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A Min-Max Algorithm for Solving the Linear Complementarity Problem

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Abstract The Linear Complementarity Problem $LCP(M,q)$ is to find a vector x in IR^n satisfying $x \geq 0$, $Mx+q \geq 0$ and $x^T(Mx+q)=0$, where M as a matrix and q as a vector, are given data. In this paper we show that the linear complementarity problem is completely equivalent to finding the fixed point of the map $x = \max(0, (I-M)x-q)$; to find an approximation solution to the second problem, we propose an algorithm starting from any interval vector $X^{(0)}$ and generating a sequence of the interval vector $(X^{(k)})_{k=1}$ which converges to the exact solution of our linear complementarity problem. We close our paper with some examples which illustrate our theoretical results.

Keywords: *linear complementarity problem, min-max algorithm, fixed point, Brouwer theorem, interval vector, closed bounded convex set*

1. Introduction

The Complementarity Problem noted (CP) is a classical problem from optimization theory of finding $x \in \mathrm{IR}^n$ such that

$$\begin{cases} x^T f(x) = 0 \\ f(x) \geq 0 \\ x \geq 0 \end{cases} \quad (1)$$

where f, a continuous operator from IR^n into itself, is given data.

The contraint $x^T f(x) = 0$ is called the complementarity condition since for any i, $1 \leq i \leq n$, $x_i = 0$ if $f_i(x) > 0$, and vice versa. It may be the case that $x_i = f_i(x) = 0$ however.

This problem becomes in present the subject of much research important because it arises in many areas and it includes important fields, we cite for example the linear programming (LP), the nonlinear programming (NLP), the convex quadratic programming and the variational inequalities problems.

In the case that the function f is a nonlinear continuous operator from IR^n into itself, so it is called a NonLinear Complementarity Problem associated with the function f and noted $(NLCP)$. Numerous methods and algorithms exist to solve nonlinear complementarity problems such as the homotopy methods of Merrill [23] and several other authors (see for example Eaves et al. [3,5], Saigal [24]), using a reformulation of the $NLCP(f)$ due to Mangasarian [22] in which the zero finding problem can be made as smooth as desired, Watson [26] applied the homotopy or continuation method of Chow, Mallet-Paret and Yorke [1] to solve the problem. Instead of reformulating the $NLCP(f)$ as a zero finding problem, other authors adjusted simplicial fixed point algorithms to solve the $NLCP(f)$ directly, see e.g. Fisher and Gould [17], Garcia [18], Kojima [19] or Lüthi [21] and recently our modest work Elfoutayeni and Khaladi [14], In this paper we have given a new method for solving this problem which converges very rapidly relative to most of the existing methods and does not require a lot of arithmetic operations to converge. For this we have showed that solving the $CP(f)$ is equivalent to solving $F(x)=0$ where F is a function from IR^n into itself defined by $F(x) = f(|x|-x) - |x| - x$. After that we have built a sequence of smooth functions $F^{(k)} \in C^\infty$ which is uniformly convergent to the function F and we have showed that an approximation of the solution of the $CP(f)$ is obtained by solving $F^{(k)}(x)=0$ for a parameter k large enough. For solving the system of nonlinear equations $F^{(k)}(x)=0$ we have used the Gauss-Seidel-He algorithm. The numerical results obtained in this paper are very favorable and showed that our method works well for the problems tested.

