Journal of Finance and Economics, 2024, Vol. 12, No. 1, 1-14
Available online at http://pubs.sciepub.com/jfe/12/1/1
Published by Science and Education Publishing
DOI:10.12691/jfe-12-1-1

Univariate and Multivariate Volatility Models for Portfolio Value at Risk

Jingyi Xiao, Siqi Mao*, Xufeng Niu, Yixin Kang

Department of Statistics, Florida State University, Tallahassee, USA
*Corresponding author: sqmao17@gmail.com

Received December 21, 2023; Revised January 25, 2024; Accepted February 02, 2024

Abstract In modern day financial risk management, modeling and forecasting stock return movements via their conditional volatilities, particularly predicting the Value at Risk (VaR), became increasingly more important for a healthy economical environment. In this paper, we evaluate and compare two main families of models for the conditional volatilities - GARCH - in terms of their VaR prediction performance of 5 major US stock indices. We calculate GARCH-type model parameters via Quasi Maximum Likelihood Estimation (QMLE). Since financial volatilities are moving together across assets and markets, it becomes apparent that modeling the volatilities in a multivariate framework of modeling is more appropriate. However, existing studies in the literature do not present compelling evidence for a strong preference between univariate and multivariate models. In this paper we also address the problem of forecasting portfolio VaR via multivariate GARCH models versus univariate GARCH models. We construct 3 portfolios with stock returns of 3 major US stock indices, 6 major banks and 6 major technical companies respectively. For each portfolio, we model the portfolio conditional covariances with GARCH, EGARCH and MGARCH-BEKK, MGARCH-DCC, and GO-GARCH models. For each estimated model, the forecast portfolio volatilities are further used to calculate (portfolio) VaR. The ability to capture the portfolio volatilities is evaluated by MAE and RMSE; the VaR prediction performance is tested through a two-stage backtesting procedure and compared in terms of the loss function. The results of our study indicate that even though MGARCH models are better in predicting the volatilities of some portfolios, GARCH models could perform as well as their multivariate (and computationally more demanding) counterparts.

Keywords: *Value at Risk (VaR), GARCH-type Model, Multivariate GARCH models*

Cite This Article: Jingyi Xiao, Siqi Mao, Xufeng Niu, and Yixin Kang, “Univariate and Multivariate Volatility Models for Portfolio Value at Risk.” *Journal of Finance and Economics*, vol. 12, no. 1 (2024): 1-14. doi: 10.12691/jfe-12-1-1.

1. Introduction

In the financial markets, volatilities of assets are often the major components of risk [1], thus one main objective of modeling and forecasting the conditional volatilities in asset returns is risk management. While risk is associated with the probabilities about the future, a risk measure is required to summarize the risks into a single number. Value at risk (VaR) is one of the most commonly used risk measure in risk management [2], which is defined as the maximum potential loss of a risky asset with a given confidence interval over a determined time period [3]. One approach of VaR calculation is determined by conditional volatilities [4], thus two main types of volatility models GARCH-type models and Stochastic Volatility models are investigated and compared in terms of VaR forecasting in this paper.

1.1. Univariate Volatility Models

ARCH (Autoregressive conditional heteroskedasticity) model was originally introduced by [5], he expressed the conditional variance as a deterministic function of the past squared return innovations and the model was highly successful in capturing the financial time series data characteristics. After that, many different ARCH-type models were developed, of which the most famous one was Generalized ARCH (GARCH) model introduced by [6]. In order to capture the asymmetry exhibited in the financial data, a new class of ARCH-type models was introduced. The most popular model to parameterize the asymmetric effects is Exponential GARCH (EGARCH) by [6] model. Stochastic Volatility model works as an alternative to the GARCH-type models. It is fundamentally different from the GARCH-type models and it contains an additional innovation term to account for the financial market innovations beside the returns' innovations.

1.2. Multivariate Volatility Models

It is well accepted that the asset returns in a portfolio are moving together with time-varied conditional covariance and dynamic conditional correlations.

Therefore, predicting and modeling the conditional volatilities of the portfolio return process is of crucial importance in risk management, e.g., the computation of the portfolio VaR. Existing univariate volatility models consider only single time series which can't provide information about the dependency between assets returns within a portfolio. In order to capture the time-varied co-movements of assets in a portfolio, the univariate volatility models have been extended to the multivariate volatility models.

Reference [8] proposed a direct extension of univariate GARCH models to multivariate ones in terms of VEC representation, in which each element of the covariance matrix is determined by the squared errors and cross-products of errors and lagged values of the elements. To reduce the number of parameters, [8] also imposed diagonal to the model, names DVEC model, that each element of the covariance matrix depends only on its own lag and the previous value of the cross-products of the errors. However, even under the diagonal assumption, the number of parameters increases with $O(N^2)$ as the number of assets increase. Moreover, the positiveness of the covariance matrix is difficult to guarantee in the VEC/DVEC representation without imposing strong restrictions on the parameters.

Reference [9] proposed the BEKK model, which can impose positiveness on the covariance matrix easily and it is a specific case of VEC model. However, the practical usefulness of both VEC and BEKK models can be affected by "the curse of dimensionality" [10]. That is, even after imposing restrictions, the number of parameters of both VEC and BEKK increases with $O(N^2)$ as the number of assets increases. Thus, it's not surprising they are rarely used when the number of series is greater than 3 [11].

To solve the dimensions problem without losing the generality, [12] proposed the GO-GARCH model, which is based on the idea that the co-movements of the stock returns in a portfolio are driven by a small number of common factors. According to [12], the observed return process is governed by a linear combination of uncorrelated unobserved economic components where each component can be described by a GARCH(1,1) process. Therefore, it is still feasible to fit GO-GARCH model when the number of assets is large.

The covariance of multivariate time series can be represented in terms of correlations, thus the conditional variance can be modeled in a hierarchical way by first choosing a univariate GARCH model for each of the conditional variance, then model the conditional correlation matrix based on the residuals from the conditional covariance step. Reference [13] proposed an MGARCH model that assumes constant conditional correlation (CCC) over time. While the constant conditional correlation is empirically not possible, thus another class of MGARCH models that allows for dynamic conditional correlations (DCC) are proposed by [14,15]. Both the CCC and DCC models are less greedy in parameters therefore they are more easily estimable [11].

In this paper, we compare the performances of forecasting a portfolio's VaR using multivariate GARCH models versus using univariate GARCH models. We study 3 portfolios of stock returns from 3 major US stock indices, 6 major banks and 6 major technical companies respectively. For each portfolio, we model the portfolio conditional covariances with univariate GARCH, EGARCH and multivariate GARCH-BEKK, DCC, and GO-GARCH models. The estimated models are then used to forecast the portfolio volatilities which are further used to calculate the portfolio VaR. The abilities of capturing the portfolio volatilities are evaluated through MAE and RMSE, then the performances of VaR prediction are tested through a two-stage backtesting procedure and examined by the loss function.

The results of the paper demonstrate that in terms of capturing the portfolio volatilities, MGARCH models perform better than univariate GARCH models for the Index portfolio, while conclusion on the Tech portfolio is opposite: univariate GARCH models act better than MGARCH models. There is no straightforward conclusion for the Bank portfolio. As for the VaR prediction, GARCH works best for the Tech portfolio, DCC performs best in the Bank portfolio, and GO-GARCH acts best for the Index portfolio. With additional analysis for the correlations within the portfolio, we can conclude that multivariate models are more suitable to forecast the VaR only when the composite assets are highly correlated in a portfolio. If the composite assets are not highly correlated, extension of univariate GARCH models to multivariate GARCH models is not necessary. Thus, it typically depends on the data to decide if it is worthwhile to apply multivariate volatility models for the VaR prediction.

2. Methodologies

In this section, the univariate volatility models and the multivariate extensions of the GARCH models covering the specifications are discussed.

Let $y_t = ln S_t / S_{t-1}$ be the log-return from time $t-1$ to t, where S_t is the asset price at time $t = 1,2,3,\ldots,T$. The time series interest, y_t, can be decomposed into two components: the predictable conditional mean and the unpredictable innovation [16]. That is,

$$y_t = \mu_t + \epsilon_t$$

where I_t is all the information available at time t, $\mu_t = E(y_t \mid I_{t-1})$ is the predictable conditional mean and ϵ_t is the unpredictable innovation. The conditional mean return μ_t can be adjusted as a k-lagged autoregressive process AR(k):

$$\mu_t = E(y_t \mid I_{t-1}) = c_0 + \sum_{i=1}^{k} c_i r_{t-i}$$

where $c_i, i = (1,2,\ldots,t-1)$ is the i-th order autoregressive parameter.

