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Estimation of Natural Frequencies and Mode Shapes of a Shaft Supported by more than Three Bearings

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Abstract The present paper emphasizes on the estimation of natural frequencies and mode shapes of a shaft supported by more than three bearings . In advent of this, a counter shaft of already developed experimental setup has been considered. The natural frequencies and mode shapes of counter shaft are deteremined analytically by adopting Holzer's method and the results obtained are then compared with the results of software based approach . By adopting the same method , Stresses in the shaft and amplitudes of rotors are also estimated. Based on the obtained data the generallised mathematical models have been formulated for the prediction of stresses in shaft and amplitudes of rotors. Significant independent variables which influence the phenomenon i.e development of stresses and amplitudes of rotor have been accounted in terms of a group of pie terms. These group of pie terms are accomplished with the help of dimensional analysis technique and they are used to formulate the mathematical models. In this paper the qualitative and quantitative analyses of the established mathematical models are also carried out.

Keywords: *countershaft, natural frequency, mode shapes, shear stress, mathematical model*

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1. Introduction

Rotor dynamics have mesmerized the intellect since medieval times. In the past era rotors were most often used at slower speed for the applications like moving a bullock cart and running of oil extraction units. However, as the time passed, the interest of human in the rotors aroused and different attempts were made to understand the dynamics of rotors in the past 300 years [10]. With the advent of modern technology, it has become much easier and less time consuming to analyze the behavior of a rotor system with the help of advanced software’s. The natural frequency is an inherent property of a rotating shaft and is a major entity in determining shaft’s stability in dynamic conditions. In industrial applications, it becomes inevitable to transfer the power of a motor to the end user through various mechanical components. These mechanical components could be pulleys, gears, chains, couplings etc. The shafts may be a driving shaft, a counter shaft or a driven shaft. These shafts handle the time varying loads which ultimately act on the machine components. The bearing elements help the shaft to bear such loads. Generally it is seen that, the shafts are placed on two or three bearings but in rare cases it is supported by more than three bearings. Such shafts are referred as a counter shaft (intermediate shaft) which bears the excessive load and becomes a node member. Hence it is required to investigate its behavior under different loading condition. Also, it is essential to find its natural frequencies and mode shapes. In this regards, a countershaft of already developed experimental set-up is selected and by using analytical and software based approaches the results are obtained.

2. Description of already Developed Experimental Setup

The detailed construction of the experimental setup [7], which is shown in Figure 1 is given below, A 0.5 HP motor is used to feed the power to various machine components. In this experimental setup a V-Belt drive is used for the first stage speed reduction. For this stage the velocity ratio is taken as 2.4. A smaller pulley P1 is keyed on the motor shaft S1, whereas, a bigger pulley P2 is placed on Shaft S2. This shaft S2 is placed on two antifriction bearings B1 and B2. The second stage speed reduction is with velocity ratio as 2 and it is accomplished with gear pair PI2GE2. Pinion PI2 is placed on the same shaft, where the bigger pulley P2 is placed. The GE2 is placed on shaft S4. The third stage speed reduction is accomplished by the gear pairs PI1GE1 and PI3GE3. The velocity ratio of this reduction is 2. Pinion PI1 and PI3 are placed on shafts S3 and S5 respectively, and Gear GE1 and GE3 are placed on shaft S6 and S7. A unique assembly is known as counter shaft which is formed by connecting the three shafts S3, S4 and S5 with the help of two rigid couplings CP1 and CP2. The six different antifriction bearings B3, B4, B5, B6, B7 and B8 withstand the loads and forces acting on the countershaft. The load drums BRD1, BRD2 are placed on shafts S6 and S7, where already GE1 and GE3 are placed. These shafts are supported by antifriction bearings B9, B10, B11 and B12. The antifriction bearings B1 to B12 all are fixed with the help of the pedestal to the frame. A frame is constructed with steel angles of size 30 mm X 30 mm X 5 mm and 50 mm x 50 mm x 5 mm. These steel angles are joined by welding process. This complete assembly of experimental set-up is then fixed to the foundation by using four foundation bolts.

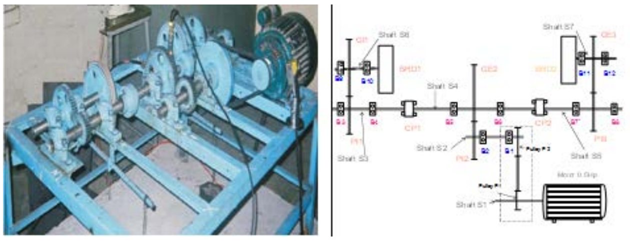

Figure 1. Pictorial and schematic view of experimental setup

2.1. Description of a Counter Shaft Assembly

The counter shaft is supported by six anti friction bearings. On this counter shaft, three powers transmitting components GE2, PI1, and PI3, are placed. The counter shaft is a unique one and it is assembled by three shafts S3, S4 and S5, which are connected by two rigid couplings CP1 and CP2. These three shafts S3, S4, S5 are having stepped sections for placing the appropriate power transmitting components.

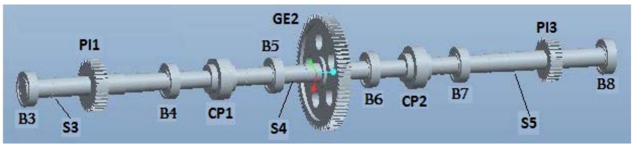

Figure 2. Schematic of countershaft in the setup

3. Estimation of Natural Frequencies and Mode Shapes for Counter Shaft by Numerical Approach

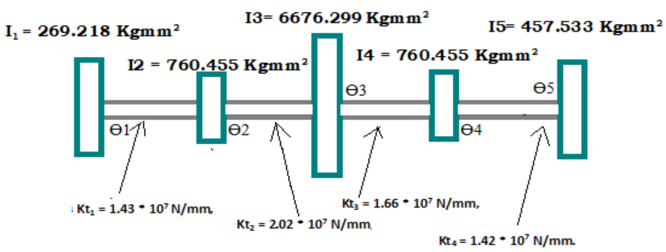

Figure 3. Equivalent system of countershaft considering intertias placed on countershaft

The counter shaft which is discussed above is used for the present analysis. It is required to convert the present counter shaft into an equivalent system. The equivalent system of a counter shaft is shown in Figure 3. In this Figure, I1, I2, I3, I4 and I5 refer to the inertias of pinion PI1, Coupling CP1, Gear GE2, Coupling CP2 and Pinion PI3. The equivalent system of counter shaft could be established by the theory of equivalent system [4].

After establishing the equivalent systems the Holzer's method [3] is then applied to find the natural frequencies and mode shapes of the shaft. Consider the systems shown in Figure 3, the corresponding sets of equations can be established, which is as under

$$I_1\ddot{\theta}_1 + K_1(\theta_1 - \theta_2) = 0 \tag{1}$$

$$I_2\ddot{\theta}_2 + K_1(\theta_2 - \theta_1) - K_2(\theta_3 - \theta_2) = 0 \tag{2}$$

$$I_3\ddot{\theta}_3 + K_2(\theta_3 - \theta_2) - K_3(\theta_4 - \theta_3) = 0 \tag{3}$$

$$I_4\ddot{\theta}_4 + K_3(\theta_4 - \theta_3) - K_4(\theta_5 - \theta_4) = 0 \tag{4}$$

$$I_5\ddot{\theta}_5 + K_4(\theta_5 - \theta_4) = 0 \tag{5}$$

In above equations I1, I2, I3, I4 & I5 are the inertias, K1, K2, K3 & K4 are torsional stiffnesses, $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ & Θ_5 are angular displacements. The equations (1-5) are the general equations of motions for torsional system refer to Figure 3. Further to obtain the approximate natural frequencies, the equations can be rewritten as.

$$T1 = 4367.8175w2 - 0.1844w4 + 2.36E-6w6 \tag{6}$$

$$T5 = 4556.1325w2 - 0.2696w4 + 4.927E-6w6 \tag{7}$$

In the above equations T1 and T2 are the residual torques at rotors I1 and I5

By putting the torque value equals to zero, one can obtain the approximate values of natural frequencies for the systems shown in Figure 3. These approximate values of natural frequencies are then used for iteration process to find out the natural frequencies of the system described in Figure 3. However the explaination of iteration process for finding out first natural frequency is given in Appendix 1.

Table 1. Values of natural frequencies for the system

W1 (Hz)	W2 (Hz)	W3 (Hz)	W4 (Hz)	W5 (Hz)
1.6E-5	26.20	28.98	35.89	76.49

4. An Estimation of Natural Frequencies and Mode Shapes for Counter Shaft by Software Approach

In this section, to have better insight, software based approach is being applied to the counter shaft for the estimation of natural frequencies and mode shapes. Here three approaches have been applied, namely 1) Matlab

Approach 2) Ansys Approach 3) Hypermesh and Radios Approach. And the results are shown in Figure 4- Figure 8 and Table 2.