In the case that the function f is affine, i.e., its accurate form as below $f(x) = q + Mx$, where q is an element of IR^n and M is a real square matrix of order n, so it is called a Linear Complementarity Problem associated with the matrix M and the vector q and noted (LCP). To solve this problem, there are several methods and algorithms, we cite for example Lemke [20] first presented a solution for this problem. His ideas were later exploited by Scarf [25] in his work on fixed point algorithms. The relationship between the $LCP(q,M)$ and the fixed point problem is well described by Eaves and Scarf[6] and by Eaves and Lemke [4]. Cottle and Dantzig's principal pivot method[2] and recently our modest works Elfoutayeni and Khaladi [7,8]. In the first one we have built an interior point method to solve a linear complementarity problem $LCP(M,q)$ for some $n\times n$ matrix M and $q\in \mathrm{IR}^n$; the convergence of this method requires $o(\sqrt{n}L)$ number of iterations where L is the length of a binary coding of the input data of the problem $LCP(M,q)$. This interior point method is globally efficient and has a good iteration complexity but it has the problem of finding a strictly feasible starting point. In the second one we have given a globally convergent hybrid method which is based on vector divisions and the secant method for solving the $LCP(M,q)$; we have given in this paper some numerical simulations to illustrate our theoretical results, and to show that this method can solve efficiently large-scale linear complementarity problems. We have to note that in our paper Elfoutayeni and Khaladi [15] we have given a general characterization of a linear complementarity problem. Furthermore, through this paper, we can provide the solution (if it exists) of this problem in a straightforward manner and according to the data. Precisely, we have demonstrated that $LCP(M,q)$ has a solution if and only if there is a set X such that the system of linear equations $M[X]x+q[X]=0$ has a nonnegative solution $x^*\ge 0$ and

$$M[\bar{X},X]x^*+q[\bar{X}]\ge 0;$$

this solution is then given by $x=(x_i)_{i\in N}$ where $x_i=x_i^*$ if $i\in X$, and $x_i=0$ if $i\in \bar{X}$. We have to note that as one of the remarks we would like to point out that we can find many sets X, but a unique solution x^* and contrary, we can find a set X but many solutions x^* (we must think of the invertibility of the submatrix in each case). As a final remark we like to stress that the LCP has a unique solution if and only if there is a unique set X such that the system of linear equations $M[X]x+q[X]=0$ has a unique nonnegative solution and

$$M[\bar{X},X](M[X])^{-1}q[X]-q[\bar{X}]\le 0.$$

In this paper we show that the linear complementarity problem is completely equivalent to finding the fixed point of the map $x=\max(0,(I-M)x-q)$; to find an approximation solution to the second problem, we propose an algorithm starting from any interval vector $X^{(0)}$ and generating a sequence of the interval vector $(X^{(k)})_{k=1,..}$ which converges to the exact solution of our linear complementarity problem. We close our paper with some examples which illustrate our theoretical results.

The paper is organized as follows. In section 2 we briefly give some definitions and notations to be used through the paper. In section 3 we write LCP in the equivalent form of finding a solution of a fixed point of the function $g(x)=\max(0,x-f(x))$. In section 4 we give some numerical examples and we give a conclusion in section5.

2. Preliminaries

In this section, we summarize some basic properties and related definitions which be used in the following discussion.

In particular, IR^n denotes the space of real $n-$dimensional vectors and $\mathrm{IR}^n_+ := \{x\in \mathrm{IR}^n : x_i\ge 0, i=1...n\}$ the nonnegative orthant of IR^n.

Let x, $y\in \mathrm{IR}^n$, x^Ty is their inner product; $\|x\|$ is the Euclidean norm and $\|x\|_\infty$ is maximum norm. The vector $e:=(1,...,1)^T$ is the vector of ones in IR^n.

For $x\in \mathrm{IR}^n$ and k a nonnegative integer, $x^{(k)}$ refers to the vector obtained after k iterations; for $1\le i\le n$, x_i refers to the i^{th} element of x, and $x_i^{(k)}$ refers to the i^{th} element of the vector obtained after k iterations.

Let x, $y\in \mathrm{IR}^n$, the expression $x\le y$ (respectively $x<y$) meaning that $x_i\le y_i$ (respectively $x_i<y_i$) for each $i=1...n$.

The transpose of a vector is denoted by super script T, such as the transpose of the vector x is given by x^T.

$X=([a_1,b_1],....,[a_n,b_n])^T$ is a n-dimensional interval vector which $a=(a_1,...,a_n)^T$ and $b=(b_1,...,b_n)^T$ are two vectors in IR^n_+; noted by: $X=([a,b])$.

$[\mathrm{IR}^n_+]$ denotes the set of all interval vector in IR^n_+.

