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Approximate Controllability of Fractional Sobolev Type Stochastic Differential Equations Driven by Mixed Fractional Brownian Motion

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Abstract In this paper, the approximate controllability of nonlinear Fractional Sobolev type with order Caputo $\frac{1}{2} < \alpha < 1$ stochastic differential equations driven by mixed fractional Brownian motion in a real separable Hilbert spaces has been studied by using contraction mapping principle, fixed point theorem, stochastic analysis theory, fractional calculus and some sufficient conditions.

Keywords: *approximate controllability, mixed fractional brownian motion, fixed point contraction principle, stochastic differential equations, mild solution, control function*

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1. Introduction

The purpose of this paper is to prove the existence and approximate controllability of mild solution for the class of fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion with Hurst $H > \frac{1}{2}$ and wiener process. The following form is the system under our consideration,

$$^{C}D_t^{\alpha}\left[\hat{S}x(t) - h(t, x(t))\right] = Lx(t) + Bu(t) + F(t, x(t)) + G_1(t, x(t))\frac{dW_1(t)}{dt} + \sigma_1(t)\frac{dW_1^H(t)}{dt} \quad (1)$$

$$^{L}D^{1-\alpha}x(t)|_{t=0} = G_2(t, x(t))\frac{dW_2(t)}{dt} + \sigma_2(t)\frac{dW_2^H(t)}{dt}$$

$t \in [0,T]$, where, $x(t) \in C([0,T]; L^2(\Omega, X))$ equipped with the sup norm $\|x\|_C = (\sup_{t\in[0,T]} E\|x(t)\|^2)^{\frac{1}{2}}$, such that X is a real separable Hilbert space. $^{C}D_t^{\alpha}$ the Caputo fractional derivative of order $\frac{1}{2} < \alpha < 1$ and $^{L}D_t^{1-\alpha}$ the Riemann-Liouville fractional derivative of order $1-\alpha$. The operators $\hat{S}$ and L are defined on domains contained in X, $D(\hat{S}) \subset D(L)$ and ranges contained in a real separable Hilbert space Z, such that $\hat{S}$ is a bijective linear operator, $\hat{S}^{-1}$ is a compact and L is a closed linear operator. The control function $u(.) \in L^2_{\mathfrak{F}}([0,T]; U)$, U is a Hilbert space and the operator B from U into Z is a bounded linear operator. The functions $F:[0,T]\times X \to Z$, $G_1: [0,T] \times X \to L_2(K, Z)$, $G_2: [0,T] \times X \to L_2(K, X)$, $\sigma_1: [0,T] \to L_2^0(Y, Z)$, $\sigma_2: [0,T] \to L_2^0(Y,X)$ and $h: [0,T] \times X \to Z$ are continuous functions such that K and Y are real separable Hilbert spaces.

$W_1 = \{ W_{1(t)}, t \in [0,T]\}$ and $W_1 = \{ W_{2(t)}, t \in [0,T]\}$ are the standard cylindrical Brownian motion (cylindrical wiener process) defined on complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$ equipped with normal filtration $\{\mathfrak{F}_t\}_{t\geq 0}$, $\mathfrak{F}_t$ is the sigma algebra generated by $\{ W_1(s), W_2(s), W_1^H(s), W_2^H(s) : 0 \leq s \leq t\}$.

Let Q be a positive, self –adjoint and trace class operator on K and let $L_2(K,X)$ be the space of all Q - Hilbert-Schmidt operators acting between K and X equipped with the Hilbert-Schmidt norm $\|.\|_{L_2}$

$W_1^H = \{W_1^H{}_{(t)}, t \in [0,T]\}$ and $W_2^H = \{ W_2^H{}_{(t)}, t \in [0,T] \}$ are the Q-fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$ with values in a real separable Hilbert space Y, such that Q is a positive, self –adjoint and trace class operator on Y and let $L_2^0(Y;X)$ be the space of all Q -Hilbert-Schmidt operators acting between Y and X equipped with the Hilbert-Schmidt norm $\|.\|_{L_2^0}$. x_0, W_1, W_2, W_1^H and W_2^H are independents which defined on a complete probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, P)$.

Approximate controllability of stochastic differential equations driven by fractional Brownian motion has been interested by many authors; Sakthivel [19] referred to future study for the approximate controllability of impulsive stochastic systems with fractional Brownian motion. Guendouzi and Idrissi, [7] established and discussed the approximate controllability result of a class of dynamic control systems described by nonlinear fractional stochastic functional differential equations in Hilbert space driven by fractional Brownian motion with

Hurst parameter $H > \frac{1}{2}$. Ahmed [2] investigate the approximate controllability problem for the class of impulsive neutral stochastic functional differential equations with finite delay and fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ in a Hilbert space. Abid, Hasan and Quaez [1] studied the Approximate controllability of fractional stochastic integro-differential equations which is derived by mixed type of fractional Brownian motion with Hurst parameter H >12 and wiener process in real separable Hilbert space. On the other hand, Sobolev type differential equations have been investigated by many authors, for example, Balachandran, Kiruthika and Trujillo [3] established the existence of solutions of nonlinear impulsive fractional integrodifferential equations of Sobolev type with nonlocal condition. Zhou, Wang and Feckan [20] investigated a class of Sobolev type semilinear fractional evolution systems in separable Banach space. Kerboua, Debbouche and Baleanu [8] sudied the approximate controllability of Sobolev type fractional stochastic nonlocal nonlinear differential equations in Hilbert spaces.

In this paper we will study the approximate controllability of nonlinear stochastic system. More precisely, we shall formulate and prove sufficient conditions for the Approximate controllability of fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion with Hurst $H > \frac{1}{2}$ and wiener process in Hilbert space.

The rest of this paper is organized as follows, in section 2, we will introduced some concepts, definitions and some lemmas of fractional stochastic calculus which are useful for us here. In section 3, we will prove our main result.

2. Preliminaries

In this section, we introduce some notations and preliminary results, which we needed to establish our results.

***Definition (2.1)*, [5]:**

Let H be a constant belonging to (0, 1). A one dimensional fractional Brownian motion $B^H = \{B^H_{(t)}, t \geq 0\}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E\left(B^H_{(t)} B^H_{(s)}\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \text{ for } t,s \geq 0.$$

- If $H = \frac{1}{2}$, then the increments of B^H are non-correlated, and consequently independent. So B^H is a Wiener Process which we denote further by B.
- If $H \in (\frac{1}{2}, 1)$ then the increments are positively correlated.
- If $H \in (0, \frac{1}{2})$ then the increments are negative correlated.

B^H has the integral representation

$$B^H_{(t)} = \int_0^t K_H(t,s)\, dB_{(s)} \tag{2}$$

where, B is a wiener process and the kernel $K_H(t,s)$ defined as

$$K_H(t,s) = c_H\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \tag{3}$$

$$\frac{\partial K}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \tag{4}$$

$c_H = \left[\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}\right]^{\frac{1}{2}}$, $t > s$ and β is a beta function.

In the case $H = \frac{1}{2}$, we shall use Ito Isometry theorem

***Lemma (2.1)*, "Ito isometry theorem", [11]:**

Let V [0,T] be the class of functions such that $f: [0,T] \times \Omega \to R$, f is measurable, $\mathfrak{F}_t$- adapted and $E\left[\int_0^T (f(t,\omega))^2 dt\right] \leq \infty$. Then for every $f \in V$ [0,T], we have

$$E\left[\int_0^T f(t,\omega) dB(t)\right]^2 = E\left[\int_0^T (f(t,\omega))^2 dt\right] \tag{5}$$

where B is a wiener process.

