduction

In 2006, Z. Mustafa and B. I. Sims [6] introduced the concept of G-metric space which is a generalization of metric space, and proved some fixed point theorems in G-metric space. Subsequently, many authors were proved fixed point theorems in G- metric space (see, eg. [3,7,11]). And B. C. Dhage [4] introduced the notion of D-metric space. In 2007, S. Sedghi, N. Shobe and H. Zhou [10] introduced D^*- metric space which is a modification of D-metric space of [4] and proved some fixed point theorems in D^*- metric space and later on many authors were proved fixed point theorems in D^*- metric space (see, e.g. [1,5]). In 2012, S. Sedghi et al. [9] introduced the notion of S-metric space which is a generalization of G-metric space of [4] and D^*- metric space of [10] and proved some fixed point theorems on S-metric space. Recently, S. Sedghi, N.V. Dung [8] proved generalized fixed point theorems in S-metric spaces which is a generalization of [9]. In this paper, we proved some fixed point results on complete S-metric spaces. Our results extended and improved the results of [8].

2. Preliminaries

2.1. [2] Definition

Let X be a nonempty set. A metric on X is a function d: $X^2 \to [0, \infty)$ if there exists a real number $b \geq 1$ such that the following conditions holds for all x, y, z ∈ X.

(i) d(x, y) = 0 if and only if x = y.
(ii) d(x, y) = d(y, x).
(ii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.
The pair (X, d) is called a B-metric space.

2.2. [9] Definition

Let X be a nonempty set. An S-metric on X is a function S: $X^3 \to [0, \infty)$ that satisfies the following conditions holds for all x, y, z, a ∈ X.

(i) S(x, y, z) = 0 if and only if x = y = z.
(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

2.3. [9] Definition

Let (X, S) be an S-metric space. For r > 0 and x ∈ X, we define the open ball $B_S(x, r)$ and the closed ball $B_S[x,r]$ with centre x and radius r as follows

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

The topology induced by the S-metric is the topology generated by the base of all open balls in X.

2.4. [9] Definition

Let (X, S) be an S-metric space. A sequence $\{x_n\} \subset X$ converges to x ∈ X if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write for $x_n \to x$.

3. Main Results

In this section, we have proved some fixed point theorems on complete S- metric spaces.

S. Sedghi, N.V. Dung [8] introduced an implicit relation to investigate some fixed point theorems on S-metric spaces.

Let $\mathcal{M}$ be the family of all continuous functions of five variables

M: $\mathbb{R}_+^5 \to \mathbb{R}_+$ for some k∈[0,1). We consider the following conditions.

(C_1) For all x, y, z ∈ $\mathbb{R}_+$, if $y \leq M(x, x, 0, z, y)$ with $z \leq 2x + y$, then $y \leq kx$.
(C_2) For all y∈ $\mathbb{R}_+$, if $y \leq M(y, 0, y, y, 0)$, then y = 0.
(C_3) If $x_i \leq y_i + z_i$ for all $x_i, y_i, z_i \in \mathbb{R}_+$, i≤5, then

$$M(x_1, x_2, x_3, x_4, x_5) \leq M(y_1, y_2, y_3, y_4, y_5) + M(z_1, z_2, z_3, z_4, z_5).$$

Moreover, for all y∈X, $M(0, 0, 0, y, 2y) \leq ky$.

The following theorem was proved in [8] (Theorem 2.6 of [8]).

3.1. [9] Theorem

Let T be a self-map on a complete S-metric space (X, S) and

$$S(Tx,Tx,Ty) \le M\begin{pmatrix} S(x,x,y), S(Tx,Tx,x), \\ S(Tx,Tx,y), \\ S(Ty,Ty,x), S(Ty,Ty,y) \end{pmatrix}$$

for all x, y, z ∈ X and some M∈$\mathcal{M}$. Then we have

(i) If M satisfies the condition (C_1), then T has a fixed point. Moreover, for any x_0∈X and the fixed point x, we have $S(Tx_n, Tx_n, x) \le (2k^n / 1\text{-}k) S(x_0, x_0, Tx_0)$.