2.1. Univariate Volatility Models

2.1.1. ARCH(q)

The ARCH(q) model expressed the conditional variance as a linear function of the past q squared innovations:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2$$

where $\alpha_0 > 0$ and $\alpha_i > 0, i = 1,2,\ldots,q$ to ensure the positiveness of the conditional variance. The unconditional variance of ARCH(q) is $\sigma = \frac{\alpha_0}{1-\sum_{i=1}^{a}\alpha_i}$ with $\sum_{i=1}^{a} \alpha_i < 1$ to guarantee the covariance stationarity. Financial times series usually exhibit volatility clustering, that is, a volatile period tends to be followed by another volatile period. The ARCH allows for time varying conditional volatility and therefore is able to capture the volatility clustering.

2.1.2. GARCH (p,q)

After its success in describing the characteristics of financial time series, many extensions of the ARCH model has been proposed, the most popular of which is the Generalized ARCH (GARCH) introduced by [6], who was Engle's Ph.D student at that time. The GARCH model considered the conditional variance as a linear function of the past q squared innovations as well as the past p observations of itself.

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

where $\alpha_0 > 0$ and $\alpha_i > 0, i = 1,2,\ldots,q$ and $\beta_j \geq 0$ for $j = 1,2,\ldots,p$ to avoid the negative variance. If $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$, then ϵ_t is co-variance stationary with unconditional variance

$$\sigma^2 = \frac{\alpha_0}{1-\sum_{i=1}^{q}\alpha_i - \sum_{j=1}^{p}\beta_j}$$

2.1.3. EGARCH (p,q)

The leverage effect suggests an asymmetry constraint on the conditional volatilities, thus a new class of models was introduced to capture the asymmetry exhibited in the data. The most popular of which is [6] Exponential GARCH or EGARCH (p,q) model.

In the EGARCH model, the conditional variance σ_t^2 is modeled as an asymmetric function of the lagged innovations ϵ_{t-i}.

$$ln\,\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q}\left(\alpha_i \left|\frac{\epsilon_{t-i}}{\sigma_{t-i}}\right| + \gamma_i \frac{\epsilon_{t-i}}{\sigma_{t-i}}\right) + \sum_{j=1}^{p} \beta_j \, ln\,\sigma_{t-j}^2$$

Where β_j is the persistence parameter, if β_j is large, then the volatility is persistent and we require $|\beta_j| < 1$ to guarantee the covariance stationarity. The logarithm transformation assures the non-negative variance thus no restrictions are needed to be imposed in the model. $z_t = \epsilon_t / \sigma_t$ is the standardized innovation, the parameter α_i measures the magnitude effect of z_t. The asymmetry parameter γ_i allows the conditional variance σ_t^2 to respond asymmetrically to rises and to falls in stock price. If $\gamma_1 = 0$ then a positive innovation $\epsilon_t > 0$ has the same effect on volatility as a negative innovation $\epsilon_t > 0$. If $\gamma > 0$, then the past negative innovations have a larger impact on conditional variance than the positive innovations. If $\gamma < 0$, then the past positive innovations have a larger impact on conditional variance than the negative innovations. The coefficient γ is typically negative [17], and the presence of leverage effect can be confirmed by testing the hypothesis if $\gamma_1 < 0$.

2.2. Multivariate Volatility Models

Let $y_t = (y_{1t}, y_{2t}, \ldots, y_{Nt})'$, be a vector of return time series with dimension N that observed by the past information I_{t-1}, and θ be the vector of parameters. Similar as the univariate case, y_t can be decomposed into two disjoint components: the conditional mean vector and the innovation vector.

$$y_t = \mu_t(\theta) + \epsilon_t$$

where $\mu_t = E(y_t \mid I_{t-1})$ traces the time evolution of the conditional mean of y_t (for convenience we leave out θ in the notation). It can be modeled through a vector autoregressive (VAR) representation, typically a VAR(1).

2.2.1. MGARCH-VEC

The VEC-GARCH model of [8] is a straightforward generalization of univariate GARCH models. They proposed that each element of H_t is a function of all lagged values of elements of H_t, as well as the lagged squared innovations and cross-products of innovations. which is written as:

$$vech(H_t) = C + \sum_{j=1}^{q} A_j vech(\epsilon_t \epsilon_t') + \sum_{i=1}^{p} G_i vech(H_{t-1})$$

where $vech(\cdot)$ is the operator that stacks the lower triangular proportion of an $N \times N$ matrix as an $N(N+1)/2 \times 1$ vector. A and G are square parameter matrices of order $N(N+1)/2$ and C is an $N(N+1)/2 \times 1$ parameter vector. For the bivariate case and $p = q = 1$, the VEC-GARCH model can be written as:

$$\begin{bmatrix} h_{11,t} \\ h_{12,t} \\ h_{22,t} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1}^2 \\ \epsilon_{1,t-1}\epsilon_{2,t-1} \\ \epsilon_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{12,t-1} \\ h_{21,t-1} \end{bmatrix}$$

In this case, the number of parameters is 21. In the VEC representation of the MGARCH process, ϵ_t is covariance stationary if and only if all the eigenvalues of A + G be less than 1 in modulus. In which case the unconditional variance matrix Σ, equal to $E(H_t)$, is given by

$$vech(\Sigma) = \left(I_{N^*} - A - G\right)^{-1} vech(C)$$

where $N^* = N(N+1)/2$.

The generalization of VEC is flexible and straightforward but has some disadvantages. First, the positiveness of H_t is not guaranteed [8]. Second, the number of parameters of VEC is $N(N+1)\left(N(N+1)+1\right)/2 = O\left(N^4\right)$, which is very large unless N is small. Third, the estimation of the parameters is computational demanding.

To overcome the shortcoming of large number of parameters, [8] simplified the VEC model by imposing diagonality on the matrices A and G. In this situation, each element of H_t will depend on its own past values and on the previous values of $\epsilon_{it}\epsilon_{jt}$, the number of parameters reduces to $N(N+5)/2$. But the positive definiteness of the covariance matrix H_t is not guaranteed without imposing strong restrictions on the parameters [18]. Thus, useful restrictions are derived by [9], who proposed a new parameterization for H_t in which the positiveness can be imposed easily by construction.

2.2.2. MGARCH-BEKK

The BEKK model can be viewed as a restricted version of VEC representation, see [9]. BEKK is named according to the synthesized work by Baba, Engle, Kraft, and Kroner. Given in GARCH(1,1) case, the model has the form:

$$H_t = C^{*'}C^* + \sum_{k=1}^{K} A_k^{*'}\epsilon_{t-1}\epsilon_{t-1}'A_k^* + \sum_{k=1}^{K} G_k^{*'}H_{t-1}G_k^*$$

The summation limit K determines the generality of the process, we will focus on K = 1 throughout the whole study. C^* is an upper triangular matrix, A_k^*, and G_k^* are $N \times N$ square matrices. For the bivariate case with $p = q = 1$ and $K = 1$, the BEKK model can be written as:

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{21,t} & h_{21,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}'$$

$$+\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \epsilon_{1,t-1}^2 & \epsilon_{1,t-1}\epsilon_{2,t-1} \\ \epsilon_{1,t-1}\epsilon_{2,t-1} & \epsilon_{2,t-1}^2 \end{bmatrix}'$$

$$+\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} h_{11,t-1} & h_{12,t-1} \\ h_{21,t-1} & h_{22,t-1} \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}'$$

The number of parameters in the BEKK reduces to 11. For the BEKK model to be covariance stationary, it is required that all the eigenvalues of $A_k^* \otimes A_k^* + G_k^* \otimes G_k^*$ be less than one in modulus. The unconditional variance matrix Σ is given by

$$E(H_t) = \left(I_{N^*} - \left(A_k^* \otimes A_k^*\right)' - \left(G_k^* \otimes G_k^*\right)'\right)^{-1} vec\left(C^{*'}C^*\right)$$

The number of parameters in BEKK(1,1) model is $N(5N+1)/2 = O\left(N^2\right)$. To further reduce the number of parameters, diagonality can be imposed to the matrices A_k^*, and G_k^*. Another way to reduce the parameters is to use Scalar BEKK model, i.e. A_k^*, and G_k^* are equal to a scalar times of matrix of ones.

If consider the portfolio returns are conditional on the full vector of the composite assets returns, then the portfolio returns are estimated through multivariate GARCH models, e.g. MGARCH BEKK, MGARCH-DCC, and GO-GARCH. For the univariate case, we will consider GARCH and EGARCH, for the multivariate case, we will consider BEKK, GO-GARCH and DCC. In both case, because of the skewness and fat tails exhibited in the empirical distribution of the portfolio returns, we chose to model the conditional distribution of the portfolio returns as a skewed Student's t distribution.