Now, we denote by ε the set of step functions on [0, T].If $\Phi \in \varepsilon$ then, we can write it the form as:

$$\Phi(t) = \sum_{k=1}^{n} a_k\, 1_{[t_k, t_{k+1}]}(t), \text{ where } t \in [0,T].$$

The integral of a step function $\Phi \in \varepsilon$ with respect to one dimensional fractional Brownian motion is defined

$$\int_0^T \Phi(t) dB_t^H = \sum_{k=1}^{n} a_k \left(B^H_{t_{k+1}} - B^H_{t_k}\right),$$

where $a_k \in R$,

$$0 = t_1 < t_2 < \ldots\ldots < t_{n+1} = T.$$

Let $\mathcal{H}$ be the Hilbert space defined as the closure of ε with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle = R_H(t,s) = E(B_t^H B_s^H)$. The mapping $1_{[0,t]} \to \{B^H(t), t \in [0,T]\}$ can be extended to an isometry between $\mathcal{H}$ and span$^{L^2(\Omega)}$ $\{B^H_{(t)}, t \in [0,T]\}$ .i.e. the mapping $\mathcal{H} \to L^2(\Omega, \mathfrak{F}, P)$, $\Phi \to \int_0^T \Phi(t) dB_t^H$ is isometry.

Remark (2.1):

- If $H = \frac{1}{2}$ and $\mathcal{H} = L^2([0,T])$ then by use Ito isometry, we have

$$E\left(\int_0^T \Phi(t) dB\right)^2 = \int_0^T (\Phi(t))^2 dt \tag{6}$$

- If $H > \frac{1}{2}$, we have

$$R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \; t, s \geq 0 \tag{7}$$

$$\frac{\partial R_H}{dt} = H\left(|t|^{2H-1} - |t-s|^{2H-1}\right) \tag{8}$$

$$\partial R_H^2 = H(2H-1)|t-s|^{2H-2} dsdt$$

***Lemma (2.2)*, [6]:**

For any functions $\Phi, \varphi \in L^2[0,T] \cap L^1[0,T]$, we have

i) $E\left(\int_0^T \Phi(t) dB^H_{(t)} \int_0^T \varphi(s) dB^H_{(s)}\right) = H(2H-1) \times \int_0^T \cdot \int_0^T \Phi(t)\, \varphi(s) |t-s|^{2H-2} dsdt$

ii) $E\left(dB^H_{(t)} dB^H_{(s)}\right) = \frac{\partial R_H^2}{dsdt} = H(2H-1)|t-s|^{2H-2} dsdt$

From this Lemma above, we obtain

$$E\left(\int_0^T \Phi(t) dB^H_{(t)}\right)^2 = H(2H-1) \int_0^T \int_0^T \left[\frac{\Phi(s)\Phi(t)}{|t-s|^{2H-2}}\right] dsdt \tag{9}$$

***Remark (2.2)*, [6]:**

The space $\mathcal{H}$ contains the set of functions Φ, such that, $\int_0^T \cdot \int_0^T \Phi(s)\Phi(t) |t-s|^{2H-2} dsdt < \infty$, which includes $L^{\frac{1}{H}}([0,T])$.

Now,

Let $\hat{\mathcal{H}}$ be the Banach space of measurable functions on [0, T], such that

$$\|\Phi\|^2_{\hat{\mathcal{H}}}=H(2H-1)\int_0^T\int_0^T \Phi(s)\Phi(t)\,|t-s|^{2H-2}\,dsdt<\infty \quad (10)$$

Lemma (2.3), [10]:

$$L([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset \hat{\mathcal{H}} \subset \mathcal{H}.$$

Suppose that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in Y. Let Q $\in$L (Y,Y) be the operator defined by $Qe_n=\lambda_n e_n$, where $\lambda_n \geq 0$ (n=1,2,….) are non-negative real numbers with finite trace $\operatorname{Tr} Q=\sum_{n=1}^{\infty}\lambda_n<\infty$.The infinite dimensional fractional Brownian motion on Y can be defined by using covariance operator Q as

$$W^H_{(t)}=W^H_Q{}_{(t)}=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\,e_n\,B^H_{n\,(t)},$$

where $B^H_{n\,(t)}$ are one dimensional fractional Brownian motions mutually independent on $(\Omega,\mathfrak{F},P)$.

In order to defined stochastic integral with respect to the Q-fractional Brownian motion. We introduce the space L_2^0 (Y,X) of all Q-Hilbert- Schmidt operators that is with the inner product $\langle\Phi,\varphi\rangle_{L_2^0}=\sum_{n=1}^{\infty}\langle\Phi e_n,\varphi e_n\rangle$ is a separable Hilbert space.

Lemma (2.4), [10]:

Let $\{\Phi(t)\}_{t\in[0,T]}$ be a deterministic function with values in L_2^0 (Y,X) The stochastic integral of Φ with respect to W^H is defined by

$$\int_0^t\Phi(s)dW^H_{(s)}=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}\,\Phi(s)e_n dB^H_{n(s)}$$
$$=\sum_{n=1}^{\infty}\int_0^t\sqrt{\lambda_n}\,(K_H^*(\Phi e_n)(s)dB_{n(s)} \quad (11)$$

Lemma (2.5), [10]:

If φ: [0, b] $\rightarrow L_2^0(Y,X)$ satisfies $\int_0^T\|\varphi(s)\|^2_{L_2^0}ds<\infty$ then the above sum in (11) is well defined as an X-valued random variable and we have

$$E\left\|\int_0^t\varphi(s)dW^H_{(S)}\right\|^2\leq 2H\,t^{2H-1}\int_0^t\|\varphi(s)\|^2_{L_2^0}\,ds \quad (12)$$

Definition (2.2), [18]:

The Riemann - Liouvill derivative of order $\alpha>0$ with lower limit zero for a function f can be written as:

$$^{L}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\alpha+1-n}} \quad (13)$$

where, $t>0,\ n-1<\alpha<n$.

Definition (2.3), [18]:

The Caputo derivative of order $\alpha>0$ with lower zero for a function f can be written as:

$$^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\,ds \quad (14)$$

where, $t>0, n-1<\alpha<n$.

Remark (2.3), [9]:

The relationship between the two definitions Riemann – Liouvill derivative and Caputo derivative gives as:

$$^{C}D_t^{\alpha}f(t)=\,^{L}D_t^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right) \quad (15)$$

where, $t>0,\ n-1<\alpha<n$.

Definition (2.4), [18]:

The Laplace transform of the Riemann-Liouville fractional derivation of order $\alpha>0$ gives as:

$$\mathcal{L}\{LD_t^{\alpha}f(t)\}=\lambda^{\alpha}\mathcal{L}(f(t))(\lambda)-\sum_{k=0}^{n-1}\lambda^k[D_t^{\alpha-k-1}f(t)]_{t=0}(\lambda) \quad (16)$$

where, n-1< α <n.

Definition (2.5), [18]:

The Laplace transform of the caputo derivation of order α $>$ 0 is given as:

$$\mathcal{L}\{^{C}D_t^{\alpha}f(t)\}=\lambda^{\alpha}\mathcal{L}(f(t))(\lambda)-\sum_{k=0}^{n-1}\lambda^{\alpha-k-1}f^{(k)}(0) \quad (17)$$

where, n-1< α <n.