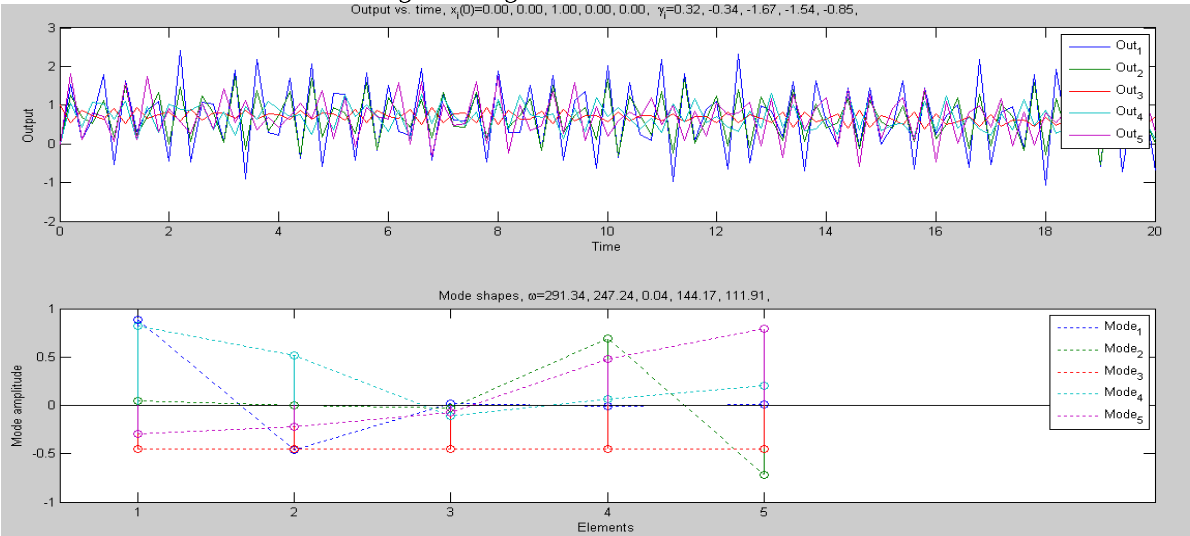

Figure 4. Matlab output for natural frequencies and modeshapes

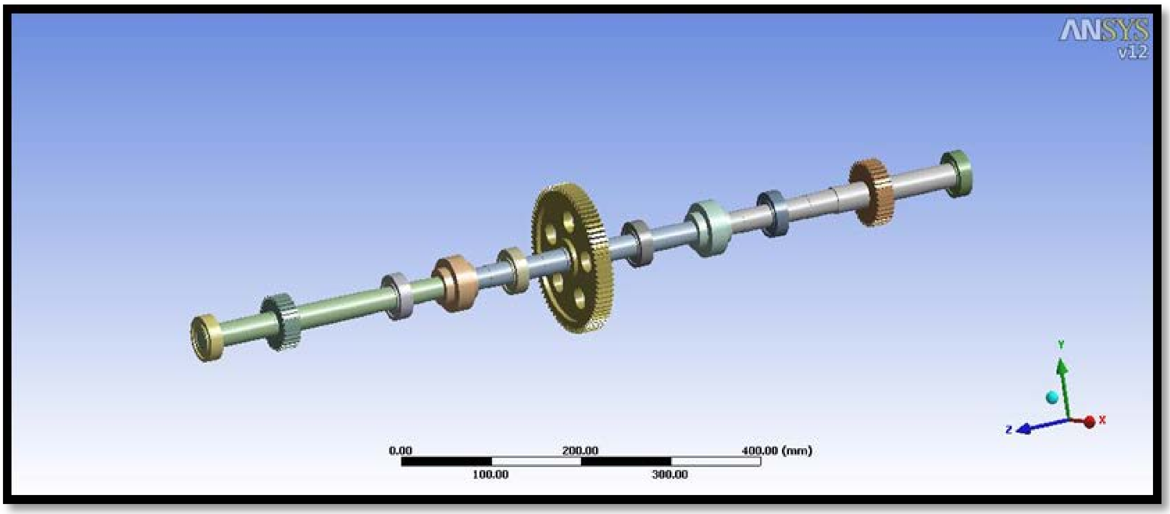

Figure 5. CAD model imported in Ansys Environment

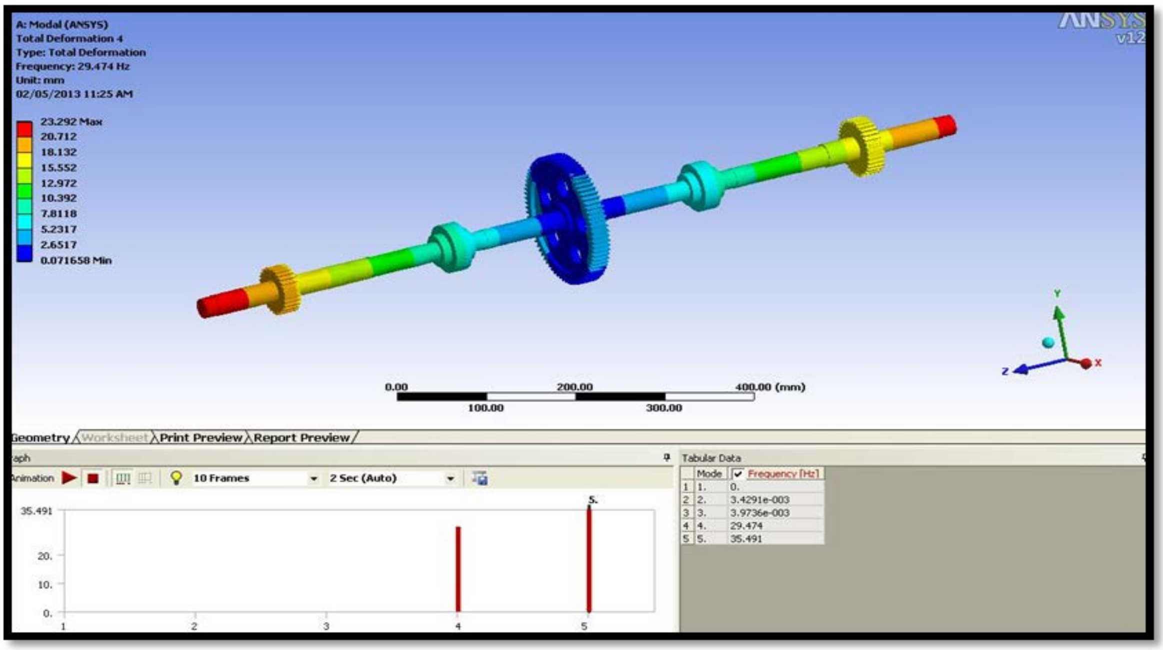

Figure 6. Ansys output for natural frequencies and mode shapes

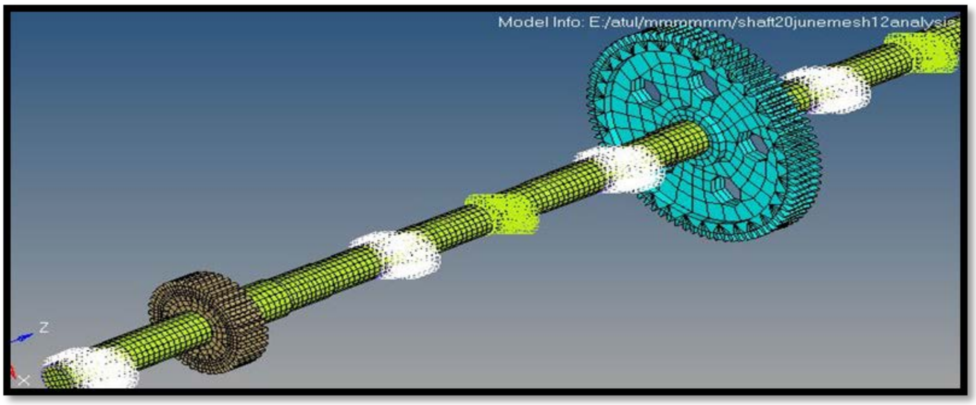

Figure 7. Meshed model in Hypermesh software

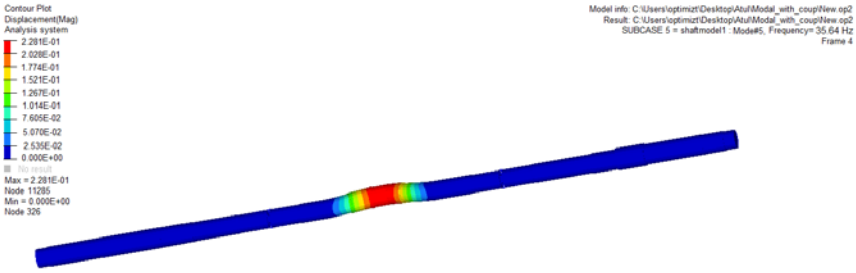

Figure 8. Radioss output for natural frequencies and mode shapes

Table 2. Values of natural frequencies from different approaches

	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)	W_n (Hz)
Holzer's Method	0.00					26.20	28.98	35.89	76.49
Matlab Software	0.006			17.82	22.95			39.36	46.39
Ansys software	0.00	0.003	0.004				29.47	35.49	
Hypermesh + Radioss software	0.006	0.053				26.43		35.64	75.26

5. An Estimation of Amplitudes of Rotors and Twist of the Counter Shaft Considering Forced Vibration

The former article focuses on the estimation of natural frequencies of a counter shaft. But if one applies a torque to the equivalent system of the counter shaft (Kindly refer Figure 3) then the case falls under forced vibration. Here the amplitudes of the rotors and angular twist have been estimated by the tabulation method same as the previously used in articles but with certain modification. The detail procedure is explained with reference to Figure 9 and Table 3 which is given as below.