(ii) If M satisfies the condition (C_2) and T has a fixed point, then the fixed point is unique.

(iii) If M satisfies the condition (C_3) and T has a fixed point, then T is continuous at x.

3.2. Theorem

Let T be a self-map on a complete S-metric space (X, S) and

$$S(Tx,Tx,Ty) \le \alpha S(x,x,y) + \beta\begin{bmatrix} S(Tx,Tx,x) \\ +S(Ty,Ty,y) \end{bmatrix}$$

for some α, β ≥0 such that α +2β<1 and for all x, y ∈ X. Then T has a unique fixed point in X. Moreover, if 2β<1, then T is continuous at the fixed point.

Proof: The following ascertain is by using the Theorem 3.1 with

$$M(x,y,z,s,t) = \alpha x + \beta(y+t)$$
$$\text{for some } \alpha, \beta \ge 0, \alpha + 2\beta < 1$$

and for all x, y, z, s, t∈$\mathbb{R}_+$. Indeed, M is continuous. First, we have,

$$M(x,x,0,z,y) = \alpha x + \beta(x+y) = \alpha x + \beta x + \beta y.$$

So, if y ≤ M(x, x, 0, z, y) with z ≤ 2x + y, then

$$y \le \alpha x + \beta x + \beta y$$
$$\le (\alpha+\beta)x + \beta y.$$
$$\Rightarrow (1-\beta)y \le (\alpha+\beta)x.$$
$$\Rightarrow y \le (a+b/1-b)x \text{ with } (a+b/1-b) < 1.$$

Therefore, T satisfies the condition (C_1).

Next, if

$$y \le M(y,0,y,y,0) = \alpha y + \beta(0+0) = \alpha y,$$

then y = 0. Since α <1.

Therefore, T satisfies the condition (C_2).

Finally, if $x_i \le y_i + z_i$ for i≤5, then

$$M(x_1,x_2,x_3,x_4,x_5) = \alpha x_1 + \beta(x_2+x_5)$$
$$= \alpha(y_1+z_1) + \beta[(y_2+z_2)+(y_5+z_5)]$$
$$\le (\alpha y_1 + \beta(y_2 + y_5)) + (\alpha z_1 + \beta(z_2+z_5))$$
$$= M(y_1,y_2,y_3,y_4,y_5) + M(z_1,z_2,z_3,z_4,z_5).$$

More over, M(0, 0, 0, y, 2y) = 0 + β(0 + 2y)= 2βy, where 2β <1.

Therefore, T satisfies the condition (C_3).

3.3. Theorem

Let T be a self-map on a complete S-metric space (X, S) and

$$S(Tx,Ty,Ty) \le \alpha\ S(x,x,y) + \beta\begin{bmatrix} S(Tx,Tx,y) \\ +S(Ty,Ty,x) \end{bmatrix}$$

for some α, β ≥ 0 such that α+β<1 and for all x, y ∈ X. Then T has a unique fixed point in X. Moreover, if 2β <1, then T is continuous at the fixed point.

Proof: The following ascertain is by using the Theorem 3.1 with M(x, y, z, s, t) = αx + β(z + t) for some α, β≥0, α+β<1 and for all x, y, z, s, t ∈ $\mathbb{R}_+$. Indeed, M is continuous. First, we have

$$M(x,x,0,z,y) = \alpha x + \beta(0+y).$$
$$= \alpha x + \beta y.$$

So, if y ≤ M(x, x, 0, z, y) with z ≤ 2x + y, then

$$y \le \alpha x + \beta y$$
$$\Rightarrow (1-\beta)y \le \alpha x.$$
$$\Rightarrow y \le (a/1-b)x, \text{ with}(a/1-b) < 1.$$

Therefore, T satisfies the condition (C_1).

Next, if

$$y \le M(y,0,y,y,0) = \alpha y + \beta(y+0)$$
$$= \alpha y + \beta y,$$
$$= (\alpha+\beta)y,$$

then y = 0. Since α +β<1.