Remarks (2.4)

i) The operator M: $D(\hat{S})\subseteq X\rightarrow Z$ is a bijective linear operator, then $\hat{S}^{-1}$: $Z\rightarrow D(\hat{S})\subseteq X$ is a a bijective linear

ii) $\hat{S}^{-1}$ is a compact linear operator, we obtain that $\hat{S}^{-1}$ is bounded.

iii) $\hat{S}^{-1}$ is a bounded and L is a closed linear operator by (closed graph theorem), we obtain the boundedness of linear operator $L\hat{S}^{-1}$:Z→ Z.

vi) The operator $L\hat{S}^{-1}$ is bounded. Then, $L\hat{S}^{-1}$ is an infinitesimal generator of semigroup $\{T(t)=e^{L\hat{S}^{-1}t}, t\geq 0\}$. Suppose that $\sup_{t\geq 0}\|T(t)\|_Z=\mathfrak{m}<\infty$. (see[4])

Definition (2.6):

An X-valued process $x(t)\in C([0,T];L^2(\Omega,X))$ is a mild solution of the stochastic differential equation driven by mixed fractional Brownian motion in (1) if, for each control function $u(t)\in L^2_{\mathfrak{F}}([0,T];U)$, the integral equation

$$\begin{aligned}x(t)=\breve{T}_{\alpha}(t)\hat{S}&\left[x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t\left[\frac{(t-s)^{-\alpha}}{G_2(t,x(s))}\right]dW_2(s)+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\right]\\&-\breve{T}_{\alpha}(t)h(0,x(0))+\hat{S}^{-1}h(t,x(t))\\&+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}Bu(s)ds\\&+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}F(s,x(s))ds\\&+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}G_1(s,x(s))dW_1(s)\\&+\int_0^t T_{\alpha}(t-s)(t-s)^{\alpha-1}\sigma_1(t)dW_1^H(s)\end{aligned} \quad (18)$$

is satisfied.

where, the operators $\breve{T}_{\alpha}(t)$ and $T_{\alpha}(t)$ are given by

$$\breve{T}_{\alpha}(t)=\int_0^{\infty}\hat{S}^{-1}M_{\alpha}(r)T(t^{\alpha}r)\,dr \quad (19)$$

$$T_{\alpha}(t)=\int_0^{\infty}\alpha\hat{S}^{-1}rM_{\alpha}(r)T(t^{\alpha}r)\,dr \quad (20)$$

$M_{\alpha}(r)$ is a Mainardi's function.

Lemma (2.6):

If $\{\breve{T}(t), t\geq 0\}$ is a strongly continuous semigroup by linear operator $L\hat{S}^{-1}: Z\rightarrow Z$, then the operators $\{\breve{T}_{\alpha}(t), t\geq 0\}$ and $\{T_{\alpha}(t), t\geq 0\}$ have the following properties:

i. For any fixed $t\geq 0$, the operators$\breve{T}_{\alpha}(t)$ and $T_{\alpha}(t)$ are linear and bounded, i.e. for any $z\in Z$, there exists $\mathfrak{m}>1$ such that

$$\|T_{\alpha}(t)z\|\leq\frac{C_1\mathfrak{m}}{\Gamma(\alpha)}\|z\|_Z,\ \|\breve{T}_{\alpha}(t)z\|\leq C_1\mathfrak{m}\|z\|_Z,$$

where, $\|\hat{S}^{-1}\|_X = C_1$.

ii. The operators$\{\breve{T}_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous, which mean that for every $z \in Z$ and $0 \leq t_1 < t_2 \leq T$, we have

$$\|\breve{T}_\alpha(t_2)z - \breve{T}_\alpha(t_1)z\|_X \to 0$$

$$\|T_\alpha(t_2)z - T_\alpha(t_1)z\|_X \to 0, \text{ if } \ t_2 \to t_1$$

iii. $T_\alpha(t)$ is a compact operator in X for each $t > 0$.

3. Main Result of the Approximately Controllable

In this section, we formulate and prove the result on approximate controllability of nonlinear fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion in (1). To establish our results, we introduce the following assumptions:

a) The semigroup $T(t), t \geq 0$ which generated by linear operator $L\hat{S}^{-1}$is a strongly continuous and it is compact for any $t > 0$.

b) The Sobolev type linear fractional order Caputo type system of corresponding to the system (1) of the following form:

$$\left.\begin{aligned} &{}^{C}D_t^{\alpha}\left[\hat{S}x(t)\right] = Lx(t) + Bu(t), t \in [0,T] \\ &x(0) = x_0, \frac{1}{2} < \alpha < 1 \end{aligned}\right\} \quad (21)$$

is approximately controllable on [0,T].

c) The function $\sigma_1 : [0,T] \to L_2^0(Y;Z)$ satisfies : for every $t \in [0, T]$, $\int_0^t \|\sigma_1(s)\|_{L_2^0}^2 ds < \infty$, and there exists $D_1 > 0$ such that $\sup_{t \in [0,T]} \|\sigma_1(t)\|_{L_2^0}^2 \leq D_1$.

d) The function $\sigma_2 : [0,T] \to L_2^0(Y;X)$ satisfies *:* for every $t \in [0, T]$, $\int_0^t \|\sigma_2(s)\|_{L_2^0}^2 ds < \infty$, and there exists $D_2 > 0$ such that $\sup_{t \in [0,T]} \|\sigma_2(t)\|_{L_2^0}^2 \leq D_2$.

e) The functions $F:[0,T] \times X \to Z$, $h:[0,T] \times X \to Z$, $G_1 : [0,T] \times X \to L_2(K, Z)$ and $G_2 : [0,T] \times X \to L_2(K, X)$ satisfy linear growth and Lipschitz conditions. This mean that, for any $x, y \in X$, there exists positive constants $K_1, K_2 > 0$, $K_3, K_4 > 0$, $K_5, K_6 > 0$ and $K_7, K_8 > 0$ such that

$$\|F(t,x) - F(t,y)\|_Z^2 \leq K_1\|x - y\|_X^2,$$

$$\|F(t,x)\|_Z^2 \leq K_2(1 + \|x\|_X^2)$$

$$\|G_1(t,x) - G_1(t,y)\|_{L_2}^2 \leq K_3\|x - y\|_X^2$$

$$\|G_1(t,x)\|_{L_2}^2 \leq K_4(1 + \|x\|_X^2)$$

$$\|G_2(t,x) - G_2(t,y)\|_{L_2}^2 \leq K_5\|x - y\|_X^2,$$

$$\|G_2(t,x)\|_{L_2}^2 \leq K_6(1 + \|x\|_X^2)$$

$$\|h(t,x) - h(t,y)\|_Z^2 \leq K_7\|x - y\|_X^2,$$

$$\|h(t,x)\|_Z^2 \leq K_8(1 + \|x\|_X^2)$$

Definition (3.1):

The system (1) is said to be approximately controllable on [0,T] if the reachable set $\mathfrak{R}(T)$ is dense in the space $L^2(\Omega, X)$. This mean that $\overline{\mathfrak{R}(T)} = L^2(\Omega, X)$. where, $\mathfrak{R}(T) = \{x(T,u) : u \in L^2_{\mathfrak{F}}([0,T]; U)\}$

Remark (3.1), [12]:

The linear fractional order system (21) of the corresponding system (1) is a natural generalization of approximate controllability of linear first order control system.