To overcome the curse of dimensionality, several dimension reduction methods have been proposed for the conditional covariances modeling. For example, we may use the orthogonal transformation models which linearly combine several univariate models. The most common used orthogonal transformation is perhaps the principal component analysis (PCA) for the Gaussian data and the independent component analysis (ICA) for the non-Gaussian data. [12] adopted ICA to propose a class of models called Generalized Orthogonal GARCH (GO-GARCH).

Models in this category take advantages of the fact that correlation matrices are easier to handle than covariance matrices [15]. Conditional covariances in this category are modeled in a hierarchical way. First, one can choose a GARCH-type model for each conditional variance. Then, based on the conditional covariances to model the conditional correlation matrix.

To understand the relation between conditional covariances and conditional correlations, we let $h_{ij,t}$ as the ij -th element of H_t, and $y_{i,t}$ and $\epsilon_{i,t}$ be the i -th element of y_t and ϵ_t, respectively. Then the coherence between $y_{i,t}$ and $y_{j,t}$ can be measured as:

$$\rho_{ij,t} = \frac{E\left(y_{i,t}y_{j,t} \mid I_{t-1}\right)}{\sqrt{E\left(y_{i,t}^2 \mid I_{t-1}\right)E\left(y_{j,t}^2 \mid I_{t-1}\right)}} = \frac{h_{ij,t}}{\sqrt{h_{ii,t}h_{jj,t}}}$$

$$h_{ii,t} = E\left(y_{i,t}^2 \mid I_{t-1}\right)$$

We can also write the return series in terms of conditional standard deviations times the standardized innovations:

$$y_{i,t} = E\left(y_{i,t} \mid I_{t-1}\right) + \sqrt{h_{ii,t}}\; z_{i,t}$$

Then the correlation between the time series can be written as:

$$\rho_{ij,t} = \frac{E\left(y_{i,t}y_{j,t} \mid I_{t-1}\right)}{\sqrt{E\left(y_{i,t}^2 \mid I_{t-1}\right)E\left(y_{j,t}^2 \mid I_{t-1}\right)}} = \frac{E\left(z_{i,t}z_{j,t} \mid I_{t-1}\right)}{\sqrt{E\left(z_{i,t}^2 \mid I_{t-1}\right)E\left(z_{j,t}^2 \mid I_{t-1}\right)}} = E\left(z_{i,t}z_{j,t} \mid I_{t-1}\right)$$

That is, the conditional correlations between the time series are the conditional covariances between the corresponding standardized innovations. $-1 \le \rho_{ij,t} \le 1$ for all t and generally, $\rho_{ij,t}$ is time varying as H_t varies over time.

2.2.3. MGARCH-CCC

Reference [13] proposed an N-dimensional MGARCH Constant Conditional Correlation (CCC) model, according to which the time varying conditional covariances are taken to be proportional to the square root of the product of the corresponding two conditional variances while leaving the conditional correlations constant over time, which is defined as:

$$H_t = D_t R D_t$$
$$D_t = diag\left(h_{11,t}^{-1/2}, \ldots, h_{NN,t}^{-1/2}\right)$$

where D_t is an $N \times N$ diagonal matrix with elements h_{iit}, the conditional variances, thus we can model each element of D_t separately through any univariate GARCH models. $R = \left\{\rho_{ij}\right\}$ is an $N \times N$ time invariant matrix with $\rho_{ii} = 1$ for all i . The CCC model contains $N\left(N+5\right)/2$ parameters. H_t will be positive definite if and only if each of the N conditional variances are well defined and R is positive definite. The correlation matrix R may be estimated preliminarily using the sample correlation matrix of standardized innovations.

$$R = \frac{\sum_{t=1}^{T} z_{i,t} z_{j,t}}{\sqrt{\left(\sum_{t=1}^{T} z_{i,t}^2\right)\left(\sum_{t=1}^{T} z_{j,t}^2\right)}}$$

However, as stated in [13], the assumption of the conditional correlations being constant throughout the time is an empirical question, it is not satisfied by many of the empirical time series data.

2.2.4. MGARCH-DCC

Reference [14] and [15] extended the CCC model by allowing the conditional correlation matrix to be time-dependent, called Dynamic Conditional Correlation (DCC) model. The DCC model of [15] models the conditional correlations as a function of the previous conditional correlations and a set of estimated correlations. More specifically,

$$H_t = D_t R_t D_t$$
$$D_t = diag\left(h_{11,t}^{-1/2}, \ldots, h_{NN,t}^{-1/2}\right)$$
$$R_t = \left(1-\theta_1-\theta_2\right)R + \theta_1 \Psi_{t-1} + \theta_2 R_{t-1}$$

where R is an $N \times N$ constant, positive definite parameter matrix with ones on the diagonal. θ_1 and θ_2 are non-negative parameters satisfying $\theta_1 + \theta_2 < 1$. Ψ_{t-1} is the sample correlation matrix of the past M standardized innovations $z_{t-1}, \ldots, z_{t-M}$, where $z_t = D_t^{-1}\epsilon_t$. The ij -th element of Ψ_{t-1} is the local correlation matrix given by

$$\psi_{ij,t-1} = \frac{\sum_{m=1}^{M} z_{i,t-m} z_{j,t-m}}{\sqrt{\left(\sum_{m=1}^{M} z_{i,t-m}^2\right)\left(\sum_{m=1}^{M} z_{j,t-m}^2\right)}}$$

The matrix Ψ_{t-1} can be expressed as

$$\psi_{t-1} = B_{t-1}^{-1} L_{t-1} L_{t-1}' B_{t-1}^{-1}$$

where B_{t-1} is an $N \times N$ diagonal matrix with ith diagonal element given by $\left(\sum_{h=1}^{M} z_{i,t-h}^2\right)^{1/2}$ and $L_{t-1} = \left(z_{t-1}, \ldots, z_{t-M}\right)$ is a $N \times M$ matrix with $z_t = \left(z_{1t}, z_{2t}, \ldots, z_{Nt}\right)'$.

2.3. Value at Risk

2.3.1. VaR Estimation

Let $y_t = ln S_t / S_{t-1}$ denote the log-return series at time t which is expressed as percentage, S_t is the asset price at time $t = 1, 2, \ldots T$. VaR at significance level α is defined as the α -quantile of the return distribution [19]; therefore the α -quantile of the conditional distribution, representing the estimation of VaR at $\left(1-\alpha\right)\%$ confidence interval can be computed as :

$$VaR_t = \mu_t + F^{-1}\left(\alpha\right)\sigma_t$$

Here μ_t is the conditional mean which can be modeled with an $AR\left(k\right)$ process, while σ_t^2 is the estimated conditional variance which can be modeled with GARCH, EGARCH. $F^{-1}\left(\alpha\right)$ is the α -quantile of the cumulative distribution of the standardized returns and the distributions F can be normal, Student's t, skewed Student's t, etc.

2.3.2. Portfolio VaR Estimation

VaR can also be defined in terms of the return of a portfolio. Denote $Y_t = \left(y_{1t}, y_{2t}, \ldots, y_{Nt}\right)'$ as the vector of returns at time t of the N assets that compose a portfolio and $W = \left(w_1, \ldots, w_N\right)'$ as the vector of portfolio weights, then the portfolio return at time t can be calculated as:

$$y_{p,t} = W_{t-1}' Y_t$$

The portfolio VaR can be considered as the α-quantile of the conditional distribution of the portfolio return $y_{p,t}$, and it can be expressed as

$$VaR_{p,t} = \mu_{p,t} + \sigma_{p,t} F^{-1}\left(\alpha\right)$$

where $F^{-1}(\alpha)$ is the α-quantile of the distribution function of the centered and standardized portfolio returns. The conditional mean process $\mu_{p,t}$ can be modeled with a vector autoregressive process with order k, that is, VAR (k). The conditional mean part is not our main focus in this study so we can model it according to a simple specification, e.g. assume $\mu_{p,t}=0$ or a VAR(1) model. We also suppose $F^{-1}(\alpha)$ is time invariant and then we will focus on the specification of $\sigma_{p,t}^2$.

There are two alternative ways for the specification of $\sigma_{p,t}^2$: univariately and multivariately. If we consider the portfolio return $y_{p,t}$ is conditional on a linear combination of the past values of the composite assets returns, that is, $W_{t-h-1}'y_{t-h}$, then we will apply a univariate model for $\sigma_{p,t}^2$ given by

$$\sigma_{p,t}^2 = E\left(y_{p,t}^2 \mid y_{p,1},\ldots,y_{p,t-1}\right)$$

The conditional distribution of $y_{p,t}$ in this situation can be normal, Student-t, skewed Student-t, etc. $F^{-1}(\alpha)$ is the α-quantile of the conditional distribution of $y_{p,t}$ given $\left\{y_{p,1},\ldots,y_{p,t-1}\right\}$.