Let X be an interval vector in $[\mathrm{IR}^n_+]$, $x\in X$ meaning that $x_i\in X_i$ for each $i=1,...,n$.

Let h be a function from IR^n to $[\mathrm{IR}^n]$ then

$$h(X):=([\min_{x\in X}h(x);\max_{x\in X}h(x)])\in \mathrm{IR}^n_+]$$

$$\min h(X):=\min_{x\in X}h(x)\in \mathrm{IR}^n;$$

$$\max h(X):=\max_{x\in X}h(x)\in \mathrm{IR}^n.$$

For any $X=([a,b])$ an interval vector in $[\mathrm{IR}^n_+]$, we define the function Δ from $[\mathrm{IR}^n_+]$ into IR^n_+ by: $\Delta(X):=b-a$.

The notation $\lim_{k\to+\infty} X^{(k)} = x^*$ meaning that $\lim_{k\to+\infty} a^{(k)} = x^*$ and $\lim_{k\to+\infty} b^{(k)} = x^*$.

Recall that the spectrum S of the matrix A is the set of its values and its spectral radius ρ is given by $\rho(A) := \sup\{|\lambda| \text{ such that } \lambda \in S\}$.

3. Equivalent Reformulation of CP and Algorithm

Our first objective in this section is to show that if x^* is a solution of (CP) then x^* is a fixed point of the map

$$g(x) := \max(0, x - f(x)). \qquad (2)$$

and vise versa.

Proposition 1 *Solving the complementarity problem* $CP(f)$ *is equivalent to finding the solution of* $g(x) = x$.

Proof. Let x^* be a solution of (CP) and let's consider the set

$$I := \{1 \le i \le n : x_i^* = 0\}$$

and the complementarity of I defined by

$$\bar{I} := \{1 \le i \le n : x_i^* \ne 0\} = \{1 \le i \le n : f_i(x^*) = 0\}$$

then we have for each $i \in I$

$$\begin{aligned} g_i(x^*) &= \max(0, x_i^* - f_i(x^*)) \\ &= \max(0, -f_i(x^*)) \\ &= 0 \\ &= x_i^* \end{aligned}$$

and for each $i \in \bar{I}$

$$\begin{aligned} g_i(x^*) &= \max(0, x - f_i(x^*)) \\ &= \max(0, x_i^*) \\ &= x_i^* \end{aligned}$$

then x^* is a fixed point of the function g.

Now let x^* be a fixed point of the function g, then we have $x^* \ge 0$.

Now if $i \in I$ then we have $x_i^* = 0$ and $g_i(x^*) = 0$, therefore $x_i^* - f_i(x^*) \le 0$ i.e., $f_i(x^*) \ge 0$.

else if $i \in \bar{I}$ then we have $f_i(x^*) = 0$.

In the two cases we have $x^* \ge 0$, $f(x^*) \ge 0$ and $(x^*)^T f(x^*) = 0$, therefore x^* is a solution of (CP). This concludes the proof.

Now we propose an algorithm generating a sequence $(X^{(k)})_{k=1,..}$ of the interval vector in $[\mathrm{IR}_+^n]$ and tending to the fixed point of the function g.

For this, we choice $X^{(0)} = ([a^{(0)}, b^{(0)}])$ an interval vector in $[\mathrm{IR}_+^n]$ so large to ensure that $x^* \in X^{(0)}$.

We define the interval vector in $[\mathrm{IR}_+^n]$ at iteration k by $X^{(k)} := ([a^{(k)}, b^{(k)}])$ such that

$$\begin{cases} a^{(k+1)} := \max(a^{(k)}, \max(0, a^{(k)} - \max\ f(X^{(k)}))) \\ b^{(k+1)} := \min(b^{(k)}, \max(0, b^{(k)} - \min\ f(X^{(k)}))) \end{cases} \qquad (3)$$

It is easy to show that if $x^* \in X^{(k)}$ is a solution of $g(x) = x$, then $x^* \in g(X^{(k)})$.