Therefore, T satisfies the condition (C_2).

Finally, if $x_i \le y_i + z_i$ for i≤5, then

$$M(x_1,x_2,x_3,x_4,x_5) = \alpha x_1 + \beta(x_3+x_5)$$
$$= \alpha(y_1+z_1) + \beta[(y_3+z_3)+(y_5+z_5)]$$
$$\le (\alpha y_1 + \beta(y_3+y_5)) + (\alpha z_1 + \beta(z_3+z_5))$$
$$= M(y_1,y_2,y_3,y_4,y_5) + M(z_1,z_2,z_3,z_4,z_5).$$

More over, M(0, 0, 0, y, 2y) = 0 + β(0 + 2y)= 2βy, where 2β <1.

Therefore, T satisfies the condition (C_3).

3.4. Theorem

Let T be a self-map on a complete S-metric space (X, S) and

$$S(Tx,Ty,Tz) \le \alpha S(x,x,y) + \beta\begin{bmatrix} S(Tx,Tx,x) \\ +S(Ty,Ty,y) \end{bmatrix}$$
$$+\gamma[(S(Tx,Tx,y) + S(Ty,Ty,x)]$$

for some α, β, γ ≥0 such that α+2β+3γ<1 and for all x, y ∈ X. Then T has a unique fixed point in X. Moreover, if 2β + γ<1, then T is continuous at the fixed point.

Proof: The following ascertain is by using the Theorem 2.1 with

$$M(x,y,z,s,t) = \alpha x + \beta(y+t) + \gamma(z+s)$$

for some α, β, γ ≥0, α+2β+3γ<1.

Indeed, M is continuous. First, we have

$$M(x,x,0,z,y) = \alpha x + \beta(x+y) + \gamma(0+z)$$
$$= \alpha x + \beta(x+y) + \gamma z.$$

So, if y≤ M(x, x, 0, z, y) with z ≤ 2x + y, then

$$y \le \alpha x + \beta x + \beta y + \gamma z$$
$$\le \alpha x + \beta x + \beta y + \gamma(2x+y)$$
$$\le (\alpha+\beta+2\gamma)x + (\beta+\gamma)y$$
$$\Rightarrow (1-(\beta+\gamma))y \le (\alpha+\beta+2\gamma)x$$
$$\Rightarrow y \le (a+b+2g/1-b-g)x,$$
$$\text{with } (a+b+2g/1-b-g) < 1.$$

Therefore, T satisfies the condition (C_1).
Next, if

$$y \le M(y,0,y,y,o) = \alpha y + \beta(0+0) + \gamma(y+y)$$
$$= \alpha y + 2\gamma y$$
$$= (\alpha + 2\gamma)y.$$

Then, y = 0. Since $\alpha + 2\gamma < 1$.
Therefore, T satisfies the condition (C_2).
Finally, if $x_i \le y_i + z_i$ for $i \le 5$, then

$$M(x_1,x_2,x_3,x_4,x_5) = \alpha x_1 + \beta(x_2+x_5) + \gamma(x_3+x_4)$$
$$= \alpha(y_1+z_1) + \beta[(y_2+z_2)+(y_5+z_5)]$$
$$+\gamma[(y_3+z_3)+(y_4+z_4)]$$
$$\le [\alpha y_1 + \beta(y_2+y_5) + \gamma(y_3+y_4)]$$
$$+[\alpha z_1 + \beta(z_2+z_5) + \gamma(z_3+z_4)]$$
$$= M(y_1,y_2,y_3,y_4,y_5) + M(z_1,z_2,z_3,z_4,z_5).$$

More over,

$$M(0,0,0,y,2y) = 0 + \beta(0+2y) + \gamma(0+y)$$
$$= 2\beta y + \gamma y$$
$$= (2\beta+\gamma)y, \text{ where } 2\beta+\gamma<1.$$

Therefore, T satisfies the condition (C_3).

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