If we consider that the portfolio returns $y_{p,t}$ is conditional on the whole vector of the previous values of the assets returns, Y_{t-h}, then we will apply a multivariate model for $\sigma_{p,t}^2$ given by

$$\sigma_{p,t}^2 = E\left(y_{p,t}^2 \mid Y_1,\ldots,Y_{t-1}\right) = W_{t-1}'H_t W_{t-1}$$

$$= \begin{bmatrix} w_1 & w_2 & \ldots w_N \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}$$

where H_t is the positive definite conditional covariance matrix. In this case, $F^{-1}(\alpha)$ is the α-quantile of the distribution of the linear combination of $W_{t-1}'Y_t$, given $\left\{Y_1,Y_2,\ldots,Y_{t-1}\right\}$. Conditional distribution in this case is generally unknown. However, as pointed in [20], standardized multivariate normal and Student's t distribution take tractable forms when the distribution of returns is closed under linear transformations, i.e. all linear combination of Y have the same distribution as the marginal distribution of returns.

For the univariate case, we will consider GARCH and EGARCH, for the multivariate case, we will consider BEKK, GO-GARCH and DCC. In both case, because of the skewness and fat tails exhibited in the empirical distribution of the portfolio returns, we chose to model the conditional distribution of the portfolio returns as a skewed Student's t distribution.

2.4. Model Evaluation Measures

2.4.1. MAE and RMSE

Before we use the forecasted volatilities as input for the VaR calculation, we first evaluate the out-of-sample forecasting performances of the different volatility models. The evaluation is based on two error measures: mean absolute error (MAE) and root mean squared error (RMSE), which are defined as:

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|y_t^2 - \hat{\sigma}_t^2\right|$$

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(y_t^2 - \hat{\sigma}_t^2\right)^2}$$

$\hat{\sigma}_t^2$ is the predicted variance of the portfolio. The actual volatility σ_t^2 is not observable, we substitute the squared return y_t^2 as the actual conditional variance, see [21].

2.4.2. Backtesting

After the estimation of VaR, we need to evaluate the quality of the VaR forecasts for the different models. The statistic procedure for examining the appropriate estimation of VaR is called backtesting which implements both unconditional and conditional coverage tests for the correct number of exceedances. The unconditional coverage test checks if the frequency exceptions, during the selected time interval, are synchronized with the chosen confidence interval. The most commonly used test in this group is the Kupiec test. The conditional coverage test examines if the exceedances over VaR levels are serially independent and the most widely known test for this group is the Christoffersen independence test.

2.4.2.1. Kupiec Test

Let $N=\sum_{t=1}^{T} I_t$ be the observed number of exceptions over a T-days period that the portfolio loss is larger than the VaR estimate, where:

$$I_{t+1} = \begin{cases} 1 & \text{if } y_{t+1} < VaR_{t+1|t}; \\ 0 & \text{if } y_{t+1} \geq VaR_{t+1|t}. \end{cases}$$

Thus, $I=\{1,1,0,0,\ldots\}$ can be considered as random variable representing if the exceeds of y_t to VaR_t is realized or not. As argued in [22], the number of exceptions follows a binomial distribution with exception probability $p=N/T$, that is $N \sim B(T,p)$. If consider the α-quantile of the realized returns distribution as the VaR at significance level α, then the Kupiec test checks if the number of exceptions N is grater or smaller than $\alpha \times 100\%$ of the sample periods.

The null hypothesis of the test is:

$$H_0: p=\alpha$$

The appropriate likelihood ratio statistic is:

$$LR_c = 2log\left[\left(1-\frac{N}{T}\right)^{T-N}\left(\frac{N}{T}\right)^{N}\right] - 2log\left[(1-p)^{T-N} p^N\right]$$

Under the null hypothesis, the Kupiec test statistic is asymptotically distributed as a chi-square distribution with one degree of freedom. Rejection of the null hypothesis implies the model generated too many or too few exceptions. However, as stated by [22], it fails to examine the extent to which the exceptions are independent, especially at 99% confidence level. Thus conditional coverage tests, such as the Christoffersen test, can be used for further examination of VaR model reliability.

2.4.2.2. The Christoffersen Test

Reference [20] proposed a conditional coverage test to detect whether the exceptions occur independently in clusters. In other words, it can account for volatility clustering. It is important to test for volatility clustering from a practical point of view. For example, if a bank allocates capital for 20 exceedances over a period of one year it may not be able to stay liquid if the majority of the exceedances appear during a period of two weeks. The null hypothesis of the independence test is:

$$H_0 : \pi_{00} = \pi_{01}$$

The statistic testing independence is:

$$LR_{ind} = -2log\left[\left(1-\frac{N}{T}\right)^{T-N}\left(\frac{N}{T}\right)^{N}\right]$$
$$+2log\left[(1-\pi_{01})^{n_{00}} \pi_{01}^{n_{01}} (1-\pi_{11})^{n_{10}} \pi_{11}^{n_{11}}\right]$$

where n_{ij} is the amount of days that j (exception) occurred when it was i (no exception) the day before ($i, j = 0,1$). The probability of state j being observed given that state i was observed the previous day is noted by $\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$, so π_{01} is the probability of a non exception being followed by an exception, and π_{11} is the probability of an exception being followed by an exception. The probability of a non-exception or exception being followed by an exception is denoted by N/T. The independence test statistic is distributed as a χ^2 with one degrees of freedom.

Reference [20] combined this independent likelihood ratio test with the previous unconditional likelihood ratio test to create a conditional coverage test, which is also distributed as χ^2 with 2 degrees of freedom:

$$LR_{cc} = LR_{uc} + LR_{ind}$$

The advantage of the conditional coverage test is in the combination of two tests. This test can be performed separately to backtrack if the model fails due to the wrong number of exceptions or due to exception clustering.

2.3.4. Loss Function

The evaluation of VaR forecasts is obtained by a two-stage backtesting procedures under the hypothesis-testing framework. When the null hypothesis is not rejected, the model is characterized as adequate for volatility forecasting. Although the backtesting measures are used to check the adequacy of the models, they can not conclude if a model is more accurate than another, because it doesn't include the magnitudes between the forecasted VaR values and the realized returns. Therefore, a numerical score that reflects the relative performance of VaR estimates has to be assigned, such as loss functions.

Reference [19] suggested measuring the accuracy of VaR forecasts on the basis of the distance between the observed returns and the forecasted VaR values,

$$\Psi_{t+1} = \begin{cases} 1+\left(r_{t+1} - VaR_{t+1|t}\right)^2, & if y_{t+1} < VaR_{t+1|t}. \\ 0, & if y_{t+1} \geq VaR_{t+1|t}. \end{cases}$$

$\Psi = \sum_{t=1}^{T} \Psi_t$ is the loss function, which incorporates both the cumulative number and the magnitudes of the exceptions. A VaR model is penalized when an exception takes place and the quadratic term ensures that large failures are penalized more than smaller failures. The model minimizes the loss function is then preferred.

Consider the "true" VaR as the empirical distribution of the future realized returns. For example, if T observations of the returns are used for out-of-sample evaluation, then the α-quantile of them will approximate the "true" VaR. The model passes the two-state backtesting procedure with minimum loss function is preferred.

Moreover, loss values of different models are different but they may not statistically significant different. Thus we construct a hypothesis test for the difference between the loss values. Let $d_{t+1} = \Psi_{A_{t+1}} - \Psi_{B_{t+1}}$, where $\Psi_{A_{t+1}}$ and $\Psi_{B_{t+1}}$ are the loss function indicators of models A and B, respectively. If d_{t+1} is negative, then $\Psi_{A_{t+1}}$ is smaller than $\Psi_{B_{t+1}}$, that is, model A is superior to model B.

3. Empirical Data Analysis

This section discusses the empirical data analysis of the volatility models for portfolio VaR prediction. The chapter contains the VaR prediction of a portfolio contains more than one assets using both univariate and multivariate GARCH models.

This part discusses the empirical data analysis of both univariate and multivariate GARCH models for VaR prediction of a portfolio contains more than one assets. 3 equally weighted portfolios are constructed in this part and each portfolio is fitted with univariate GARCH and EGARCH, MGARCH-DCC and GO-GARCH models. The fitted models are then used to forecast the portfolio volatilities which are further used to predict the portfolio VaR at 1%. Models passing both the unconditional and independence tests are considered as appropriate models and appropriate models with smaller loss values perform better in VaR prediction.