Now,

The controllability operator Γ_s^T associated with equation (21) is defined by

$$\Gamma_s^T = \int_s^T (T - t)^{\alpha-1} T_\alpha(T - t) BB^* T_\alpha^*(T - t) dt \quad (22)$$

Also, for any $\varepsilon > 0$ and $0 \leq s < T$, the operator $R(\varepsilon, \Gamma_s^T)$ is defined by

$$R(\varepsilon, \Gamma_s^T) = (\varepsilon I + \Gamma_s^T)^{-1} \quad (23)$$

where, $B^*$and T_α^* are the adjoint operators for B and $\breve{T}_\alpha$ respectively.

Lemma (3.1), [12]**:**

The Sobolev type linear fractional order deterministic system in (21) is an approximately controllable on [0, T] if and only if the operator $\varepsilon R(\varepsilon, \Gamma_0^T) \to 0$ as $\varepsilon \to 0^+$. Moreover $\|\varepsilon R(\varepsilon, \Gamma_s^T)\| \leq 1$.

Lemma (3.2), [13]:

For any $x_T \in L^2(\Omega, \mathfrak{F}_T, X)$ there exists $\hat{H} \in L^2(\Omega; L^2([0,T]; L_2^0(Y,Z)))$ and

$$\phi \in L^2\left(\Omega; L^2\left([0,T]; L_2(K;Z)\right)\right),$$

such that

$$x_T = Ex_T + \int_0^T \phi(s) dW_{(s)} + \int_0^T \hat{H}(s) dW_{(s)}^H \quad (24)$$

where, $\sup_{t \in [0,T]} E\|\phi(t)\|_{L_2}^2 \leq \breve{C}_1$,

$$\sup_{t \in [0,T]} E\|\hat{H}(t)\|_{L_2^0}^2 \leq \breve{C}_2$$

Now, For any $\varepsilon > 0$ and any $x_T \in L^2(\Omega, F_T, X)$, we define the control function of the system (1) in the following form:

$$\begin{aligned} u^\varepsilon(t,x) &= B^* T_\alpha^*(T-t) R(\varepsilon, \Gamma_0^T) \\ &\left[\begin{aligned} &-\breve{T}_\alpha(T)\hat{S}\left(\begin{aligned} &x_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t \begin{bmatrix} (t-s)^{-\alpha} \\ G_2(s, x(s)) \end{bmatrix} dW_2(s) \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\sigma_2(s) dW_2^H(s) \end{aligned}\right) \\ &+\breve{T}_\alpha(T)h(0, x(0)) - \hat{S}^{-1}h(T, x(T)) \end{aligned}\right] \\ &+ B^* T_\alpha^*(T-t)\left(\int_0^t R(\varepsilon, \Gamma_s^T)\phi(s) dW_{(s)}\right) \\ &+ B^* T_\alpha^*(T-t)\left(\int_0^t R(\varepsilon, \Gamma_s^T)\hat{H}(s) dW_{(s)}^H\right) \\ &- B^* T_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\varepsilon, \Gamma_s^T)(T-s)^{\alpha-1} \\ T_\alpha(T-s)F(s, x(s)) \end{bmatrix} ds \\ &- B^* T_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\varepsilon, \Gamma_s^T)(T-s)^{\alpha-1} \\ T_\alpha(T-s)G_1(s, x(s)) \end{bmatrix} dW_1(s) \\ &- B^* T_\alpha^*(T-t)\int_0^t \begin{bmatrix} R(\varepsilon, \Gamma_s^T)(T-s)^{\alpha-1} \\ T_\alpha(T-s)\sigma_1(t) \end{bmatrix} dW_1^H(s) \end{aligned} \quad (25)$$

Lemma (3.3):

There exists positive real constants $Ꝁ_1$, $Ꝁ_2$ such that for all x, y∈ $C([0,T];\ L^2(\Omega,X))$, we have

i $\quad E\|u^\varepsilon(t,x) - u^\varepsilon(t,y)\|^2 \le Ꝁ_1 \|x-y\|_C^2$ (26)

ii $\quad E\|u^\varepsilon(t,x)\|^2 \le Ꝁ_2 (1 + \|x\|_C^2)$ (27)

Proof

i. Let $x, y \in C([0,T];\ L^2(\Omega,X))$ and $T > 0$ be a fixed. From the equation (25), we have:

$$E\left\|u^\varepsilon(t,x) - u^\varepsilon(t,y)\right\|^2$$
$$\le 4E\left\| \begin{matrix} B^*T_\alpha^*(T-t)R(\varepsilon,\Gamma_0^T)\breve{T}_\alpha(T)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_0^t (t-s)^{-\alpha}\begin{bmatrix} G_2(s,x(s)) \\ -G_2(s,y(s))\end{bmatrix} dW_2(s)\end{matrix}\right\|^2$$
$$+4E\left\|\begin{matrix} B^*T_\alpha^*(T-t)R(\varepsilon,\Gamma_0^T) \\ \times\hat{S}^{-1}\left(h(T,x(T)) - h(T,y(T))\right)\end{matrix}\right\|^2$$
$$+4E\left\| B^*T_\alpha^*(T-t)\int_0^t \left\{\begin{matrix} R(\varepsilon,\Gamma_s^T)(T-s)^{\alpha-1} \\ \times T_\alpha(T-s) \\ \times\begin{bmatrix} F(s,x(s)) \\ -F(s,y(s))\end{bmatrix}\end{matrix}\right\} ds\right\|^2$$
$$+4E\left\| B^*T_\alpha^*(T-t)\int_0^t \left\{\begin{matrix} R(\varepsilon,\Gamma_s^T)(T-s)^{\alpha-1} \\ \times T_\alpha(T-s) \\ \times\begin{bmatrix} G_1(s,x(s)) \\ -G_1(s,y(s))\end{bmatrix}\end{matrix}\right\} dW_1(s)\right\|^2$$

Applying Holder's inequality and from Ito isomerty theorem, we obtain

$$\begin{aligned} &E\|u^\varepsilon(t,x) - u^\varepsilon(t,y)\|^2 \\ &\le \frac{4L_B^2\, C_1^4 \mathfrak{m}^4 \|\hat{S}\| T^{2-2\alpha}}{(\varepsilon(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2} \sup_{t\in[0,T]} E\left\|G_2(t,x(t)) - G_2(t,y(t))\right\|_{L_2}^2 \\ &+ \frac{4L_B^2\, C_1^4 \mathfrak{m}^2}{\varepsilon^2} E\left\|h(T,x(T)) - h(T,y(T))\right\|_Z^2 \\ &+ \frac{4L_B^2\, C_1^4 \mathfrak{m}^4 T^{2\alpha}}{\varepsilon^2|2\alpha-1|} \sup_{t\in[0,T]} E\left\|F(t,x(t)) - F(s,y(t))\right\|_Z^2 \\ &+ \frac{4L_B^2\, C_1^4 \mathfrak{m}^4 T^{2\alpha-1}}{\varepsilon^2|2\alpha-1|} \sup_{t\in[0,T]} E\left\|G_1(s,x(s)) - G_1(s,y(s))\right\|_{L_2}^2 \end{aligned}$$

From the assumption (e), we get

$$E\|u^\varepsilon(t,x) - u^\varepsilon(t,y)\|^2 \le Ꝁ_1 \|x-y\|_C^2$$

where,

$$Ꝁ_1 = \Big[\frac{3L_B^2\, C_1^4 \mathfrak{m}^4 \|\hat{S}\| T^{2-2\alpha} K_5}{(\varepsilon(1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2} + \frac{4L_B^2\, C_1^4 \mathfrak{m}^2 K_7}{\varepsilon^2} + \frac{3L_B^2\, C_1^4 \mathfrak{m}^4 T^{2\alpha} K_1}{\varepsilon^2|2\alpha-1|} + \frac{3L_B^2\, C_1^4 \mathfrak{m}^4 T^{2\alpha-1} K_3}{\varepsilon^2|2\alpha-1|}\Big]$$

The proof of ii. similar to the proof of the Lemma (3.1) (see [1]).