Now we show that

Proposition 2 *If there is an* i *in the set* $\{1,..,n\}$ *such that*

$$\begin{cases} (\max(0, b^{(k)} - \min\ f(X^{(k)})))_i < a_i^{(k)} \\ \text{or} \\ b_i^{(k)} < (\max(0, a^{(k)} - \max\ f(X^{(k)})))_i \end{cases} \qquad (4)$$

then the solution of (2) is not existing in $X^{(k)}$.

Remark *If the solution of (*2*) is not existing in* $X^{(k)}$*, we redefine the algorithm with a new* $X^{(0)} \in \mathrm{IR}_+^n]$ *larger than the older one.*

Proof. Assume the contrary, i.e., we suppose there is a solution $x^* \in X^{(k)}$ of the (2), then $x^* \in g(X^{(k)})$, i.e., for each $1 \le i \le n$ we have

$$\begin{cases} a_i^{(k)} \le x_i^* \le b_i^{(k)} \\ (\max(0, a^{(k)} - \max\ f(X^{(k)})))_i \le x_i^* \\ x_i^* \le (\max(0, b^{(k)} - \min\ f(X^{(k)})))_i \end{cases}$$

thus

$$\begin{cases} (\max(0, b^{(k)} - \min\ f(X^{(k)})))_i \ge a_i^{(k)} \\ b_i^{(k)} \ge (\max(0, a^{(k)} - \max\ f(X^{(k)})))_i \end{cases}$$

this contradicts (4).

Now we show that

Proposition 3 *If*

$$\begin{cases} \max(0, a^{(k)} - \max\ f(X^{(k)})) \ge a^{(k)} \\ \max(0, b^{(k)} - \min\ f(X^{(k)})) \le b^{(k)} \end{cases}$$

then there is a solution of (2) in $X^{(k)}$.

Proof. To prove that let's $x^{(k)} \in X^{(k)} = ([a^{(k)}, b^{(k)}])$, then we have $g(x^{(k)}) \in X^{(k)}$ and $X^{(k)}$ is a closed bounded convex set of the IR_+^n, since the function g is a continuous function from $X^{(k)}$ to itself then from Brouwer theorem, we know that there is a fixed point $x^* \in X^{(k)}$ satisfied $x^* = g(x^*)$.

Now we show that

Lemma *For each* X *interval vector in* $[\mathrm{IR}_+^n]$ *we have*

$$\Delta(g(X)) \le \Delta(X - f(X))$$

Proof. For each $i = 1,\dots,n$ we have

$$\Delta_i(g(X)) := \max(0, b_i - \min\ f_i(X)) - \max(0, a_i - \max\ f_i(X))$$

If

$$\begin{cases} b_i - \min\ f_i(X) \geq 0 \\ a_i - \max\ f_i(X) \geq 0 \end{cases}$$

then

$$\Delta_i(g(X)) = (b_i - \min\ f_i(X)) - (a_i - \max\ f_i(X)) = \Delta_i(X - f(X)).$$

If

$$\begin{cases} b_i - \min\ f_i(X) \geq 0 \\ a_i - \max\ f_i(X) \leq 0 \end{cases}$$

then

$$\begin{aligned} \Delta_i(g(X)) &= (b_i - \min\ f_i(X)) \\ &\leq (b_i - \min\ f_i(X)) - (a_i - \max\ f_i(X)) \\ &= \Delta_i(X - f(X)). \end{aligned}$$

If

$$\begin{cases} b_i - \min\ f_i(X) \leq 0 \\ a_i - \max\ f_i(X) \leq 0 \end{cases}$$

then

$$\begin{aligned} \Delta_i(g(X)) &\leq (b_i - a_i) + (\max\ f_i(X) - \min\ f_i(X)) \\ &= (b_i - \min\ f_i(X)) - (a_i - \max\ f_i(X)) \\ &= \Delta_i(X - f(X)). \end{aligned}$$

and if

$$\begin{cases} b_i - \min\ f_i(X) \leq 0 \\ a_i - \max\ f_i(X) \geq 0 \end{cases}$$

then

$$\max\ f_i(X) \leq a_i \leq b_i \leq \min\ f_i(X)$$

this contradicts the fact that exist at least an integer $j \in \{1,\dots,n\}$ such that $\min\ f_j(X) < \max\ f_j(X)$. This completes the proof of the lemma.