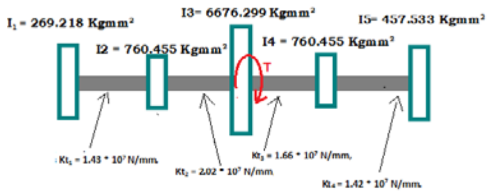

Figure 9. Five rotor system for torsion vibration

Table 3. Holzer's table for forced vibration for an equivalent system shown in Figure 9

1	2	3	4	5	6	7	8	9
w^2	S.no	I	Iw^2	Θ	Iw^2Θ	ΣIw^2Θ	K_t	$\Sigma Iw^2\Theta/K_t$
29697.63	1	269.218	7995136	Θ_1	7.9e6*Θ_1	7.9e6*Θ_1	15080996	0.5329Θ_1
29697.63	2	760.455	22583710	0.4671Θ_1	10.54e6*Θ_1	15.54e6Θ_1	20409614	0.9271Θ_1
29697.63	3	6676.299	1.98E+08	-0.460Θ_1	-91.27e6*Θ_1	-72.73e6Θ_1+T	16729192	-4.21 Θ_1+5.88e-8T
29697.63	4	760.455	22583710	3.81Θ_1-5.8e-8T	86.23e6*Θ_1-1.32T1	12.5e6Θ_1+0.32T	14372968	0.964Θ_1-2.28e-8T
29697.63	5	457.533	13587645	2.85Θ_1-8.16e-8T	38.83e6e6*Θ_1-1.11T	52.36e6Θ_1-1.43T		

After filling columns 1,2,3,4 and 8 one will proceed to fill the table row wise by adopting the generalized procedure of free vibration [3]. All quantities will be ascertained in the form of Θ1. As in the present case external torque is acting on the 3rd rotor (Figure 9) it inevitable to include external torque 'T' in the 3rd row of column 7. The procedure lasts till one gets the remainder torque as 52.36e6Θ_1-1.43T N-mm and that will be equal to zero (refer Equation 8). Thus, one can calculate Θ1 in radians and ie.. the amplitude of rotor 1.

$$52.36\times10^{6}\theta_1 - 1.43\mathrm{T} = 0 \qquad (8)$$

The amplitudes of other rotors have been obtained from column 5 by putting the values of Θ and Torque. Column 9 gives twist in the shaft and column 7 gives the twisting moment of the shaft. However, for different torque values the amplitudes of rotors, twist in the shafts and twisting moment have been calculated which are shown below. Note that, this torque variation is considered for the present analysis. Refer Appendix II.

Table 4. Amplitudes at different rotors of counter shaft

T (N-mm)	Θ1 (rad)	Θ2(rad)	Θ3(rad)	Θ4(rad)	Θ5(rad)
11953.691	0.000308	0.000144	-0.00014	0.000428	0.000409
119620.51	0.000308	0.000144	-0.00014	0.000445	0.000419
119704.22	0.000308	0.000144	-0.00014	0.000439	0.000416
120717.94	0.000308	0.000144	-0.00014	0.000446	0.00042
12253.39	0.000308	0.000144	-0.00014	0.000445	0.000419

Table 5. Angular twist in counter shaft due to varying torque

T (N-mm)	Ψ1	Ψ2	Ψ3	Ψ4
11953.691	0.000164	0.000281	-0.00057	1.85E-05
119620.51	0.000164	0.000281	-0.00058	2.58E-05
119704.22	0.000164	0.000281	-0.00058	2.3E-05
120717.94	0.000164	0.000281	-0.00058	2.6E-05
12253.39	0.000164	0.000281	-0.00058	2.56E-05

Table 6. Twisting moment of counter shaft at different torques

T (N-mm)	Mt1(N-mm)	Mt2(N-mm)	Mt3(N-mm)	Mt4(N-mm)
11953.691	2462.502	5704.172	-9394.81	263.4951
119620.51	2462.502	5704.172	-9686.15	368.2814
119704.22	2462.502	5704.172	-9576.41	328.8112
120717.94	2462.502	5704.172	-9694.51	371.2882
12253.39	2462.502	5704.172	-9677.78	365.2709

6. Estimation of Shear Stresses in Torsional Vibration

It is generally seen that shear stresses are generated in torsional vibration. Here in the present case shear stresses are induced in counter shaft due to variation in the torque. The detailed procedure is as under.

Firstly shear stresses are calculated theoretically with the basic formulas (Equation 10) given for maximum shear stress. However, angular displacements at different rotors with varying torque are estimated by the formula (Equation 9) which is given below:

$$\theta = (32/\pi)\times\left(\mathrm{TL}/\mathrm{Gd}^4\right) \text{ radians} \qquad (9)$$

Maximum shear stress is given by

$$\text{Maximum shear stress } \tau = (2\mathrm{T})/(\pi r^3)\mathrm{N/mm}^2 \qquad (10)$$

The corresponding results for angular displacements and maximum shear stresses at different torque values are tabulated below in Table 7.

Table 7. Values of maximum shear stress and angular displacement

Torque (N-mm)	0	2×10^3	5×10^3	8×10^3	10×10^3	11.87×10^3
Shear stress (Mpa)	0	0.651	1.629	2.607	3.259	3.869
Angular displacement (radian)	0	7.61E-4	1.9E-3	3.06E-3	3.8E-3	4.55E-3

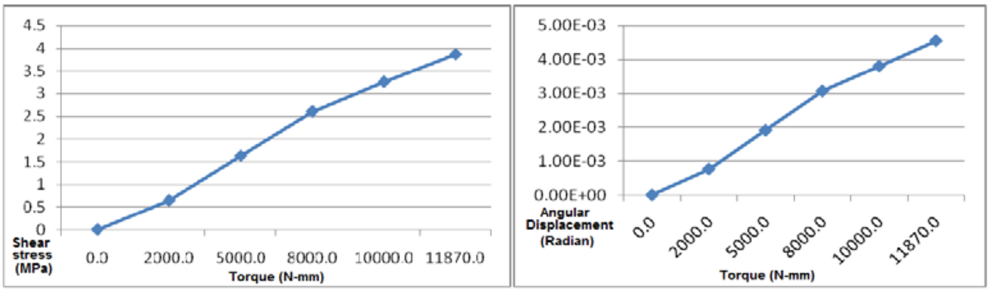

Figure 10. Graph for max shear stress and angular displacement

Secondly, Ansys software is used to estimate max shear stresses. As explained in the previous article 4, the counter shaft assembly is imported in Ansys work bench. A torque is applied with the magnitude of 11.28 N-m to the counter shaft and in the solution panel, maximum shear stresses and equivalent stresses are requested. Results for maximum shear stresses are shown below:

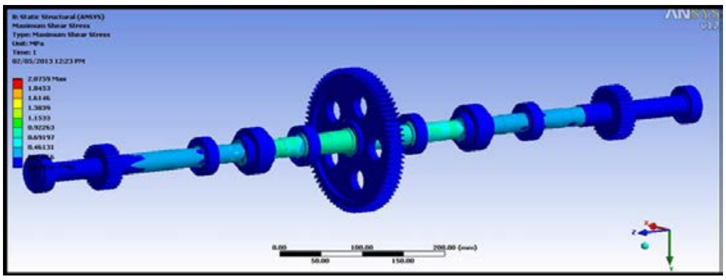

Figure 11. Results from Ansys workbench for Max shear stresses = 2.0759 MPa

7. Formulation of Approximate Generalized Mathematical Model

Eagerness in prowl of a new method for finding out the natural frequencies of a countershaft turned the present work for the formulation of mathematical models. In fact, one can represent any physical phenomenon by knowing its causes and effects. And if he further obtains the relationship amongst causes and effects, then it will be known as a mathematical model. The upcoming article gives a brief in site about the formulation of such mathematical model.

8. Need of Mathematical Model

The present work focuses on not only on the estimation of natural frequencies and mode shapes of a rotor system, but also to find the behavior of the rotor when torque is being applied. Thus the rotor system falls under forced vibration. By considering the forced vibration one can obtain the amplitudes for the rotors, the angular twist in the shafts and shear stresses induced in the counter shaft. It is seen that one has to do the rigorous calculation for it and this is time consuming and laborious. This in fact instigates to think how to find out an alternate way so that one can easily find out the amplitudes of the rotors, angular twist and shear stresses induced in the counter shaft just by putting the available physical values. Hence it is decided to formulate a mathematical model, which certainly helps to find out the amplitudes of the rotors, angular twist and shear stresses induced in the counter shaft.

9. Process of Model Formulation

The process of model formulation requires some important steps [5].

1. Identification of variables.
2. Reduction of variables through dimensional analysis.
3. Test planning.
4. Obtain the data from numerical solutions.
5. Model formulation by identifying the constant and various indices of pie terms.

9.1. Identification of Variables

Generally the model formulation process starts with the identification of variables. These variables are (1) Independent variables or causes (2) Dependent variables or effects and (3) Extraneous variable. The independent variables may vary according to the choice of designer while the dependent variable may only vary if there is a variation in the independent variable. The extraneous variables are random and one does not have any control over it. For the present work following vantage variables have been identified.