3.1. Data Description

The data is downloaded from Yahoo Finance, which consists daily close stock prices of 3 major stock indices, 6 largest banks and 6 largest technical companies with totally 15 US stocks. The covering period is from 2008-01-01 to 2018-12-31 with 2767 observations for each stock. Since the prices are non-stationary, time series interest is the daily log returns, which represent the logarithm levels of prices on two successive days. The log returns are expressed by percentage.

We construct 3 portfolios based on the log returns of the stocks, namely Index, Bank and Tech for convenience; the symbols of the composite stocks of each portfolio can be seen in Table 1. Full sample descriptions of the series of each portfolio are summarized in Table 2 and Table 3. The corresponding returns are graphically shown in Figures 1, 2, and 3.

According to Figures 1, 2, and 3, all the 15 series are moving around near zero with time-varied clustering volatilities. The volatilities of the indices are moving approximately between -10 and -10, the volatilities of the banks are moving around between -50 and 50, and the volatilities of the tech companies are moving roughly between -20 and 20.

The series in the Bank portfolio are also highly correlated with the correlations between 0.6 and 0.8. The correlations in the Tech portfolio are not as high, staying in the range of 0.4 and 0.6.

The descriptive analysis in Table 2 and Table 3 allow us to conclude that the sample daily average returns of all the 15 time series are very close to zero. Hypothesis test about the means fail to reject null hypothesis for all the 15 time series, which confirms the unconditional mean of the returns are 0. Of the 15 time series, 12 series have positive daily average returns while 3 series have negative daily average returns, and all the 3 negative mean return series are from the Bank portfolio. All 15 series, without exceptions, have kurtosis much greater than 3 (kurtosis of normal distribution is less than 3). The skewness coefficients suggest that 10 series are negative skewed and 5 series are positive skewed, which suggest distributions of the series are not symmetric. Both the high kurtosis and non-zero skewness suggest that distributions of the series are far from being Gaussian, which are further confirmed by the zero p-values of Shapiro-Wilk normality test. The stationarity of the series are examined by the ADF test. The null hypothesis states the series have a unit root, i.e. the series are not stationary, versus the alternative hypothesis that the series don't have a unit root, i.e. the series are stationary. The p-values of the test statistics don't exceed 0.01 for all the series, which suggest the rejection of the null hypothesis, therefore all the series are stationary.

The existence of ARCH (conditional heteroskedasticity) effects are tested by applying the Lagrange multiplier (LM) test of [5], which test the second order of the return series. In all cases, the p-values of LM tests allow us to conclude that all return series exhibit conditional heteroskedasticity, which confirmed that the volatilities for all the series are not constant but time-varied. Thus the use of GARCH-type models proves its adequacy.

Table 1. Symbols and descriptions of the 15 time series

Portfolio	Composite Stocks	Symbols
	S&P 500	GSPC
Index	Nasdaq Composite	IXIC
	Dow Jones Industrial	DJI
	JP Morgan	JPM
	Bank of America	BAC
Bank	Wells Fargo	WFC
	Citi Group	C
	Goldman Sachs	GS
	PNC Financial Services	PNC
	Apple	AAPL
	Microsoft	MSFT
Tech	Google	GOOGL
	Intel	INTC
	IBM	IBM
	Oracle	ORCL

Table 2. Full sample descriptive statistics of the series (Part 1)

Series	Mean	Minimum	Maximum	Skewness
GSPC	0.020	-9.469	10.957	-0.348
IXIC	0.033	-9.588	11.159	-0.287
DJI	0.053	-8.200	10.508	-0.119
JPM	0.030	-23.228	22.392	0.321
BAC	-0.018	-34.206	30.210	-0.307
WFC	0.016	-27.210	28.341	0.853
C	-0.062	-49.469	45.632	-0.497
GS	-0.008	-21.022	23.482	0.307
PNC	0.021	-53.436	31.547	-1.608
AAPL	0.062	-19.747	13.019	-0.494
MSFT	0.038	-12.458	17.063	0.133
GOOGL	0.040	-12.340	18.225	0.561
INTC	0.022	-12.221	11.199	-0.099
IBM	0.003	-8.642	10.899	-0.217
ORCL	0.025	-12.393	12.283	-0.243

Table 3. Full sample descriptive statistics of the series (Part 2)

Series	Kurtosis	Shapiro-Wilk	LM	ADF
GSPC	10.828	(0.000)	(0.000)	(0.01)
IXIC	7.288	(0.000)	(0.000)	(0.01)
DJI	10.335	(0.000)	(0.000)	(0.01)
JPM	15.494	(0.000)	(0.000)	(0.01)
BAC	20.315	(0.000)	(0.000)	(0.01)
WFC	22.304	(0.000)	(0.000)	(0.01)
C	33.744	(0.000)	(0.000)	(0.01)
GS	16.571	(0.000)	(0.000)	(0.01)
PNC	67.334	(0.000)	(0.000)	(0.01)
AAPL	7.902	(0.000)	(0.000)	(0.01)
MSFT	9.239	(0.000)	(0.000)	(0.01)
GOOGL	11.870	(0.000)	(0.000)	(0.01)
INTC	5.295	(0.000)	(0.000)	(0.01)
IBM	6.019	(0.000)	(0.000)	(0.01)
ORCL	6.998	(0.000)	(0.000)	(0.01)

Notes: Values of ADF, Shapiro-Wilk and LM are the p-values of the corresponding test statistics.

Table 4. Multivariate ARCH test results

Portfolio	Index	Bank	Tech
Q(12)	1115.323	3655.389	2334.079
p-values	0.000	0.000	0.000

The existence of multivariate ARCH effects in a portfolio are checked by the MarchTest of [23], which is asymptotically equivalent to the multivariate generalization of the LM test of [5] for conditional heteroscedasticity. The test statistic is (is the number of lags) and the results are shown in Table 4. The zero probabilities of the test statistics support a multivariate ARCH effect in all portfolios. Thus, extension to the multivariate GARCH-type models is probably necessary.

Therefore, we consider two approaches for modeling the portfolio returns: both univariate and multivariate GARCH models. [24] suggests that equally weighted portfolio strategy (called 1/N) consistently outperforms almost other optimization strategies. Therefore, the portfolios are all equally weighted with weight vector W = (1/N, . . . , 1/N)′ in the analysis, N is the number of series in the portfolio. For the conditional distribution of the series, we will fit skewed Student's t distribution to the univariate cases and multivariate skewed Student's t distribution to the multivariate cases to better capture the fat tails and skewness in the series.

For the univariate case, we apply GARCH and EGARCH, while for the multivariate case, we apply GO-GARCH and DCC. We also fit BEKK model to the Index portfolio alone, since BEKK loses its predictive accuracy when the dimension is greater than 3, we don't fit it to the Bank or Tech portfolios. Therefore, there are 4 models for the Bank and Tech: GARCH, EGARCH, GO-GARCH, DCC and 5 models for Index: GARCH, EGARCH, BEKK, GO-GARCH, and DCC. The first 2200 observations of each series are used for estimation and the remaining observations are used for out-of-sample prediction.

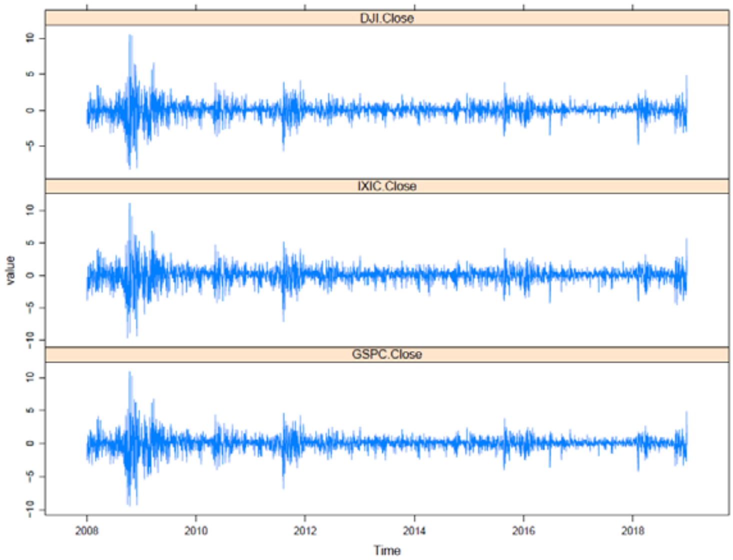

Figure 1. Log returns of each series in portfolio Index from 2008-01-01 to 2018-12-31