Now we provide a theorem to prove the convergence of the algorithm in the case where the function f is linear, i.e., $f(x) = Mx + q$; with M is $n \times n$ matrix and $q \in \mathrm{IR}^n$.

Theorem *Let M be a matrix satisfy that $\rho(I - M) < 1$, if there is a solution x^* of (2) in $X^{(0)}$ then $\lim_{k\to+\infty} X^{(k)} = x^*$.*

Proof. Using the fact (3) we have

$$\Delta(X^{(k+1)}) \leq \Delta(g(X^{(k)})), \tag{5}$$

and by previous lemma it follows that

$$\begin{aligned} \Delta(X^{(k+1)}) &\leq \Delta(X - f(X)) \\ &\leq (I - M)\Delta(X^{(k)}) \end{aligned}$$

Further, the matrix M satisfied $\rho(I - M) < 1$ then

$$\Delta(X^{(k+1)}) \to 0. \tag{6}$$

On the other hand if there is a solution x^* of (2) in $X^{(0)}$ then $x^* \in g(X^{0)})$, and from (3) we have $x^* \in X^{(1)}$; if we use the simple principle, we can deduct that $x^* \in X^{(k)}$.

Therefore from (6) we have $\lim_{k\to+\infty} X^{(k)} = x^*$.

With the above ideas, we suggest the following algorithm for solving the linear complementarity problem.

Algorithm

- **Initialization**

$k = 0$

$X^{(0)} = ([a^{(0)}, b^{(0)}])$ an interval vector in $[\mathrm{IR}^n_+]$;

tolerance ε;

- **Iterative step**

Step 1 Compute

$$\begin{cases} \min\ f(X^{(k)}) := \min_{x\in X^{(k)}} f(x) \\ \max\ f(X^{(k)}) := \max_{x\in X^{(k)}} f(x) \end{cases}$$

Step 2 Compute

$$\begin{cases} a^{(k+1)} := \max(a^{(k)}, \max(0, a^{(k)} - \max\ f(X^{(k)}))) \\ b^{(k+1)} := \min(b^{(k)}, \max(0, b^{(k)} - \min\ f(X^{(k)}))) \end{cases}$$

Step 3 If there is an i in the set $\{1,..,n\}$ such that

$$\begin{cases} (\max(0, b^{(k)} - \min\ f(X^{(k)})))_i < a_i^{(k)} \\ \text{or} \\ b_i^{(k)} < (\max(0, a^{(k)} - \max\ f(X^{(k)})))_i \end{cases}$$

then the solution is not existing in $X^{(0)}$; and therefore we redefine the algorithm with a new $X^{(0)} \in \mathrm{IR}^n_+]$ larger than the older one and go to **step 1**.

Else go to **step 4**.

Step 4 If $\| \Delta(X^{(k+1)} \| < \varepsilon$ then we obtain the solution $X^{(k+1)}$ and terminate the algorithm.

Otherwise $k{+}{+}$ and return to **step 1**.

4. Numerical tests

In this section, we provide numerical examples to demonstrate the efficiency of our algorithm. To test the efficiency of our proposed algorithm, we conducted the numerical experiments on some test problems.

In the following, we will implement our algorithm in Matlab 7.14 and run it on a personal computer with a 2.5 GHZ CPU processor and 512 MB memory. We stop the iterations if the condition $\|\Delta(X^{(k+1)}\| \leq 10^{-6}$ is satisfied.