Now, for any ε > 0, we define the operator Ψ_ε on the space $C([0,T];\ L^2(\Omega,X))$ by

$$\begin{aligned} &\Psi_\varepsilon x(t) \\ &= \breve{T}_\alpha(t)\hat{S}\left[\begin{matrix} x_0 + \dfrac{1}{\Gamma(1-\alpha)}\int_0^t \begin{bmatrix}(t-s)^{-\alpha} \\ G_2(s,x(s))\end{bmatrix} dW_2(s) \\ + \dfrac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{matrix}\right] \\ &-\breve{T}_\alpha(t)h(0,x(0)) + \hat{S}^{-1}h(t,x(t)) \\ &+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} Bu^\varepsilon(s,x)ds \\ &+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} F(s,x(s))ds \\ &+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} G_1(s,x(s))dW_1(s) \\ &+\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1} \sigma_1(t)dW_1^H(s) \end{aligned} \quad (25)$$

Lemma (3.4):

For any $x \in C([0,T];\ L^2(\Omega,X))$, the operator $(\Psi_\varepsilon x)(t)$ is a continuous on [0, T] in the space $L^2(\Omega,X)$.

Proof:

Let $t_1, t_2 \in [0,\ T]$ such that $t_1 < t_2$. Then for any $x \in C([0,T];\ L^2(\Omega,X))$, from (28), we have

$$\begin{aligned} &E\left\|(\Psi_\varepsilon x)(t_2) - (\Psi_\varepsilon x)(t_1)\right\|_X^2 \\ &\le 21E\left\|\left(\breve{T}_\alpha(t_2) - \breve{T}_\alpha(t_1)\right)\hat{S}x_0\right\|_X^2 \\ &+21E\left\|\begin{matrix}\left(\breve{T}_\alpha(t_2) - \breve{T}_\alpha(t_1)\right)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_0^{t_1}(t_1-s)^{-\alpha}G_2(s,x(s))dW_2(s)\end{matrix}\right\|_X^2 \\ &+21E\left\|\begin{matrix}\left(\breve{T}_\alpha(t_2) - \breve{T}_\alpha(t_1)\right)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_0^{t_1}(t_1-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{matrix}\right\|_X^2 \\ &+21E\left\|\begin{matrix}\breve{T}_\alpha(t_2)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_0^{t_1}\begin{bmatrix}(t_2-s)^{-\alpha} \\ -(t_1-s)^{-\alpha}\end{bmatrix}G_2(s,x(s))dW_2(s)\end{matrix}\right\|_X^2 \\ &+21E\left\|\begin{matrix}\breve{T}_\alpha(t_2)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_0^{t_1}\hat{S}\begin{bmatrix}(t_2-s)^{-\alpha} \\ -(t_1-s)^{-\alpha}\end{bmatrix}\sigma_2(s)dW_2^H(s)\end{matrix}\right\|_X^2 \\ &+21E\left\|\begin{matrix}\breve{T}_\alpha(t_2)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_{t_1}^{t_2}(t_2-s)^{-\alpha}G_2(s,x(s))dW_2(s)\end{matrix}\right\|_X^2 \\ &+21E\left\|\begin{matrix}\breve{T}_\alpha(t_2)\hat{S}\dfrac{1}{\Gamma(1-\alpha)} \\ \times\int_{t_1}^{t_2}(t_2-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{matrix}\right\|_X^2 \end{aligned}$$

$$+21E\left\|\left(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\right)h(0,x(0))\right\|_X^2$$

$$+21E\left\|\hat{S}^{-1}\left[h(t_2,x(t_2))-h(t_1,x(t_1))\right]\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}\begin{bmatrix}T_\alpha(t_2-s)\\-T_\alpha(t_1-s)\end{bmatrix}(t_1-s)^{\alpha-1}Bu^\varepsilon(s,x)ds\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}T_\alpha(t_2-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\-(t_1-s)^{\alpha-1}\end{bmatrix}Bu^\varepsilon(s,x)ds\right\|_X^2$$

$$+21E\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}Bu^\varepsilon(s,x)ds\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}\begin{bmatrix}T_\alpha(t_2-s)\\-T_\alpha(t_1-s)\end{bmatrix}(t_1-s)^{\alpha-1}F(s,x(s))ds\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}T_\alpha(t_2-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\-(t_1-s)^{\alpha-1}\end{bmatrix}F(s,x(s))ds\right\|_X^2$$

$$+21E\left\|\int_1^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}F(s,x(s))ds\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}\begin{bmatrix}T_\alpha(t_2-s)\\-T_\alpha(t_1-s)\end{bmatrix}(t_1-s)^{\alpha-1}G_1(s,x(s))dW_1(s)\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}T_\alpha(t_2-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\-(t_1-s)^{\alpha-1}\end{bmatrix}G_1(s,x(s))dW_1(s)\right\|_X^2$$

$$+21E\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}G_1(s,x(s))dW_1(s)\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}\begin{bmatrix}T_\alpha(t_2-s)\\-T_\alpha(t_1-s)\end{bmatrix}(t_1-s)^{\alpha-1}\sigma_1(t)dW_1^H(s)\right\|_X^2$$

$$+21E\left\|\int_0^{t_1}T_\alpha(t_2-s)\begin{bmatrix}(t_2-s)^{\alpha-1}\\-(t_1-s)^{\alpha-1}\end{bmatrix}\sigma_1(t)dW_1^H(s)\right\|_X^2$$

$$+21E\left\|\int_{t_1}^{t_2}T_\alpha(t_2-s)(t_2-s)^{\alpha-1}\sigma_1(t)dW_1^H(s)\right\|_X^2$$

Applying Holder's inequality on the last inequality and by using Ito isometry, Lemma (2.5), Lemma (3.3), Lemma (2.6) and Lemma (3.1), we obtain