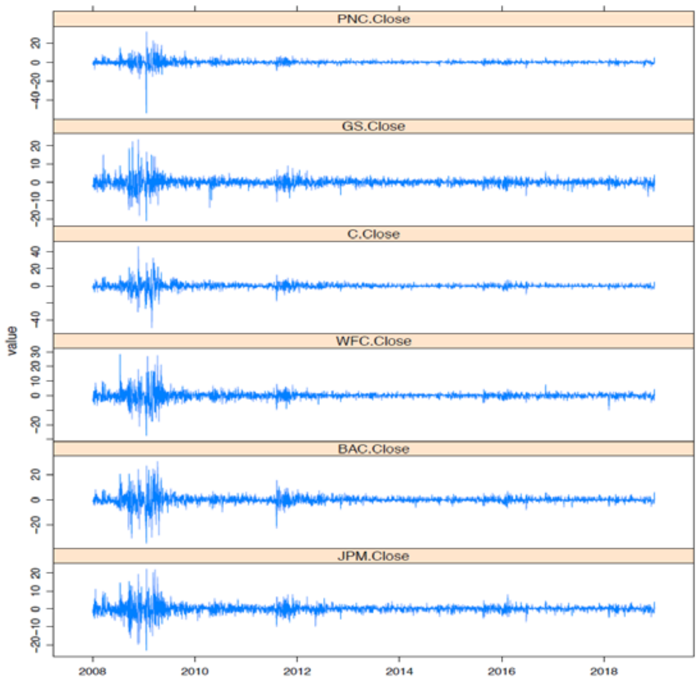

Figure 2. Log returns of each series in portfolio Bank from 2008-01-01 to 2018-12-31

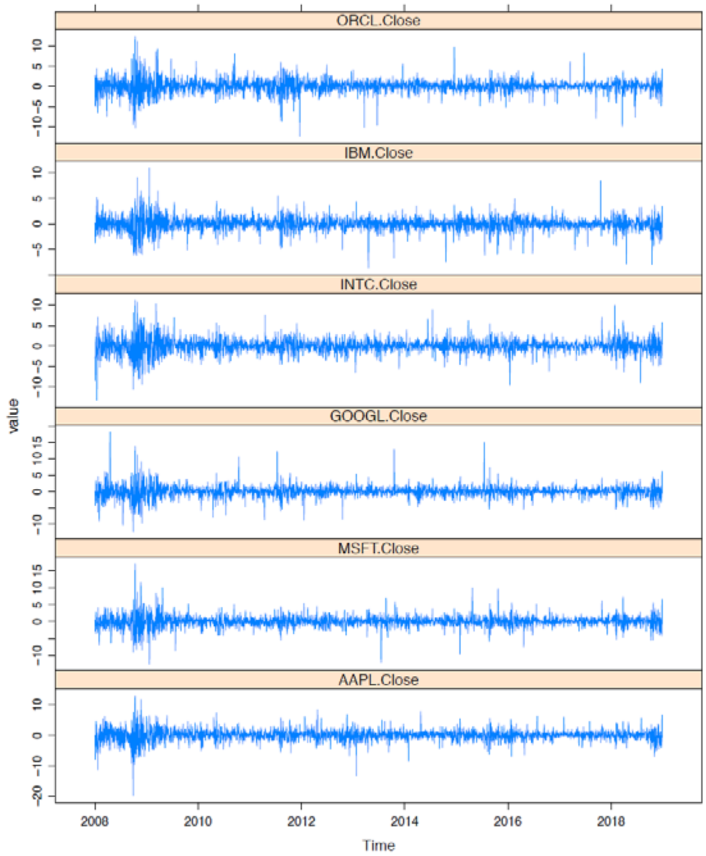

Figure 3. Log returns of each series in portfolio Tech from 2008-01-01 to 2018-12-31

3.2. Fitting Univariate and Multivariate Models

If consider the portfolio returns as a linear combination of the composite asserts returns, then the portfolio volatilities are approximated with univariate GARCH and EGARCH models. The results of [25] show the importance of allowing for heavy-tails and skewness in the distributional assumption, they proposed that the skewed Student's-t outperforms the other distributions. Thus, distribution of the portfolio return is assumed to be skewed Student's t to better capture the empirical distribution of the portfolio returns. Parameters of the GARCH and EGARCH models are estimated via MLE/QMLE and the estimated parameters are shown in Table 5 and 6. AICs and log-likelihoods of the models are also included in the tables.

If consider the portfolio returns are conditional on the full vector of the composite assets returns, then the portfolio returns are estimated through multivariate GARCH models, e.g. MGARCH-BEKK, MGARCH-DCC, and GO-GARCH. Distribution of the portfolio is fitted as multivariate Student's t to capture the fat tails exhibit in the marginal return distributions.

Parameters of BEKK model are estimated using MLE/QMLE, and are shown in Table 7. While parameters of GO-GARCH model are also estimated under MLE and listed in Table 8. The mapping matrix M for the GO-GARCH models is obtained through Independent Component Analysis (ICA), prove and analysis of ICA can be found in [12]. Parameters of the DCC model are estimate in two-steps: the first step is to fit the conditional variances of each series in the portfolio with a univariate GARCH model, then use the standardized residuals as input to further fit a DCC model. The parameters of DCC model are listed in Table 9, while parameters of each univariate model are not listed in the table.

Table 5. Estimated parameters of GARCH

Portfolio	Index	Bank	Tech
α_0	0.062 (0.004)	0.032 (0.005)	0.032 (0.000)
α	0.112 (0.000)	0.089 (0.000)	0.103 (0.000)
β	0.887 (0.000)	0.906 (0.000)	0.883 (0.000)
$\alpha + \beta$	0.999	0.995	0.986
shape	5.489 (0.000)	6.215 (0.000)	7.772 (0.000)
skew	0.976 (0.000)	0.968 (0.000)	0.950 (0.000)
AIC	4.561	4.016	3.135
loglikelihood	-5012	-4411	-3443

Notes: Values in the brackets are the p-values of the estimated parameters. $\alpha + \beta < 1$ for all the 3 portfolio returns means the returns are stationary but the volatilities are highly persistent.

Table 6. Estimated parameters of EGARCH

Portfolio	Index	Bank	Tech
α_0	0.004 (0.119)	-0.003 (0.006)	0.007 (0.123)
α	-0.223 (0.000)	-0.081 (0.000)	-0.145 (0.000)
β	0.969 (0.000)	0.991 (0.000)	0.974 (0.000)
γ	0.134 (0.000)	0.149 (0.000)	0.159 (0.000)
shape	6.916 (0.000)	6.479 (0.000)	0.913 (0.000)
skew	0.829 (0.000)	0.967 (0.000)	6.603 (0.000)
AIC	2.786	4.005	3.107
loglikelihood	-3059	-4398	-3410

Notes: Values in the brackets are the p-values of the estimated parameters. $\beta < 1$ confirms the portfolio returns are stationary but the volatilities are highly persistent. γ suggests effects of positive innovations and negative innovations on the future volatilities are asymmetric.

Table 7. Estimated parameters of BEKK for portfolio Index

	Parameter Matrix
W	$\begin{bmatrix} 0.169 & 0.179 & 0.186 \\ 0 & 0.084 & -0.062 \\ 0 & 0 & 0.002 \end{bmatrix}$
A	$\begin{bmatrix} 0.194 & -0.159 & 0.046 \\ 0.119 & 0.376 & 0.054 \\ -0.019 & 0.045 & 0.199 \end{bmatrix}$
B	$\begin{bmatrix} 0.811 & -1.216 & 1.226 \\ -0.129 & 0.666 & 0.083 \\ 0.266 & 1.649 & -0.487 \end{bmatrix}$
AIC	3.813
Log-likelihood	-3765

Notes: W is the constant matrix, A is the ARCH effect matrix, B is the GARCH effect matrix. BEKK is fitted to portfolio Index only since BEKK is only feasible for data with dimension less or equal to 3.