Example 1 *Let us consider the following linear complementarity problem* $LCP(M,q)$, *find a vector* x *satisfying* $Mx+q \geq 0, \quad x \geq 0$ *and* $x^T(Mx+q)=0$ *where*

$$M = \begin{bmatrix} \frac{8}{5} & -\frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & \frac{8}{5} & -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} & \frac{8}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{1}{5} & \frac{8}{5} \end{bmatrix} \text{ and } q = \begin{bmatrix} -4 \\ 3 \\ -4 \\ 2 \end{bmatrix}$$

The exact solution of this problem is $x^* = (\frac{5}{2}, 0, \frac{5}{2}, 0)^T$. We have to note that $\rho(I-M) = \frac{7}{10} + \frac{1}{10}\sqrt{5} < 1$.

When looking for an approximation with six significant digits, we obtain that, our algorithm requires

CPU time = 0.1210014 s.

The test results of this example are summarized in Table 1.

Table 1. Numerical results for the first example with k=20

$X^{(0)}$		$X^{(k)}$	
$a^{(0)}$	$b^{(0)}$	$a^{(k)}$	$b^{(k)}$
0.000000	b 3.000000	a 2.499997	b 2.499999
a 0.000000	b 3.000000	a 0.000017	b 0.000025
a 0.000000	b 3.000000	a 2.499999	b 2.499999
a 0.000000	b 3.000000	a 0.000078	b 0.000099

The Table 1 shows that the numerical results using a min-max algorithm to solve this linear complementarity problem.

Example 2 *Consider the following class of linear complementarity problems: For a given integer* n, *find a vector* x *in* IR^n *satisfying*

$$x^T(Mx+q)=0,\ Mz+q \geq 0 \text{ and } x \geq 0$$

where $m_{ij} = i\delta_{ij}/n$ where δ is the Kronecker's delta ($\delta_{ii}=1$ and $\delta_{ij}=0$ if $i \neq j$) and $q=(q_i)_{1\leq i\leq n}$ where $q_i = -1$.

We have to note that $\rho(I-M) = \frac{n-1}{n} < 1$. This example is used by Elfoutayeni and Khaladi [14].

The exact solution of this problem is $x^* = (\frac{n}{1}, \frac{n}{2}, \ldots, \frac{n}{n})^T$.

When looking for an approximation with six significant digits, we obtain that, our algorithm requires:

CPU time = 0,077805 s, for n=3
CPU time = 0,169023 s, for n=5
CPU time = 0,275408 s, for n=7

The test results of this example are summarized in Table 2.

Table 2. Numerical results for the second example

for n=3

k	$X^{(0)}$	X^*
07	[0.000000,3.000000]	[2.999992,2.999997]
	[0.000000,3.000000]	[1.444446,1.444449]
	[0.000000,3.000000]	[0.999994,0.999997]

for n=5

K	$X^{(0)}$	X^*
13	[0.000000,5.000000]	[4.999992,4.999995]
	[0.000000,5.000000]	[2.555552,2.555556]
	[0.000000,5.000000]	[1.666666,1.666667]
	[0.000000,5.000000]	[1.250000,1.250005]
	[0.000000,5.000000]	[1.000005,1.000009]

for n=7

k	$X^{(0)}$	X^*
27	[0.000000,7.000000]	[7.000002,7.000009]
	[0.000000,7.000000]	[3.499994,3.500045]
	[0.000000,7.000000]	[2.333333,2.333334]
	[0.000000,7.000000]	[1.750012,1.750017]
	[0.000000,7.000000]	[1.399992,1.400004]
	[0.000000,7.000000]	[1.166665,1.166669]
	[0.000000,7.000000]	[1.000003,1.000011]

The Table 2 shows that the numerical results using a min-max algorithm to solve this linear complementarity problem.

5. Conclusion

In this paper we have demonstrated that, on the one hand, solving the linear complementarity problem $LCP(M,q)$ is equivalent to finding the fixed point of the system $x = \max(0,(I-M)x-q))$; on the other hand, we have described an algorithm for finding an approximation solution to the fixed point of the second problem; this algorithm starting from any interval vector $X^{(0)}$ and generating a sequence of the interval vector $(X^{(k)})_{k=1,..}$ satisfying $\lim_{k\to+\infty} X^{(k)} = x^*$ where x^* is the exact solution of linear complementarity problem. The numerical results indicate that our algorithm works reliably and efficiently.

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