$$E\|(\Psi_\varepsilon x)(t_2)-(\Psi_\varepsilon x)(t_1)\|_X^2$$
$$\leq 21\left\|\left(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\right)\right\|_X^2\|\hat{S}\|^2E\|x_0\|_X^2$$
$$+21\left\|\left(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\right)\right\|_X^2\frac{\|\hat{S}\|^2T^{-2\alpha+2}K_6}{\Gamma(1-\alpha))^2}(1+\|x\|_C^2)$$
$$+21\left\|\left(\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\right)\right\|_X^2\frac{\|\hat{S}\|^2T^{-2\alpha+2H+1}D_2}{\Gamma(1-\alpha))^2}$$
$$+21\frac{C_1^2m^2\|\hat{S}\|^2K_6T}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\left(\int_0^{t_1}[(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}]^2ds\right)(1+\|x\|_C^2)$$
$$+21\frac{C_1^2m^2\|\hat{S}\|^2T^{2H}D_2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\left(\int_0^{t_1}[(t_2-s)^{-\alpha}-(t_1-s)^{-\alpha}]^2ds\right)$$
$$+21\frac{C_1^2m^2\|\hat{S}\|^2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\left(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}ds\right)\times\left(\int_{t_1}^{t_2}\|G_2(s,x(s))\|_{L_2}^2ds\right)$$
$$+21\frac{2HC_1^2m^2\|\hat{S}\|^2T^{2H-1}D_2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\times\left(\int_{t_1}^{t_2}(t_2-s)^{-2\alpha}ds\right)$$
$$+21\|\breve{T}_\alpha(t_2)-\breve{T}_\alpha(t_1)\|_X^2E\|h(0,x(0))\|_Z^2$$
$$+21C_1^2E\|h(t_2,x(t_2))-h(t_1,x(t_1))\|_Z^2$$
$$+21L_B^2\int_0^{t_1}(t_1-s)^{2\alpha-2}ds$$
$$\times\int_0^{t_1}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|_X^2E\|u^\varepsilon(s,x)\|^2ds$$
$$+21L_B^2C_1^2m^2TK_2$$
$$\times\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^2ds\right)(1+\|x\|_C^2)$$
$$+21L_B^2C_1^2m^2TK_2\left(\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}ds\right)(1+\|x\|_C^2)$$
$$+21K_2T^{2\alpha-1}\int_0^{t_1}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|_X^2(1+\|x(s)\|_X^2)ds$$
$$+21K_2C_1^2m^2T\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^2ds\right)(1+\|x\|_C^2)$$
$$+21K_2C_1^2m^2T\left(\int_{t_1}^{t_2}(t_2-s)^{2\alpha-2}ds\right)\times(1+\|x\|_C^2)$$
$$+21T^{2\alpha-1}K_4\left(\int_0^{t_1}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|_X^2(1+\|x(s)\|_X^2)ds\right.$$

Hence, by using the strong continuity of $\breve{T}_\alpha(t)$ and $T_\alpha(t)$ in Lemma (2.6) and Lebesgue's dominated convergence theorem, we conclude that the right-hand side of the above inequalities tends to zero as $t_2\to t_1$. Thus, we conclude $(\Psi_\varepsilon x)(t)$ is a continuous from the right in $[0,T)$. A similar argument shows that it is also continuous from the left in $(0,T]$. Thus $(\Psi_\varepsilon x)(t)$ is continuous on $[0,T]$ in the $L^2(\Omega,X)$.

Lemma (3.5):

For each $\varepsilon>0$, the operator Ψ_ε maps from $C([0,T];\ L^2(\Omega,X))$ into itself. i.e. $\Psi_\varepsilon\left(C([0,T];L^2(\Omega,X))\right)\subset C([0,T];\ L^2(\Omega,X))$.

Proof:

From Lemma (3.4), for any $x\in C([0,T];\ L^2(\Omega,X))$, the operator $(\Psi_\varepsilon x)(t)$ is a continuous on [0, T] in the space $L^2(\Omega,X)$. To prove that for $x\in C([0,T];\ L^2(\Omega,X))$ implies $E\|\Psi_\varepsilon x(t)\|_X^2<\infty$.

$$E\|\Psi_\varepsilon x(t)\|_X^2$$
$$\leq 6E\left\|\breve{T}_\alpha(t)\hat{S}\begin{bmatrix}x_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t\begin{bmatrix}(t-s)^{-\alpha}\\G_2(s,x(s))\end{bmatrix}dW_2(s)\\+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{bmatrix}\right\|_X^2$$
$$+6E\left\|\breve{T}_\alpha(t)h(0,x(0))+\hat{S}^{-1}h(t,x(t))\right\|_X^2$$
$$+6E\left\|\int_0^tT_\alpha(t-s)(t-s)^{\alpha-1}Bu^\varepsilon(s,x)ds\right\|_X^2$$
$$+6E\left\|\int_0^tT_\alpha(t-s)(t-s)^{\alpha-1}F(s,x(s))ds\right\|_X^2$$
$$+6E\left\|\int_0^tT_\alpha(t-s)(t-s)^{\alpha-1}G_1(s,x(s))dW_1(s)\right\|_X^2$$
$$+6E\left\|\int_0^tT_\alpha(t-s)(t-s)^{\alpha-1}\sigma_1(t)dW_1^H(s)\right\|_X^2$$

Applying Holder's inequality and by using Ito isometry theoem, Lemma (2.5), Lemma (3.3), Lemma (2.6) and Lemma (3.1), we obtain

$$E\|\Psi_\varepsilon x(t)\|_X^2\leq 8\frac{C_1^2m^2\|\hat{S}\|^2}{(\Gamma(\alpha))^2}[E\|x_0\|_X^2$$
$$+\frac{T^{2-2\alpha}K_6}{(\Gamma(1-\alpha)(1-\alpha))^2}(1+\|x\|_C^2)+\frac{2HT^{2H-2\alpha+1}D_2}{(\Gamma(1-\alpha)(1-\alpha))^2}]$$

$$+12\frac{C_1^2 m^2 K_8}{(\Gamma(\alpha))^2}(1+\|x(0)\|_X^2)$$
$$+C_1^2 K_8(1+\|x\|_C^2)+6\frac{C_1^2 m^2 L_B^2 T^{2\alpha} K_2}{|2\alpha-1|}(1+\|x\|_C^2)$$
$$+6\frac{C_1^2 m^2 T^{2\alpha} K_2}{|2\alpha-1|}(1+\|x\|_C^2)$$
$$+6\frac{C_1^2 m^2 T^{2\alpha-1} K_4}{|2\alpha-1|}(1+\|x\|_C^2)$$
$$+12H\frac{C_1^2 m^2 T^{2\alpha+2H-2} D_1}{|2\alpha-1|}$$

Hence, the last inequality imply that $E\|\Psi_\varepsilon x(t)\|_X^2<\infty$. Moreover, for $x\in C\left([0,T];\ L^2(\Omega,X)\right)$ then $\Psi_\varepsilon x\in C\left([0,T];\ L^2(\Omega,X)\right)$. Thus for each $\varepsilon>0$, the operator Ψ_ε maps from $C\left([0,T];\ L^2(\Omega,X)\right)$ into itself.

Theorem (3.1):

Let the assumptions (a)-(e) be satisfied. Then for each $\varepsilon>0$, the system (1) has a mild solution on [0, T].

Proof:

To prove the existence of a fixed point of the operator Ψ_ε which is defined in (28) by using the contraction mapping principle.