Table 8. Inverse of linear map M of GO-GARCH for all portfolios

Portfolio	Mixing matrix	AIC	Log-likelihood
Index	$\begin{bmatrix} -0.35 & 0.19 & 1.20 \\ -0.70 & 0.01 & 1.17 \\ -0.35 & 0.40 & 0.39 \end{bmatrix}$	3.245	-5007
Bank	$\begin{bmatrix} -0.53 & 0.13 & -0.61 & -1.93 & 0.92 & -1.25 \\ -1.77 & -1.19 & -0.00 & -1.68 & 1.37 & -1.81 \\ 0.01 & -0.59 & 0.05 & -1.26 & 1.06 & -2.22 \\ -0.33 & -2.19 & -0.29 & -2.63 & 0.92 & -0.89 \\ -0.38 & 0.05 & 1.01 & -1.85 & 0.88 & -0.59 \\ -0.12 & -0.30 & -0.40 & -1.04 & 2.17 & -1.08 \end{bmatrix}$	3.774	-4438
Tech	$\begin{bmatrix} -0.06 & -1.30 & 0.01 & -0.07 & -1.44 & -0.21 \\ -1.27 & 0.00 & -0.13 & -0.04 & -1.19 & -0.15 \\ -0.28 & -0.16 & -0.17 & -1.37 & -1.12 & -0.18 \\ 0.00 & 0.53 & 0.25 & 0.15 & -1.74 & 0.00 \\ -0.13 & 0.11 & -0.20 & 0.00 & -0.82 & -1.12 \\ -0.14 & 0.07 & -1.24 & -0.01 & -1.21 & -1.41 \end{bmatrix}$	3.849	-4725

Notes: The N dimensional observed portfolio return vectors are governed as N dimensional unobserved uncorrelated components, linked by the linear map M, which is the mixing matrix in the above table. M is obtained through ICA. The index portfolio consists 3 stocks so the dimension of M is 3×3, while the Bank and Tech portfolios consist 6 stocks, thus the dimensions of their mixing matrices M are both 6×6.

Table 9. Estimated parameters of DCC for all portfolios

Portfolio	Index	Bank	Tech
Joint dcca1 θ_1	0.044 (0.000)	0.011 (0.000)	0.019 (0.000)
Joint dccb1 θ_2	0.9170 (0.000)	0.981 (0.000)	0.950 (0.000)
mshape	9.2523 (0.000)	4.933 (0.000)	4.396 (0.000)
AIC	3.151	2.992	3.194
Log-likelihood	-3921	-4156	-4405

Notes: Values in the brackets are the p-values of the estimated parameters. Parameters of the N univariate GARCH(1,1) process of the composite series are not listed. According to the AICs listed in the tables in this section, we are able to conclude that GO-GARCH best fit the conditional volatilities of the Index portfolio, MGARCH-DCC best capture the conditional volatilities of the Bank portfolio, and EGARCH best describe the conditional volatilities of the Tech portfolio.

3.3. Volatility Forecasting

The estimated models for each portfolio are then used to forecast the portfolio volatilities. Forecasting performances are evaluated by the measures MAE and RMSE, the results are displayed in Table 10.

Of the 5 models for the Index portfolio, BEKK, DCC and GO-GARCH have smaller MAEs and RMSEs than GARCH and EGARCH. That is, all the multivariate GARCH models outperform univariate GARCH models. Thus, MGARCH models are better than GARCH models in forecasting the portfolio volatilities. One reason for the better performance of MGARCH models is that these models capture the high correlation, on average more than 0.9, among the series in the portfolio. Of the three MGARCH models, GO-GARCH performs better than the other two, as seen in Figure 4. The reason probably also goes to GO-GARCH predicting higher average correlations among the series in the portfolio than DCC and BEKK.

Table 10. MAE and RMSE

Portfolio		GARCH	EGARCH	GO-GARCH	DCC	BEKK
Index	MAE	0.842	0.831	0.756	0.827	0.825
	RMSE	1.975	1.971	1.920	1.954	1.959
Bank	MAE	1.880	1.844	1.871	2.045	-
	RMSE	3.571	2.502	2.553	3.559	-
Tech	MAE	1.234	1.217	1.263	1.366	-
	RMSE	2.576	2.496	2.599	2.630	-

Notes: BEKK is applicable only for low dimensional data, so it is fitted to the portfolio Index only.

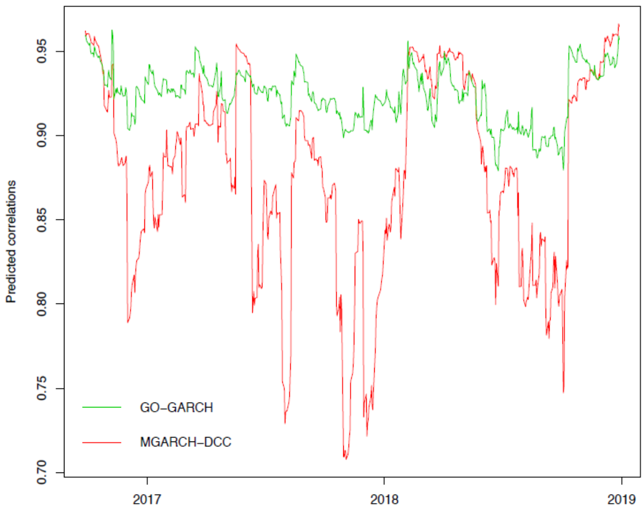

Figure 4. Predicted average correlations between assets in portfolio Index

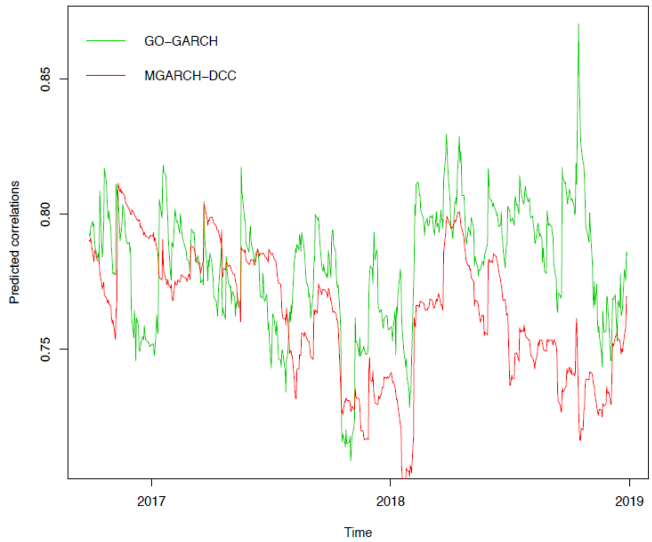

Figure 5. Predicted average correlations between assets in portfolio Bank

Of the 4 models for the Bank portfolio, the order of MAE is EGARCH < GO-GARCH < GARCH < DCC, while the order of RMSE is EGARCH < GO-GARCH < GARCH < DCC. EGARCH has the smallest MAE and

RMSE, while DCC has the largest MAE and RMSE, no conclusion on the MAEs and RMSEs of GARCH and GO-GARCH. MGARCH models don't provide extra accuracy in predicting the portfolio volatilities than a simple and parsimonious univariate GARCH model. Figure 5 displays the average predicted correlations among the series in the portfolio, ranging between 0.7 and 0.85.

For the Tech portfolio, GARCH and EGARCH models have smaller MAEs and RMSEs than their multivariate counterparts. That is, with extra information about the dependency among the series, the MGARCH models even loss accuracy in predicting the portfolio volatilities. One explanation is their lower correlations among the series in the portfolio. Since the series have low pairwise correlations thus it is not necessary to bring extra information about their co-movements.

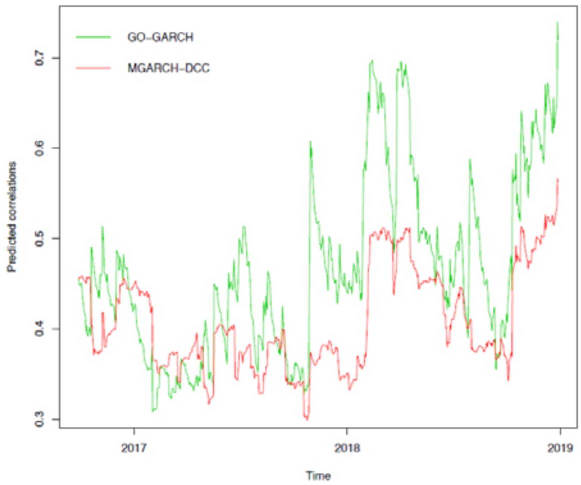

Figure 6. Predicted average correlations between assets in portfolio Tech

According to the Figure 4, 5 and 6, the volatilities predicted by the univariate models and multivariate models are not obviously different. Thus we run a hypothesis test on the pairwise differences between the fitted models of each portfolio. The test results show that the MAE of GO-GARCH for portfolio Index is significant smaller than other models fitted to Index and the MAE of EGARCH for portfolio Tech is significant smaller than the MAE of DCC.

From the empirical analysis of the three portfolios, we can conclude that, in terms of capturing the movements of the portfolio volatilities, it will be better to apply GARCH-type models when the correlations among the compositions are not high. When the portfolio composite series are highly correlated, MGARCH models will perform better. And when the correlations are moderate, GARCH and MGARCH models perform similarly, but GARCH models are simpler and more parsimony so GARCH type models are preferred.