Table 8. List of independent and dependent variables

Sr. No.	Dependent/ Independent Variable	Name of Variable	MLT form	Representation
1	Dependent variable	Amplitude of rotors	$M^0L^0T^0$	Θ
2	Dependent variable	Angular twist in the shafts	$M^0L^0T^0$	Ψ
3	Dependent variable	Shear stress induced in the shafts	$ML^{-1}T^{-2}$	T
4	Independent variable	Mass moment of Inertias of rotors	ML^2	I_1, I_2, I_3, I_4, I_5
5	Independent variable	Second moment of area of shaft	L^4	I_s
6	Independent variable	Modulus of elasticity	$ML^{-1}T^{-2}$	$E_1, E_2, E_3, E_4, E_5, E_s$
7	Independent variable	Equivalent distances	L	C_1, C_2, C_3, C_4
8	Independent variable	Torque	ML^2T^{-2}	T
9	Independent variable	Acceleration due to gravity	LT^{-2}	G
10	Independent variable	Poisons ratio	$M^0L^0T^0$	μ

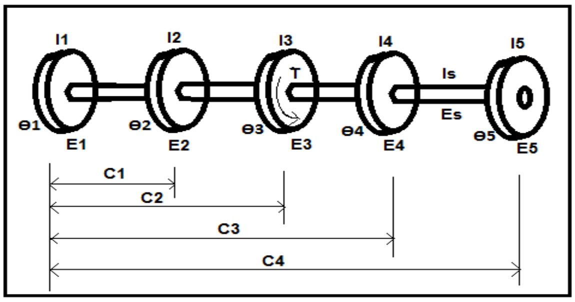

Figure 12. Five rotors system considered for dimension analysis

9.2. Reduction of Variables through Dimensional Analysis

In the second stage, all the independent variables have been reduced down into a group of Pie terms [5]. One can do it with the help of a dimensional analysis technique. This technique gives the ability to reduce any form of variables with less time. The basic steps indulged in this technique are as follows.

9.2.1. Identification of Variables which are Independent from each Others.

9.2.2. By Using the Rayleigh's Method, to Get the Dependent and Independent Pie terms [5].

For amplitude and angular twist the dimensional analysis is given as below

$$[\theta, \Psi] = \Phi \begin{pmatrix} I_1{}^a, I_2{}^b, I_3{}^c, I_4{}^d, I_5{}^e, I_s{}^f, \\ E_1{}^g, E_2{}^h, E_3{}^i, E_4{}^j, E_5{}^k, E_s{}^l, \\ C_1{}^m, C_2{}^n, C_3{}^o, C_4{}^p, T^q, N^r, g^s \end{pmatrix} \quad (11)$$

$$M^0L^0T^0 = \Phi \begin{bmatrix} (ML^2)^a, (ML^2)^b, (ML^2)^c, (ML^2)^d, \\ (ML^2)^e, (L^4)^f, (ML^{-1}T^{-2})^g, (ML^{-1}T^{-2})^h, \\ (ML^{-1}T^{-2})^i, (ML^{-1}T^{-2})^j, (ML^{-1}T^{-2})^k, \\ (ML^{-1}T^{-2})^l, (L)^m, (L)^n, (L)^o, (L)^p, \\ (ML^{-2}T^{-2})^q, (T^{-1})^r, (LT^{-2})^s \end{bmatrix} \quad (12)$$

For M:

$$0 = a + b + c + d + e + g + h + i + j + k + l + r \quad (i)$$

For L:

$$\begin{aligned} 0 &= 2a+2b+2c+2d+2e+4f-g \\ &-h-i-j-k-l+m+n+o+p-2q+2r \end{aligned} \qquad \text{(ii)}$$

For T

$$0 = -2g-2h-2i-2j-2k-2l-2q-2r-s \qquad \text{(iii)}$$

Now

From (i)

$$a = -b-c-d-e-f-g-h-i-j-k-q \qquad \text{(iv)}$$

From (ii)

$$p = 3f+3g+3h+3i+3j+3k-4l-m-n-o-s \qquad \text{(v)}$$

From (iii)

$$r = -2f-2g-2h-2i-2j-2k-2q-2s \qquad \text{(vi)}$$

The resultant equation is

$$[\theta, \Psi] = \Phi \begin{pmatrix} I_1^{-b-c-d-e-f-g-h-i-j-k-q}, \\ I_2^{b}, I_3^{c}, I_4^{d}, I_5^{e}, I_s^{f}, E_1^{g}, E_2^{h}, \\ E_3^{i}, E_4^{j}, E_5^{k}, E_s^{l}, C_1^{m}, C_2^{n}, C_3^{o}, \\ C_4^{3f+3g+3h+3i+3j+3k-4l-m-n-o-s}, \\ T^{q},\ N^{-2f-2g-2h-2i-2j-2k-2q-2s},\ g^{s} \end{pmatrix} \qquad (13)$$

Now by simplifying the above function one will get

$$(\theta) = \left[\left(\frac{I_2 I_4}{I_3 I_5}\right)\right]^a, \left[\left(\frac{E_1 E_3 E_5}{E_2 E_4 E_s}\right)\right]^b, \left[\left(\frac{C_1}{C_2}\right)\right]^c \left[\left(\frac{Tg}{I_1 C_4 N^4}\right)\right]^d \qquad \text{(14 (a))}$$

The above equation can be rewritten as

$$\pi_0 = \pi^a{}_1, \pi^b{}_2, \pi^c{}_3, \pi^d{}_4 \qquad \text{(14 (b))}$$

Where

$$\begin{aligned} &\pi_0 = (\theta), \left[\left(\pi_1 = \frac{I_2 I_4}{I_3 I_5}\right)\right], \left[\left(\pi_2 = \frac{E_1 E_3 E_5}{E_2 E_4 E_s}\right)\right], \\ &\left[\left(\pi_3 = \frac{C_1}{C_2}\right)\right], \left[\left(\pi_4 = \frac{Tg}{I_1 C_4 N^4}\right)\right] \\ &(\psi) = \left[\left(\frac{I_2 I_4}{I_3 I_5}\right)\right]^a, \left[\left(\frac{E_1 E_3 E_5}{E_2 E_4 E_s}\right)\right]^b, \\ &\left[\left(\frac{C_1}{C_2}\right)\right]^c \left[\left(\frac{Tg}{I_1 C_4 N^4}\right)\right]^d \end{aligned} \qquad \text{(14 (c))}$$

The above equation can be rewritten as

$$\pi_0 = \pi^a{}_1, \pi^b{}_2, \pi^c{}_3, \pi^d{}_4 \qquad \text{(14 (d))}$$

Where

$$\begin{aligned} &\pi_0 = (\psi), \left[\left(\pi_1 = \frac{I_2 I_4}{I_3 I_5}\right)\right], \left[\left(\pi_2 = \frac{E_1 E_3 E_5}{E_2 E_4 E_s}\right)\right], \\ &\left[\left(\pi_3 = \frac{C_1}{C_2}\right)\right], \left[\left(\pi_4 = \frac{Tg}{I_1 C_4 N^4}\right)\right] \end{aligned}$$

Similarly for shear stress dimensional analysis carried out as

$$\tau = \Phi \begin{pmatrix} I_1^{a}, I_2^{b}, I_3^{c}, I_4^{d}, I_5^{e}, \\ E_1^{f}, E_2^{g}, E_3^{h}, E_4^{i}, E_5^{j}, E_s^{k}, \\ I_s^{l}, C_1^{m}, C_2^{n}, C_3^{o}, C_4^{p}, \\ T^{q}, N^{r}, g^{s}\mu^{t} \end{pmatrix} \qquad (15)$$

Dimensionless form of the above function can be written as

$$M^0L^{-1}T^{-2} = \Phi \begin{bmatrix} \left(ML^2\right)^a, \left(ML^2\right)^b, \left(ML^2\right)^c, \left(ML^2\right)^d, \\ \left(ML^2\right)^e, \left(ML^{-1}T^{-2}\right)^f, \left(ML^{-1}T^{-2}\right)^g, \\ \left(ML^{-1}T^{-2}\right)^h, \left(ML^{-1}T^{-2}\right)^i, \left(ML^{-1}T^{-2}\right)^j, \\ \left(ML^{-1}T^{-2}\right)^k, \left(L^4\right)^l, (L)^m, (L)^n, (L)^o, (L)^p, \\ \left(ML^{-2}T^{-2}\right)^q, \left(T^{-1}\right)^r, \left(LT^{-2}\right)^s, \left(M^0L^0T^0\right)^t \end{bmatrix} \qquad (16)$$

For M,

$$1 = a+b+c+d+e+f+g+h+i+j+k+q \qquad \text{(vii)}$$

For L,

$$\begin{aligned} -1 &= 2a+2b+2c+2d+2e-f-g-h \\ &-i-j-k+4l+m+n+o+p+2q+s \end{aligned} \qquad \text{(viii)}$$

For T,

$$-2 = -2f-2g-2h-2i-2j-2k-2q-r-2s \qquad \text{(ix)}$$

From equation (vii)

$$f = 1-a-b-c-d-e-g-h-i-j-k-q \qquad \text{(x)}$$

From equation (ix)

$$r = 2a+2b+2c+2d+2e-2s \qquad \text{(xi)}$$

From equation (x), substituting value of term ‘f’ in equation (vii)

$$m = -3a-3b-3c-3d-3e-4l-n-o-p-3q-s \qquad \text{(xii)}$$

The resultant function is

$$[\tau] = \Phi \begin{pmatrix} I_1^{a}, I_2^{b}, I_3^{c}, I_4^{d}, I_5^{e}, \\ E_1^{1-a-b-c-d-e-g-h-i-j-k-q}, \\ E_2^{g}, E_3^{h}, E_4^{i}, E_5^{j}, E_s^{k}, I_s^{l}, \\ C_1^{-3a-3b-3c-3d-3e-4l-n-o-p-3q-s}, \\ C_2^{n}, C_3^{o}, C_4^{p}, T^{q}, \\ N^{2a+2b+2c+2d+2e-2s},\ g^{s}\mu^{t} \end{pmatrix} \qquad (17)$$

By simplifying the above function

$$\begin{aligned} \left(\frac{\tau}{E_1}\right) &= \left[\left(\frac{I_1 I_3}{I_2 I_4}\right)\right]^a, \left[\left(\frac{E_2 E_4 \mu}{E_3 E_5}\right)\right]^b \\ &, \left[\left(\frac{I_5 N^3 C_4 C_3}{I_s E_s C_2 C_1^4}\right)\right]^c \left[\left(\frac{TN^6}{g E_1 C_1^2}\right)\right]^d \end{aligned} \qquad \text{(18 (a))}$$

The above equation can also be written as,

$$\pi_0 = \pi^a{}_1, \pi^b{}_2, \pi^c{}_3, \pi^d{}_4 \qquad \text{(18 (b))}$$

Where

$$\pi_0 = \left(\frac{\tau}{E_1}\right), \left[\left(\pi_1 = \frac{I_1 I_3}{I_2 I_4}\right)\right], \left[\left(\pi_2 = \frac{E_2 E_4 \mu}{E_3 E_5}\right)\right],$$

$$\left[\left(\pi_3 = \frac{I_5 N^3 C_4 C_3}{I_s E_s C_2 C_1^4}\right)\right], \left[\left(\pi_4 = \frac{TN^6}{gE_1 C_1^2}\right)\right]$$

9.3. Test Planning

This step is very prominent to minimize the error in mathematical modeling and maximize the use of data. It has the following steps.