Let $x, y\in C([0,T];\ L^2(\Omega,X))$. for any $t\in[0,T]$, we have

$$E\left\|(\Psi_\varepsilon x)(t)-(\Psi_\varepsilon y)(t)\right\|_X^2$$
$$\le 6E\left\|\begin{matrix}\breve{T}_\alpha(t)\hat{S}\dfrac{1}{\Gamma(1-\alpha)}\\ \times\int_0^t (t-s)^{-\alpha}\begin{bmatrix}G_2(s,x(s))\\ -G_2(s,y(s))\end{bmatrix}dW_2(s)\end{matrix}\right\|_X^2$$
$$+6E\left\|\breve{T}_\alpha(t)\left[h(0,x(0))-h(0,y(0))\right]\right\|_X^2$$
$$+6E\left\|\hat{S}^{-1}\left[h(t,x(t))-h(t,y(t))\right]\right\|_X^2$$
$$+6E\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}B\begin{bmatrix}u^\varepsilon(s,x(s))\\ -u^\varepsilon(s,y(s))\end{bmatrix}ds\right\|_X^2$$
$$+6E\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\begin{bmatrix}F(s,x(s))\\ -F(s,y(s))\end{bmatrix}ds\right\|_X^2$$
$$+6E\left\|\int_0^t T_\alpha(t-s)(t-s)^{\alpha-1}\begin{bmatrix}G_1(s,x(s))\\ -G_1(s,y(s))\end{bmatrix}dW_1(s)\right\|_X^2$$

Applying Holder's inequality on the last inequality, by using Ito isometry and Lemma (2.5), we obtain

$$E\|(\Psi_\varepsilon x)(t)-(\Psi_\varepsilon y)(t)\|_X^2$$
$$\le\frac{6C_1^2 m^2\|\hat{S}\|^2 T^{2-2\alpha}}{((1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2}\sup_{s\in[0,t]}E\left\|G_2(s,x(s))-G_2(s,y(s))\right\|_{L_2}^2$$
$$+\frac{6C_1^2 m^2}{(\Gamma(\alpha))^2}E\left\|h(0,x(0))-h(0,y(0))\right\|_Z^2$$
$$+6C_1^2 E\left\|h(t,x(t))-h(t,y(t))\right\|_Z^2$$
$$+\frac{6C_1^2 m^2 L_B^2 T^{2\alpha}}{|2\alpha-1|}\sup_{s\in[0,t]}E\|u^\varepsilon(s,x(s))-u^\varepsilon(s,y(s))\|^2$$
$$+\frac{6C_1^2 m^2 T^{2\alpha}}{|2\alpha-1|}\sup_{s\in[0,t]}E\left\|F(s,x(s))-F(s,y(s))\right\|_Z^2$$
$$+\frac{6C_1^2 m^2 T^{2\alpha-1}}{|2\alpha-1|}\sup_{s\in[0,t]}E\left\|G_1(s,x(s))-G_1(s,y(s))\right\|_{L_2}^2$$

From the assumptions (a)-(e) and Lemma (3.3), we obtain

$$E\|(\Psi_\varepsilon x)(t)-(\Psi_\varepsilon y)(t)\|_X^2$$
$$\le\frac{6C_1^2 m^2\|\hat{S}\|^2 T^{2-2\alpha}K_5}{((1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2}\sup_{s\in[0,t]}E\|x(s)-y(s)\|_X^2$$
$$+\frac{6C_1^2 m^2 T^{2-2\alpha}K_5K_7}{((1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2}\sup_{s\in[0,t]}E\|x(s)-y(s)\|_X^2$$
$$+6C_1^2K_7E\|x(t)-y(t)\|_X^2+\frac{6C_1^2 m^2 L_B^2 T^{2\alpha}}{|2\alpha-1|}K_1\|x-y\|_C^2$$
$$+\frac{6C_1^2 m^2 T^{2\alpha}K_1}{|2\alpha-1|}\sup_{s\in[0,t]}E\|x(s)-y(s)\|_X^2$$
$$+\frac{6C_1^2 m^2 T^{2\alpha-1}K_3}{|2\alpha-1|}\sup_{s\in[0,t]}E\|x(s)-y(s)\|_X^2$$

Taking supremum over $t\in[0,T]$ for both sides, we get

$$\|\Psi_\varepsilon x-\Psi_\varepsilon y\|_C^2\le\gamma(T)\|x-y\|_C^2$$

where,

$$\gamma(T)=\frac{6C_1^2 m^2\|\hat{S}\|^2 T^{2-2\alpha}K_5}{((1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2}+\frac{6C_1^2 m^2 T^{2-2\alpha}K_5K_7}{((1-\alpha)\Gamma(\alpha)\Gamma(1-\alpha))^2}$$
$$+6C_1^2K_7+\frac{6C_1^2 m^2 L_B^2 T^{2\alpha}K_1}{|2\alpha-1|}$$
$$+\frac{6C_1^2 m^2 T^{2\alpha}K_1}{|2\alpha-1|}+\frac{6C_1^2 m^2 T^{2\alpha-1}K_3}{|2\alpha-1|}$$

Then, there exists $T_1\in(0,T]$ such that $0<\gamma(T_1)<1$ and Ψ_ε is a contraction mapping on $C([0,T_1];\ L^2(\Omega,X))$ and therefore has a unique fixed point, which is a mild solution of equation (1) on $[0,T_1]$. This procedure can be repeated in order to extend the solution to the entire interval [0,T] in finitely many steps. This completes the proof.

Theorem (3-2):

Assume that the assumptions (a) – (e) are satisfied, Further, if the functions F, G_1 and G_2 are uniformly bounded, then the system (1) is approximately controllable on [0, T].

Proof:

For every $\varepsilon>0$,let x_ε be a fixed point of the operator Ψ_ε in $([0,T];\ L^2(\Omega,X))$, which is a mild solution under the control function in (25) of the system (1). Then from (28), we have:

$$x_\varepsilon(T)=x_T-\varepsilon R(\varepsilon,\Gamma_0^T)\times\begin{bmatrix}Ex_T-\breve{T}_\alpha(T)\hat{S}\\ \times\left(\begin{matrix}x_0+\dfrac{1}{\Gamma(1-\alpha)}\int_0^T\begin{bmatrix}(T-s)^{-\alpha}\\ G_2(s,x_\varepsilon(s))\end{bmatrix}dW_2(s)\\ +\dfrac{1}{\Gamma(1-\alpha)}\int_0^T(T-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{matrix}\right)\\ +\breve{T}_\alpha(T)h(0,x_\varepsilon(0))-\hat{S}^{-1}h(T,x_\varepsilon(T))\end{bmatrix}\tag{29}$$
$$-\int_0^T\varepsilon R(\varepsilon,\Gamma_s^T)\hat{S}(s)dW_{(s)}-\int_0^t\varepsilon R(\varepsilon,\Gamma_s^T)\hat{S}(s)dW_{(s)}^H$$
$$+\int_0^T\varepsilon R(\varepsilon,\Gamma_s^T)(T-s)^{\alpha-1}T_\alpha(T-s)F(s,x_\varepsilon(s))ds$$
$$+\int_0^T\begin{bmatrix}\varepsilon R(\varepsilon,\Gamma_s^T)(T-s)^{\alpha-1}\\ T_\alpha(T-s)G_1(s,x_\varepsilon(s))\end{bmatrix}dW_1(s)$$
$$+\int_0^T\varepsilon R(\varepsilon,\Gamma_s^T)(T-s)^{\alpha-1}T_\alpha(T-s)\sigma_1(s)dW_1^H(s)$$

It follows from the assumptions on F, G_1 and G_2 that there exists $\widetilde{D}_1 > 0$, $\widetilde{D}_2 > 0$, $\widetilde{D}_3 > 0$ such that, $\|F(s,x_\varepsilon(s))\|_Z^2 \le \widetilde{D}_1$, $\|G_1(s,x_\varepsilon(s))\|_{L_2}^2 \le \widetilde{D}_2$, $\|G_2(s,x_\varepsilon(s))\|_{L_2}^2 \le \widetilde{D}_3$, for all $s\in[0,T]$. Then, there is a subsequences still denoted by $\{F(s,x_\varepsilon(s))\}$, $\{G_1(s,x_\varepsilon(s))\}$ and $\{G_2(s,x_\varepsilon(s))\}$ which converges weakly to $\{F(s)\}$, $\{G_1(s)\}$ and $\{G_2(s)\}$ in Z, $L_2(K,Z)$ and $L_2(K,X)$ respectively.

Now, from the equation (29), we have

$$
\begin{aligned}
&E\left\|x_\varepsilon(T)-x_T\right\|_X^2\\
&\le 12E\left\|\varepsilon R(\varepsilon,\Gamma_0^T)[Ex_T-\breve{T}_\alpha(T)\hat{S}(x_0)]\right\|_X^2\\
&+12E\left\|\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_0^T\right)\breve{T}_\alpha(T)\hat{S}\dfrac{1}{\Gamma(1-\alpha)}\\ \times\int_0^T(T-s)^{-\alpha}\left[G_2(s,x_\varepsilon(s))-G_2(s)\right]dW_2(s)\end{matrix}\right\|_X^2\\
&+12E\left\|\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_0^T\right)\breve{T}_\alpha(T)\hat{S}\dfrac{1}{\Gamma(1-\alpha)}\\ \times\int_0^T(T-s)^{-\alpha}G_2(s)dW_2(s)\end{matrix}\right\|_X^2\\
&+12E\left\|\begin{matrix}\varepsilon R(\varepsilon,\Gamma_0^T)\breve{T}_\alpha(T)\hat{S}\dfrac{1}{\Gamma(1-\alpha)}\\ \times\int_0^T(T-s)^{-\alpha}\sigma_2(s)dW_2^H(s)\end{matrix}\right\|_X^2\\
&+12E\left\|\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_0^T\right)[\breve{T}_\alpha(T)h(0,x_\varepsilon(0))\\ -\hat{S}^{-1}h(T,x_\varepsilon(T))]\end{matrix}\right\|_X^2\\
&+12E\left\|\int_0^T\varepsilon R\left(\varepsilon,\Gamma_s^T\right)\phi(s)dW_{(s)}\right\|_X^2\\
&+12E\left\|\int_0^T\varepsilon R\left(\varepsilon,\Gamma_s^T\right)\hat{H}(s)dW_{(s)}^H\right\|_X^2\\
&+12E\left\|\int_0^T\left\{\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}\\ \times T_\alpha(T-s)\left[F(s,x_\varepsilon(s))-F(s)\right]\end{matrix}\right\}ds\right\|_X^2\\
&+12E\left\|\int_0^T\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}T_\alpha(T-s)F(s)ds\right\|_X^2\\
&+12E\left\|\int_0^T\left\{\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}T_\alpha(T-s)\\ \times\left[G_1(s,x_\varepsilon(s))-G_1(s)\right]\end{matrix}\right\}dW_1(s)\right\|_X^2\\
&+12E\left\|\int_0^T\left[\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}\\ \times T_\alpha(T-s)G_1(s\end{matrix}\right])dW_1(s)\right\|_X^2\\
&+12E\left\|\int_0^T\left[\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}\\ \times T_\alpha(T-s)\sigma_1(s)\end{matrix}\right]dW_1^H(s)\right\|_X^2
\end{aligned}
$$

Using Ito isometry and Lemma (2.5) and Lemma (4-2), we obtain

$$
\begin{aligned}
&E\|x_\varepsilon(T)-x_T\|_X^2\\
&\le 12E\left\|\varepsilon R(\varepsilon,\Gamma_0^T)[Ex_T-\breve{T}_\alpha(T)\hat{S}(x_0)]\right\|_X^2\\
&+\frac{12C_1^2\mathfrak{m}^2\|\hat{S}\|^2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\|\varepsilon R(\varepsilon,\Gamma_0^T)\|_X^2\\
&\times\int_0^T(T-s)^{-\alpha}E\left\|G_2(s,x_\varepsilon(s))-G_2(s)\right\|_{L_2}^2ds\\
&+\frac{12C_1^2\mathfrak{m}^2\|\hat{S}\|^2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\|\varepsilon R(\varepsilon,\Gamma_0^T)\|_X^2\\
&\times\int_0^T(T-s)^{-2\alpha}E\|G_2(s)\|_{L_2}^2ds\\
&+\frac{24HT^{2H-1}C_1^2\mathfrak{m}^2\|\hat{S}\|^2}{(\Gamma(\alpha)\Gamma(1-\alpha))^2}\|\varepsilon R(\varepsilon,\Gamma_0^T)\|_X^2\\
&\times\int_0^T(T-s)^{-2\alpha}\|\sigma_2(s)\|_{L_2^0}^2ds\\
&+12E\left\|\varepsilon R\left(\varepsilon,\Gamma_0^T\right)\left[\begin{matrix}\breve{T}_\alpha(T)h(0,x_\varepsilon(0))\\ -\hat{S}^{-1}h(T,x_\varepsilon(T))\end{matrix}\right]\right\|_X^2\\
&+12\int_0^T\left\|\varepsilon R\left(\varepsilon,\Gamma_0^T\right)\right\|_X^2E\left\|\phi(s)\right\|_{L_2}^2ds\\
&+24HT^{2H-1}\int_0^T\left\|\varepsilon R\left(\varepsilon,\Gamma_0^T\right)\right\|_X^2E\left\|\hat{H}(s)\right\|_{L_2^0}^2ds\\
&+12E\left\|\int_0^T\left\{\begin{matrix}\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}T_\alpha(T-s)\\ \times\left[F(s,x_\varepsilon(s))-F(s)\right]\end{matrix}\right\}ds\right\|_X^2\\
&+12E\left\|\int_0^T\varepsilon R\left(\varepsilon,\Gamma_s^T\right)(T-s)^{\alpha-1}T_\alpha(T-s)F(s)ds\right\|_X^2\\
&+12\int_0^T\left[\begin{matrix}\left\|\varepsilon R\left(\varepsilon,\Gamma_s^T\right)\right\|_X^2(T-s)^{2\alpha-2}T_\alpha(T-s)\\ \times E\left\|G_1(s,x_\varepsilon(s))-G_1(s)\right\|_{L_2}^2\end{matrix}\right]ds\\
&+12\int_0^T\left\{\begin{matrix}\left\|\varepsilon R\left(\varepsilon,\Gamma_s^T\right)\right\|_X^2(T-s)^{2\alpha-2}\\ \times T_\alpha(T-s)E\left\|G_1(s)\right\|_{L_2}^2\end{matrix}\right\}ds\\
&+24HT^{2H-1}\int_0^T\left\{\begin{matrix}\varepsilon\left\|R\left(\varepsilon,\Gamma_s^T\right)\right\|_X^2(T-s)^{2\alpha-2}\\ T_\alpha(T-s)\left\|\sigma_1(s)\right\|_{L_2^0}^2\end{matrix}\right\}ds
\end{aligned}
$$

On the other hand, by the assumption (b), and Lemma (3.1), for all $0\le s<T$, we have the operator $\varepsilon R(\varepsilon,\Gamma_s^T)\to 0$ strongly as $\varepsilon\to 0^+$ and moreover $\|\varepsilon R(\varepsilon,\Gamma_s^T)\|\le 1$. By using the Lebesgue dominated convergence theorem, the compactness of $T_\alpha(t)$ implies that we obtain $E\|x_\varepsilon(T)-x_T\|^2\to 0$ as $\varepsilon\to 0^+$. This gives the approximate controllability.

4. Conclusions

An approximate controllability result for nonlinear Fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion is obtained by means of contraction principle fixed point theorems under the compactness assumption. It is also proven that the approximate controllability of linear deterministic system implies the approximate controllability of nonlinear Fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion in Hilbert spaces.

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