1. Test envelope
2. Test points
3. Test sequence

9.3.1. Test Envelope

Test envelope is the range over which one can vary the independent variables. This in fact affects the dependent variables. In the present case the selection of test envelope is totally dependent on the choice of designer ie.. the selection of Torque, Speed, Mass moment of inertia of rotors, Elasticity and Equivalent distances etc. However the following is the test envelope for different values of Pie terms.

Table 9. The values of different range of test points for π1, π2, π3, π4 for dependent pie term for various models

Terms	π1	π2	π3	π4
	Amplitude Θ1			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Amplitude Θ2			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Amplitude Θ3			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Amplitude Θ4			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Amplitude Θ5			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Angular twist Ψ1			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Angular twist Ψ2			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Angular twist Ψ3			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Angular twist Ψ4			
Minimum	0.187972	0.980464	0.378731	0.1989
Maximum	0.187972	0.980464	0.378731	0.2991
	Shear stress τ			
Minimum	3.108084	0.333275	0.003637	3.5256
Maximum	3.108084	0.333275	0.003825	3.6140

9.3.2. Test Points

While solving numerical solutions, some discrete values of independent variables have been taken. These values are known as test points, indeed test points are generally bounded in the domain of test envelope. The following are test points of the present work.

Table 10. The values of different test points for π1, π2, π3 and π4

Test Points	π1	π2	π3	π4
For amplitudes Θ1 to Θ5 and angular twists Ψ1 to Ψ4	0.187972	0.980464	0.378731	0.1989
	0.187972	0.980464	0.378731	0.2520
	0.187972	0.980464	0.378731	0.2661
	0.187972	0.980464	0.378731	0.2776
	0.187972	0.980464	0.378731	0.2991
Test Points	π1	π2	π3	π4
For Maximum Shear stress τ	3.108084	0.333275	0.003902	3.5256
	3.108084	0.333275	0.003896	3.6114
	3.108084	0.333275	0.003811	3.5786
	3.108084	0.333275	0.003825	3.6140
	3.108084	0.333275	0.003637	3.6089

9.3.3. Test Sequence

The method of test sequence is of two types (a) Irreversible and, (2) Reversible. This is quite tedious to get desired test condition at any time in irreversible. In reversible method one can attain the desired test point at any instant. While in irreversible technique one cannot attend the previous test points, this is applicable in experimentation. But in the present case the data are obtained from numerical solution, hence one can use both the methods as needed. In the present case the irreversible method is adopted.

9.4. Data Generation

This step is already covered in article 5.

9.5. Model Formulation by Identifying the Constant and Various Indices of Pie Terms

The indices of different pie terms aimed at model can be identified by using multiple regression analysis. By considering four independent pie terms and one dependent pie term, Let model aimed at be of the form,

$$(\pi) = K*(\pi_1)^a*(\pi_2)^b*(\pi_3)^c*(\pi_4)^d \quad (19)$$

The regression equations become asunder.

$$\begin{aligned}
&\sum Y = nK1 + a\sum A + b\sum B + c\sum C + d\sum D \\
&\sum YA = K1\sum A + a\sum A^2 + b\sum AB + c\sum AC + d\sum AD \\
&\sum YB = K1\sum B + a\sum AB + b\sum B^2 + c\sum BC + d\sum BD \quad (20) \\
&\sum YC = K1\sum C + a\sum AC + b\sum BC + c\sum C^2 + d\sum CD \\
&\sum YD = K1\sum D + a\sum AD + b\sum BD + c\sum CD + d\sum D^2
\end{aligned}$$

In the former equations n is the number of sets of readings, A,B,C,D depicts the independent pie terms π1,π2,π3 and π4, whilst Y represents, dependent pie term. Afterwards, estimate the values of independent pie terms for corresponding dependent pie term, which helps to form the equations in matrix form.

The following matrix represents the equations, which is used for programming.

Then, [Y] = [X] x [a]

By solving, the above matrix in MATLAB software, the values of different indices have been found out

K = 1, a = 4.8036, b = 3.6, c = -0.0001, d = 0.06

Thus, the mathematical model for Θ1 is formed as

$$\theta_1 = 1\times(\Pi_1)^{4.8}\times(\Pi_2)^{3.6}\times(\Pi_3)^{-0.0001}\times(\Pi_4)^{0.06} \quad (21)$$

Likewise the mathematical models for other terms like amplitudes Θ2 to Θ5, angular twist in the shafts Ψ1 to Ψ4 and shear stress τ have been formed which are given in the next article.

10. Qualitative Analysis of Mathematical Model (Interpretation)

In this article, the qualitative analyses of models have been given. It is nothing but the interpretation of obtaining mathematical models. The interpretation of model is reported in terms of several aspects viz (1) order of influence of various inputs (causes) on the outputs (effects) (2) Interpretation of curve fitting constant K. The mathematical models which are established in the previous article are considered here for the sake of further explanation. These models would predict the amplitudes of the rotors, angular twist and shear stress induced in the shaft.

Following are the models which are in the prediction of amplitudes (Θ)

$$\theta_1 = 1\times(\Pi_1)^{4.8}\times(\Pi_2)^{3.6}\times(\Pi_3)^{-0.0001}\times(\Pi_4)^{0.06} \quad (22)$$

$$\theta_2 = 1\times(\Pi_1)^{5.25}\times(\Pi_2)^{3.22}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{-0.0001} \quad (23)$$

$$\theta_3 = 1\times(\Pi_1)^{5.26}\times(\Pi_2)^{4.87}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{0.0002} \quad (24)$$

$$\theta_4 = 1\times(\Pi_1)^{4.51}\times(\Pi_2)^{2.73}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{0.0002} \quad (25)$$

$$\theta_5 = 1\times(\Pi_1)^{4.56}\times(\Pi_2)^{3.6}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{0.06} \quad (26)$$

Following are the models which are for the prediction of angular twist (Ψ)

$$\Psi_1 = 1\times(\Pi_1)^{5.1}\times(\Pi_2)^{4.919}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{0.0002} \quad (27)$$

$$\Psi_2 = 1\times(\Pi_1)^{4.8}\times(\Pi_2)^{4.9}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{-0.0001} \quad (28)$$

$$\Psi_3 = 1\times(\Pi_1)^{4.347}\times(\Pi_2)^{4.396}\times(\Pi_3)^{0.0001}\times(\Pi_4)^{0.0754} \quad (29)$$

$$\Psi_4 = 1\times(\Pi_1)^{3.625}\times(\Pi_2)^{3.1302}\times(\Pi_3)^{-0.0001}\times(\Pi_4)^{0.8309} \quad (30)$$

Following are the models which are for the prediction of shear stress (τ)

$$\tau = 0.99\times(\Pi_1)^{-616}\times(\Pi_2)^{-628}\times(\Pi_3)^{-0.0002}\times(\Pi_4)^{3.16} \quad (31)$$

10.1. Order of Influence of Various Inputs and Their Relative Influence for Mathematical Model of Θ1

Equation (22) is formed based on data obtained from numerical solution. From this equation it is seen that π_1 term, which relates to the inertia of the total rotor system, has the highest influence as 4.8 on effect i.e. amplitude for the first rotor of the system. The least influence is seen for π3 as -0.0001, which relates to geometrical entities of the system. The π2 terms which relate to the stiffness of the system has an influence of 3.6. And π4 term indicates the dynamics of the system has an influence of 0.06. Similarly an influence for models like Θ2 to Θ5 can be estimated.

10.2. Order of Influence of Various Inputs and Their Relative Influence for Mathematical Model of Ψ1

Equation (27) is formed based on data obtained from numerical solution. From this equation it is seen that π_1 term, which relates to inertia of the total rotor system, has the highest influence as 5.1 on effect i.e. amplitude for the first rotor of the system. The least influence is seen for π3 as 0.0001, which relates to geometrical entities of system. The π2 terms which relate to the stiffness of the system has an influence of 4.919. And π4 term indicates the dynamics of the system has an influence of 0.0002. Similarly influence for models like Ψ 2 to Ψ 4 can be estimated.

10.3. Order of Influence of Various Inputs and Their Relative Influence for Mathematical Model of T

Equation (31) is formed based on data obtained from numerical solution. From this equation it is seen that π_1 term, which relates to the inertia of the total rotor system, has influence as -616 on effect i.e. amplitude for the first rotor of the system. The least influence is seen for π3 as -0.0002, which relates to geometrical entities of the system. The π2 terms which relate to the stiffness of the system has influence of -6.28. And π4 term indicates the dynamics of the system has the highest influence of 3.16

11. The Quantitative Analysis of Mathematical Model

The main aim of this section is to carry out quantitative analysis of mathematical models. This analysis comprises of Sensitivity, Optimization and Reliability of the model.

11.1. Sensitivity Analysis

This sensitivity analysis helps the investigator to check the sensitiveness of each independent Pie terms involved in mathematical modelling. This analysis is inevitable when one would like to calculate the relative influence. The procedure for the sensitivity analysis is described as given below.

One could observe the fair change in dependent pie terms by substituting the percentage change of independent pie terms.

Hence, for the present case a change of +/- 10 % is being carried out in individual pie terms (one at a time) there by seeing the effect on the dependent variable. Thus, there is a total 20 % change is being introduced in individual pie terms.

By introducing 20% change in each pie terms the percentage change in dependent pie terms is obtained for all mathematical models. However the discussion of sensitivity of mathematical model referred to Θ1is given below, whereas sensitivity graphs for other mathematical models are given in Appendix III.

11.1.1. The Effect of Introducing Change in the Dependent Pie Term for Θ1

When a 20% change is being introduced in independent pie term π1 referred to the model (Equation 22), the change of 97.78 % is seen in dependent pie terms. On the other side of the flip 57.10 % of change is observed in dependent pie term due to 20% change in independent pie term π2. Similarly, 0.0017% and 0.002 % of changes have been found in dependent pie terms due to 20% change in independent π3 and π4 respectively.

If one minutely observes about the change in dependent pie terms then he could conclude that the highest change in dependent pie terms occured due to the change in independent pie terms π1. Whilst the least change in the dependent pie terms is cognized due to independent pie term π3. Hence, in this regards one can say that π1 is the most sensitive and π3 is the least sensitive pie term.

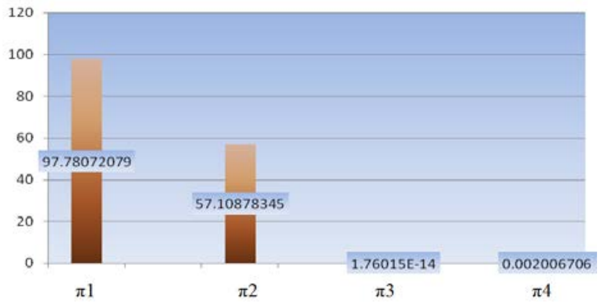

Figure 13. Sensitivity Graph of independent pie terms of model Θ1

11.2. Optimization of Model

The main intention of the present work is not merely to come out with a mathematical model, but to provide the best set of independent variables. This in turn will help us to find out the maximum or minimum value of the dependent pie term aiming with objective function. As far as the present case is concerned, the objective is to minimize the amplitudes in the rotors, twist in the shaft and shear stresses in the shaft. The present model depicted in a nonlinear form and for the optimization of this model, it is to be converted into linear form. It is carried out by taking log on both the sides. For the minimization of linear function one may use linear programming techniques as detailed below.

For the dependent π term, we have

$$(\pi) = K*(\pi 1)a\times(\pi 2)b\times(\pi 3)c\times(\pi 4)d \quad (32)$$

Taking Log on both the side of this equation,

$$\text{Log}(Y) = \text{Log } K + a \text{ Log}(\pi 1) + b \text{ Log}(\pi 2) + c \text{ Log}(\pi 3) + d \text{ Log}(\pi 4) \quad (33)$$

Let, Log (Y) =Z; Log K=K1; Log (π1)=X1; Log (π2)=X2; Log (π3)=X3; and Log(π4) = X4

Then the linear model in the form of first degree of polynomial can be written as,

$$Z = k + a*X_1 + b*X_2 + c*X_3 + d*X_4 \quad (34)$$

In this case, amplitudes of rotors, angular twist in the shaft and shear stresses induced in the shaft are the objective functions for the optimization with the specific target of minimization in view of linear programming problem. Secondly, it is required to apply the constraints to the problem. While gathering a numerical data certain range of independent pie terms are achieved. In fact this range has a minimum and maximum value. Therefore, this range can be taken as constrains for this problem. Thus, there are two constraints for each independent variable.

Let, π1 max and π1 min, are the maximum and minimum value of independent pie term, Thus the first two constraints for the problem will be obtained by taking Log of these quantities and by substituting the value of multipliers of all other variables except the one under consideration equal to zero. Let the log limits be defined as C1 and C2 (i.e. C1= Log π1 max), (i.e. C2= Log π1 min).

Hence the equation of constraints becomes.

$$1*X_1 + 0*X_2 + 0*X_3 + 0*X_4 \le C_1 \quad (35)$$

$$1*X_1 + 0*X_2 + 0*X_3 + 0*X_4 \ge C_2 \quad (36)$$

The other constraints are also found to be.

$$0*X_1 + 1*X_2 + 0*X_3 + 0*X_4 \le C_3 \quad (37)$$

$$0*X_1 + 1*X_2 + 0*X_3 + 0*X_4 \ge C_4 \quad (38)$$

$$0*X_1 + 0*X_2 + 1*X_3 + 0*X_4 \le C_5 \quad (39)$$

$$0*X_1 + 0*X_2 + 1*X_3 + 0*X_4 \ge C_6 \quad (40)$$

$$0*X_1 + 0*X_2 + 0*X_3 + 1*X_4 \ge C_7 \quad (41)$$

$$0*X_1 + 0*X_2 + 0*X_3 + 1*X_4 \ge C_8 \quad (42)$$

By solving the above linear programming one can get the minimum value of Z, and the best set of values of independent pie terms to acquire this minimum value. However, the values of dependent pie term and independent pie terms could be acquired by taking antilog of Z, X1, X2, X3 and X4. The present linear programming problem is solved by MS Solver. This function is available in Microsoft Excel office. By solving the problem in MS solver the optimized values of dependent and independent pie terms are given below.

Hence, $Z_{min}=(Y)_{min}$=antilog (-3.557)=2.77e-4 and corresponding values of independent pie terms are obtained by taking the antilog of X1, X2, X3 and X4, These values are 0.187,0.980,0.378,0.198 etc. Similarly optimization of mathematical models for amplitudes Θ2 to Θ4, angular twist Ψ4 to Ψ4 and shear stress τ can be estimated.

11.3. Reliability of Model

The reliability term is associated with the quality of the model. In the present work the reliability of the model is evaluated asunder.

In the present model the known values of independent pie terms are submitted and thus obtained dependent pie terms. This is known as calculated values of dependent pie term. Now, one could find the error by subtracting the calculated values from the observed values of the dependent pie term. Once the error is estimated, then reliability can be estimated by calculating the mean error.

This can be done by using following formula,

$$\text{Reliability} = 1 - \text{Mean error} \quad (43)$$

Where, Mean error= ΣXIFI/ ΣFI

Where, ΣXIFI= Summation of the product for percentage of error and frequency of error occurrence and ΣFI= Summation of frequency of error occurrence. Hence for the reliability for present models can be calculated.

Reliability of mathematical model for the prediction of amplitudes ϴ 1 = 99.982995%, ϴ2 = 99.999852%, ϴ3 = 99.999995%, ϴ4 = 99.999691%, ϴ5 = 99.999667%, for of angular twist Ψ1 = 99.999873%, Ψ2= 99.999848%, Ψ3= 99.999174%, Ψ4= 99.999863%, for maximum shear stress τ = 99.948303%.

12. Conclusions

1. It has been seen that for certain running frequency the counter shaft of experimental setup shows more vibration, hence to investigate real cause, the present analysis is carried out and it is found that some running frequencies lies near by the natural frequency and system goes into resonance condition.

2. Results obtained from different approaches (Refer Table 4) are approximately same. The small variations in results are due to assumptions and constraints which are considered.

3. As the natural frequency is an inherent quality of the system and it is a function of inertia and the stiffness, it is not affected by any change in dynamic characteristics like torque and speed. So, if stiffness or inertia of the system is changed, the natural frequencies of the system will be changed.

4. Results obtained for stresses generated in the counter shaft and amplitudes of rotors, shows that the system is running under safe condition. Refer Table 7

5. To ascertain the ampltiudes of rotors and shear stresses for such a countershaft requires regourus calculation, hence through this paper a solution is provided in terms of mathematical model.

6. Mathematical models which have been formulated can predict the amplitudes of rotors, the angular twist and shear stresses induced in the shaft.

7. The reliability of the obtained models is fairly good. However, some variables of the mathematical model are more sensitive, whereas some are less sensitive. Refer Figure 13.

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Appendix I

Table 11. Holzer's table for first natural frequency estimation

			w	0.04	rad/sec	0.006369	Hz	
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
w^2	**S.no.**	**I**	**Iw^2**	**ϴ**	**$Iw^2ϴ$**	**$ΣIw^2ϴ$**	**K_t**	**$ΣIw^2ϴ/K_t$**
0.0016	1	269.218	0.430749	1	4.31E-07	1.23E-05	15.081	7.9E-07
0.0016	2	760.455	1.216728	1	1.22E-06	1.19E-05	20.4096	5.2E-07
0.0016	3	6676.299	10.68208	1	1.07E-05	1.07E-05		
0.0016	4	760.455	1.216728	1	1.22E-06	1.19E-05	16.7292	6.4E-07
0.0016	5	457.533	0.732053	1	7.32E-07	1.26E-05	14.373	8.3E-07
			w	0.05	rad/sec	0.007962	Hz	
1	2	3	4	5	6	7	8	9
w2	S.no.	I	Iw2	ϴ	Iw2ϴ	ΣIw2ϴ	Kt	ΣIw2ϴ/Kt
0.0025	1	269.218	0.673045	1	6.73E-07	1.93E-05	15.081	1.2E-06
0.0025	2	760.455	1.901138	1	1.9E-06	1.86E-05	20.4096	8.2E-07
0.0025	3	6676.299	16.69075	1	1.67E-05	1.67E-05		
0.0025	4	760.455	1.901138	1	1.9E-06	1.86E-05	16.7292	1E-06
0.0025	5	457.533	1.143833	1	1.14E-06	1.97E-05	14.373	1.3E-06
			w	0.06	rad/sec	0.009554	Hz	
1	2	3	4	5	6	7	8	9
w2	S.no.	I	Iw2	ϴ	Iw2ϴ	ΣIw2ϴ	Kt	ΣIw2ϴ/Kt
0.0036	1	269.218	0.969185	1	9.69E-07	2.77E-05	15.081	1.8E-06
0.0036	2	760.455	2.737638	1	2.74E-06	2.68E-05	20.4096	1.2E-06
0.0036	3	6676.299	24.03468	1	2.4E-05	2.4E-05		
0.0036	4	760.455	2.737638	1	2.74E-06	2.68E-05	16.7292	1.4E-06
0.0036	5	457.533	1.647119	1	1.65E-06	2.84E-05	14.373	1.9E-06
			w	0.01	rad/sec	0.001592	Hz	
1	2	3	4	5	6	7	8	9
w2	S.no.	I	Iw2	ϴ	Iw2ϴ	ΣIw2ϴ	Kt	ΣIw2ϴ/Kt
0.0001	1	269.218	0.026922	1	2.69E-08	7.71E-07	15.081	4.9E-08
0.0001	2	760.455	0.076046	1	7.6E-08	7.44E-07	20.4096	3.3E-08
0.0001	3	6676.299	0.66763	1	6.68E-07	6.68E-07		
0.0001	4	760.455	0.076046	1	7.6E-08	7.44E-07	16.7292	4E-08
0.0001	5	457.533	0.045753	1	4.58E-08	7.89E-07	14.373	5.2E-08

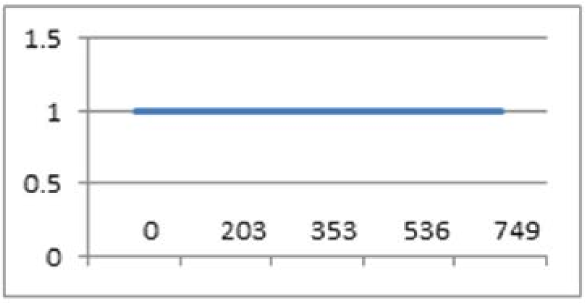

Figure 14. Mode shape for wn=1.6 E-5 Hz

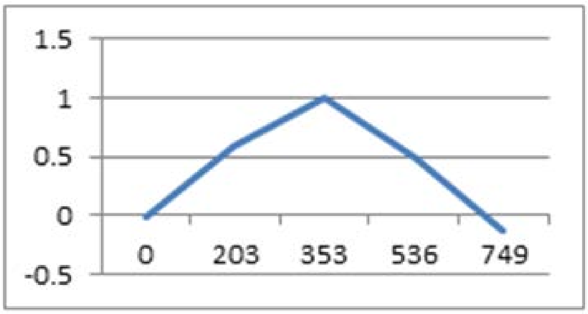

Figure 15. Mode shape for wn=26.20 Hz

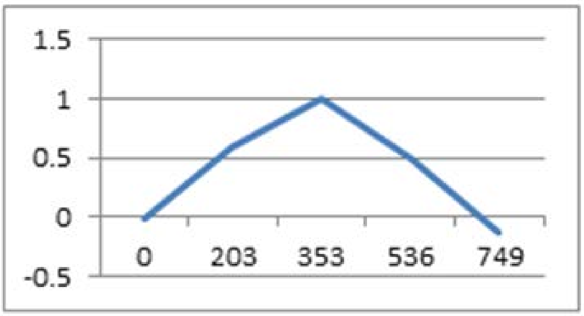

Figure 16. Mode shape for wn=28.98 Hz

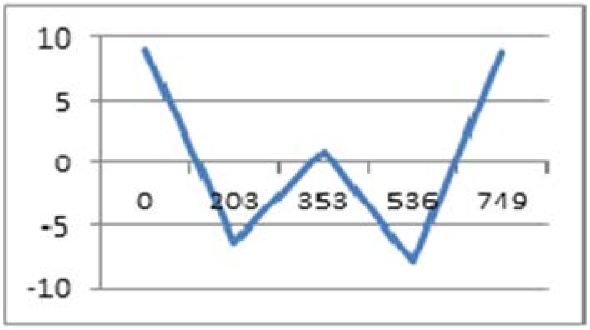

Figure 17. Mode shape for wn=35.89 Hz

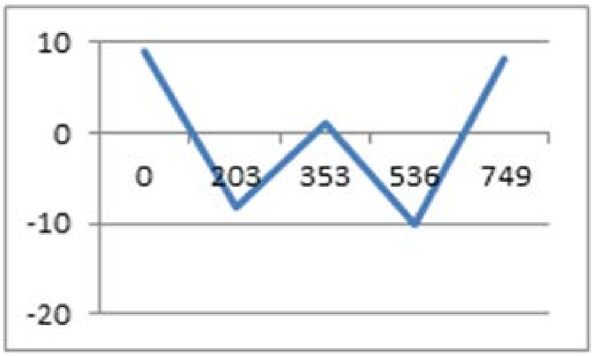

Figure 18. Mode shape for wn=76.49 Hz

Appendix II

Table 12. Holzer's Table for forced vibration considering Torque = 11953.69 N-mm

1	2	3	4	5	6	7	8	9
w^2	**S.no.**	**I**	**Iw^2**	**Θ**	**Iw^2Θ**	**ΣIw^2Θ**	**K_t**	**ΣIw^2Θ/K_t**
29697.63	1	269.218	7995136	0.000308	2462.501967	2462.502	14973276.82	0.0001645
29697.63	2	760.455	22583710	0.000144	3241.670524	5704.172	20263834.64	0.0002815
29697.63	3	6676.299	1.98E+08	-0.00014	-27352.3734	-9694.51	16609700.52	-0.000584
29697.63	4	760.455	22583710	0.000446	10065.79781	371.2879	14270306.08	0.000026
29697.63	5	457.533	13587645	0.00042	5702.630497	6073.918		

Table 13. Holzer's Table for forced vibration considering Torque = 11962.05 N-mm

1	2	3	4	5	6	7	8	9
w^2	**S.no.**	**I**	**Iw^2**	**Θ**	**Iw^2Θ**	**ΣIw^2Θ**	**K_t**	**Σiw^2Θ/K_t**
29697.63	1	269.218	7995136	0.000308	2462.501967	2462.502	14973276.82	0.0001645
29697.63	2	760.455	22583710	0.000144	3241.670524	5704.172	20263834.64	0.0002815
29697.63	3	6676.299	1.98E+08	-0.00014	-27352.3734	-9686.15	16609700.52	-0.000583
29697.63	4	760.455	22583710	0.000445	10054.43097	368.281	14270306.08	0.000026
29697.63	5	457.533	13587645	0.000419	5698.654559	6066.936		

Table 14. Holzer's Table for forced vibration considering Torque = 11970.42 N-mm

1	2	3	4	5	6	7	8	9
w^2	**S.no.**	**I**	**Iw^2**	**Θ**	**Iw^2Θ**	**Σiw^2Θ**	**K_t**	**Σiw^2Θ/K_t**
29697.63	1	269.218	7995136	0.000308	2462.501967	2462.502	14973276.82	0.0001645
29697.63	2	760.455	22583710	0.000144	3241.670524	5704.172	20263834.64	0.0002815
29697.63	3	6676.299	1.98E+08	-0.00014	-27352.3734	-9677.78	16609700.52	-0.000583
29697.63	4	760.455	22583710	0.000445	10043.04917	365.2702	14270306.08	0.000026
29697.63	5	457.533	13587645	0.000419	5694.673389	6059.944		

Table 15. Holzer's Table for forced vibration considering Torque = 121071.79 N-mm

1	2	3	4	5	6	7	8	9
w^2	**S.no.**	**I**	**Iw^2**	**Θ**	**Iw^2Θ**	**ΣIw^2Θ**	**K_t**	**Σiw^2Θ/K_t**
29697.63	1	269.218	7995136	0.000308	2462.501967	2462.502	14973276.82	0.0001645
29697.63	2	760.455	22583710	0.000144	3241.670524	5704.172	20263834.64	0.0002815
29697.63	3	6676.299	1.98E+08	-0.00014	-27352.3734	-9576.41	16609700.52	-0.000577
29697.63	4	760.455	22583710	0.000439	9905.21671	328.8098	14270306.08	0.000023
29697.63	5	457.533	13587645	0.000416	5646.461808	5975.272		

Table 16. Holzer's Table for forced vibration considering Torque = 12253.39 N-mm

1	2	3	4	5	6	7	8	9
w^2	**S.no.**	**I**	**Iw^2**	**Θ**	**Iw^2Θ**	**ΣIw^2Θ**	**K_t**	**ΣIw^2Θ/K_t**
29697.63	1	269.218	7995136	0.000308	2462.501967	2462.502	14973276.82	0.0001645
29697.63	2	760.455	22583710	0.000144	3241.670524	5704.172	20263834.64	0.0002815
29697.63	3	6676.299	1.98E+08	-0.00014	-27352.3734	-9394.81	16609700.52	-0.000566
29697.63	4	760.455	22583710	0.000428	9658.306089	263.4951	14270306.08	0.000018
29697.63	5	457.533	13587645	0.000409	5560.096438	5823.592		

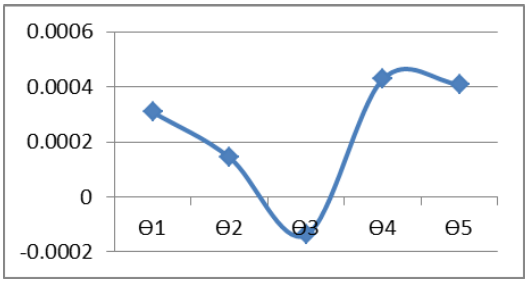

Figure 19. Mode shape at T =11953.69 N-mm 20 : Mode shapes at T=11962.05 N-mm

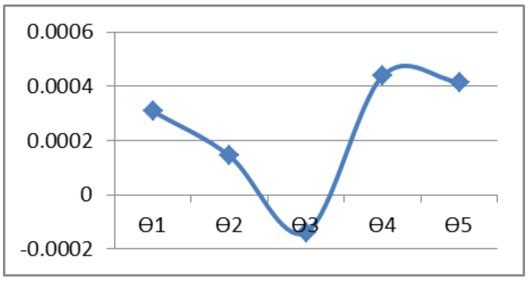

Figure 21. Mode shape at T=11970.42 N-mm 22: Mode shape at T=121071.79 N-mm

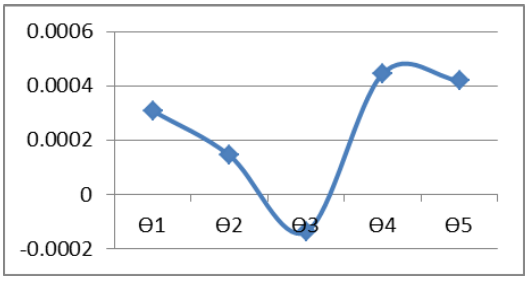

Figure 20. Mode shapes at T=11962.05 N-mm

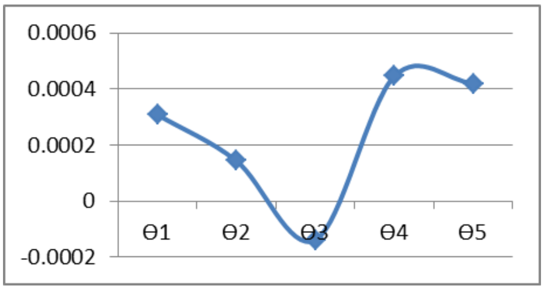

Figure 22. Mode shape at T=12253.39 N-mm

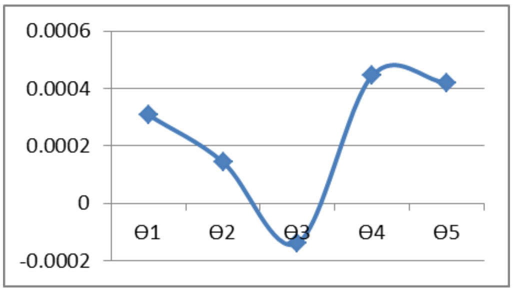

Figure 23. Mode shape at T=12253.39 N-mm

Figure 19- Figure 23: Amplitudes of rotors at different torque values.

Appendix III

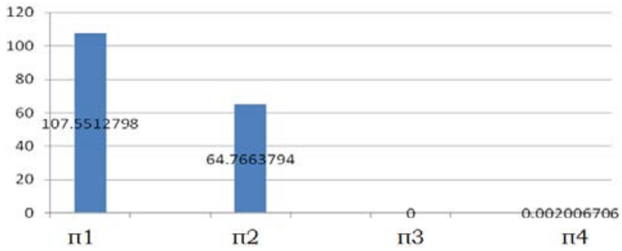

Figure 24. Sensitivity Graph of independent pie terms of model Θ2

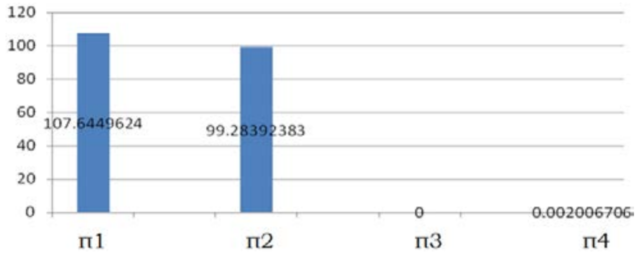

Figure 25. Sensitivity Graph of independent pie terms of model Θ3

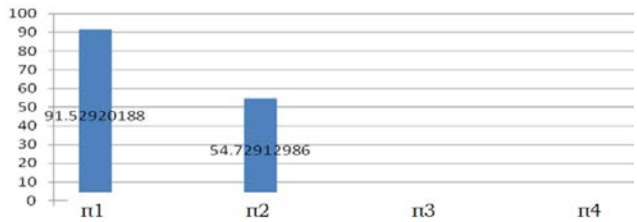

Figure 26. Sensitivity Graph of independent pie terms of model Θ4

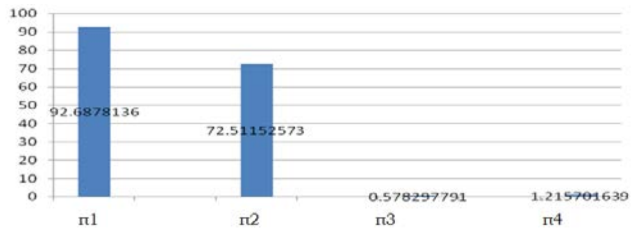

Figure 27. Sensitivity Graph of independent pie terms of model Θ5

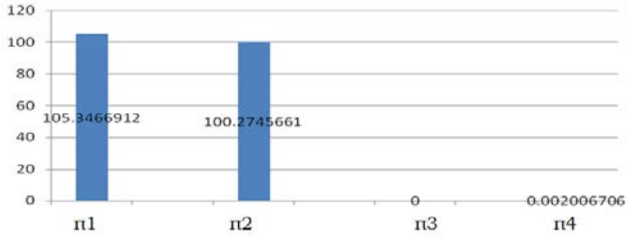

Figure 28. Sensitivity Graph of independent pie terms of model Ψ1

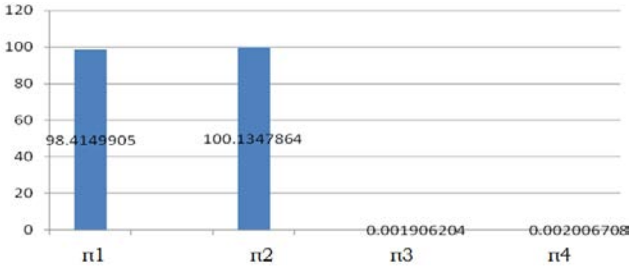

Figure 29. Sensitivity Graph of independent pie terms of model Ψ2

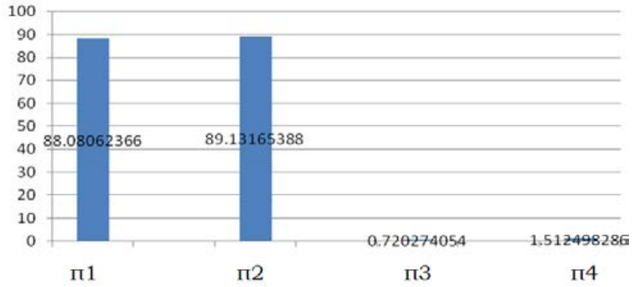

Figure 30. Sensitivity Graph of independent pie terms of model Ψ3

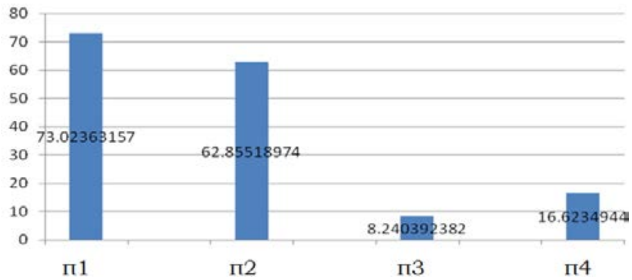

Figure 31. Sensitivity Graph of independent pie terms of model Ψ4

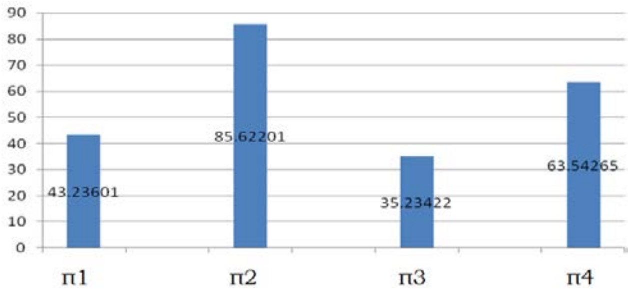

Figure 32. Sensitivity Graph of independent pie terms of model τ_{max}