3.4. VaR Prediction

The conditional volatilities forecasted by each estimated model are then used to calculate the portfolio VaR. We will use the forecasted volatilities to calculate the portfolio VaR at 1% confidence level. We calculate VaR at level 1% because of the BCBS requirement since 1996. The estimated VaR has to undergo a two-stage backtesting procedure which includes an unconditional coverage test and a conditional coverage test. The results are shown in Tables 11, 12, and 13.

The sample size of the out-of-sample realized returns is 561, and 1% of the sample size is 5.61. The floor of 5.61 is 5, which is the expected exceeds of each portfolio. Predicted exceeds are the VaR exceptions predicted by the fitted models. Unconditional Coverage (UC) test will reject the models which generated too many or too few exceptions. Conditional Coverage (CC) test can further test if the rejection of the model is due to the wrong exceptions or due to exception clustering. Values in the brackets are p-values of the corresponding test statistics. Low p-value suggests a rejection of the null hypothesis, which means the model fails the test. Loss values are calculated according to Lopez loss function, the smaller the better.

Table 11. Backtesting results and loss values at 1% for portfolio Index

	GARCH	EGARCH	BEKK	GO-GARCH	DCC
Expected exceeds	5	5	5	5	5
Predicted exceeds	8	9	11	11	7
UC	0.582 (0.445)	0.866 (0.126)	4.994 (0.046)	3.989 (0.045)	1.688 (0.194)
CC	9.568 (0.008)	9.67 (0.007)	6.479 (0.059)	5.628 (0.120)	4.033 (0.133)
Loss values	1831.69	1971.55	1897.67	1753.22	1905.37

Notes: Values in the brackets are the p-values of the corresponding test statistics. BEKK and GO-GARCH generate too many exceptions. Exceptions generated by GARCH are not independent.

Table 12. Backtesting results and loss values at 1% for portfolio Bank

	GARCH	EGARCH	DCC	GO-GARCH
Expected exceeds	5	5	5	5
Predicted exceeds	9	9	8	6
UC	1.689 (0.194)	1.688 (0.194)	0.866 (0.352)	0.020 (0.887)
CC	1.979 (0.372)	4.033 (0.133)	1.096 (0.578)	0.149 (0.928)
Loss values	1814.22	1869.40	1797.28	1970.07

Notes: Values in the brackets are the p-values of the corresponding test statistics. All the p-values of the test statistics are not low which means all the models are appropriate.

Table 13. Backtesting results and loss values at 1% for portfolio Tech

	GARCH	EGARCH	DCC	GO-GARCH
Expected exceeds	5	5	5	5
Predicted exceeds	8	8	7	1
UC	0.866 (0.352)	0.870 (0.194)	0.298 (0.585)	5.891 (0.015)
CC	3.657 (0.161)	3.657 (0.161)	3.616 (0.164)	5.895 (0.005)
Loss values	1967.26	2006.90	2019.60	2163.79

Notes: Values in the brackets are the p-values of the corresponding test statistics.

For the Index portfolio, both MGARCH-BEKK and GO-GARCH fail to predict the correct number of exceeds, since they predict too many exceptions. While both

GARCH and EGARCH predict the correct number of exceptions but the exceptions are not independent. Thus, DCC is the only model that appropriately forecasts the VaR of the Index portfolio.

For the Bank portfolio, all the four fitted models pass the two-stage backtesting procedure and therefore can be considered appropriate for the VaR prediction. DCC has the smallest loss value while GO-GARCH has the largest loss value, GARCH and EGARCH in between. The backtesting and loss values can't provide conclusion between the univariate and multivariate GARCH models.

For the Tech portfolio, GO-GARCH is not adequate for VaR estimation due to its failing in both the unconditional and conditional tests. Of the adequate models GARCH, EGARCH and DCC, GARCH model has the smallest loss value.

However, when check the loss values of the models fitted to each portfolio, we can find the loss value of each model is not that different. Thus, we further run a hypothesis t test on the pairwise differences between each two models for each portfolio. Results of the tests indicate that the loss value of GO-GARCH in the Index portfolio is significant smaller than the other 4 fitted models. While for the models fitted to the portfolios Bank and Tech, the differences between each fitted models are not significant.

From the results of the backtesting, associated with the pairwise correlations of the composite assets in the three portfolios, we can conclude that MGARCH models are better than univariate GARCH models only when the composite assets are highly correlated. When the correlations are not that high, both univariate and multivariate models can appropriately predict the portfolio VaR, while univariate models are simpler and more parsimony, the extension of univariate GARCH models to MGARCH models is not necessary.

4. Conclusion

Volatilities are usually the major input of risk, thus understanding and modeling the conditional volatilities of the return process is of crucial importance in the financial risk management. VaR is the standard measure of financial risk management due to the requirement of Basel Committee for Banking Supervision since 1996. Although the committee switched the VaR to CVaR as the official risk measure, the input of the CVaR is still VaR. The calculation of VaR depends on the conditional volatilities, thus different type of volatility models are estimated and compared in terms of VaR estimation in this paper.

In this paper, we focus on the problem if it is necessary to extend the univariate volatility models to their multivariate counterparts when calculate the VaR of a portfolio consists N assets.

4.1. Conclusion on the Multivariate Volatility Models

Since the financial volatilities are usually moving across assets in the portfolio, modeling the volatilities in the multivariate framework will lead to more relevant empirical models than working with separate univariate models. However, as the dimension of the portfolio increases, we will face the problem of "dimension of curse", that is, as the number of assets in a portfolio increases, the number of parameters in the MGARCH model will increase dramatically. Thus there comes a questions: do we really need the multivariate extension of the GARCH models? The paper addresses this kind of question in terms of the portfolio VaR prediction: which approach is better, the univariate GARCH or the multivariate GARCH? We construct 3 portfolio use the stock prices of the 3 major indices, 6 banks and 6 technical companies in the US. Our main focus is the portfolio volatilities, so the weights of the assets in the portfolios are fixed, e.g. all the portfolios are equally weighted. Then for each of the portfolio, we model the portfolio volatilities with both GARCH-type models and MGARCH models, specifically, GARCH, EGARCH, DCC, and GO-GARCH. For the Index portfolio, we also fit a BEKK model to it, since BEKK models typically not applicable when the dimension is greater than 3, we don't fit it to the Tech and Bank portfolios.

Then we use the estimated models to predict the out-of-sample volatilities for each portfolio, the predictive abilities of the models are evaluated according to the accuracy measures MAE and RMSE. Then the forecasted volatilities of each portfolio are used as inputs of the portfolio VaR estimations. Suitability of the models are examined through a two-stage backtesting procedure that includes unconditional coverage and conditional coverage test. Then accuracies of the VaR predictions are compared via the loss values.

Results of our study indicate univariate GARCH models are better in capturing the portfolio volatilities and estimating the VaR of the Tech portfolio. MGARCH models perform better than univariate GARCH models in describing the portfolio volatilities and calculating the VaR of the Index. As for the Bank portfolio, the portfolio volatilities and the VaR appear not to be very sensitive to the choices of GARCH or MGARCH estimates. Therefore, it typically depends on the data that if it is necessary to apply a multivariate volatility model, which is consistent with the conclusion of [26]: univariate or multivariate, this depends on the data.

4.2. Future Work

In the paper, we compare the univariate GARCH models with its multivariate counterparts in predicting the portfolio VaR with more than one assets. The multivariate SV model has not been studied in the paper.

In the next step, we will probably include multivariate SV model in the study. Moreover, the dimension of the portfolio is at most 6 in the paper, we will probably include more assets to create portfolio with higher dimension.

The recent integration of statistical models with neural networks provides a new way of formulating volatility models that have been widely used in time series analysis and prediction in finance. In addition, statistical and machine learning models have achieved great success. See [27,28,29,30]. These models can be generalized and further developed to the time series analysis and compare the forecasting performances with the GARCH type models. Further more, nature language processing techniques could be also helpful in predicting Value at Risk [31].

References

[1] Cappiello, Lorenzo, Robert F. Engle, and Kevin Sheppard. "Asymmetric dynamics in the correlations of global equity and bond returns." Journal of Financial econometrics 4.4 (2006): 537-572.

[2] Alexander, Gordon. (2009). From Markowitz to modern risk management. European Journal of Finance. 15. 451-461. 10.1080/13518470902853566.

[3] Philippe Jorion (2006), Value at Risk: The New benchmark to manage financial risk, second edition.

[4] Gallon and Gomez (2007). Conditional Distribution of the Colombian Exchange Rate Returns: An Empirical Exercise Using Multivariate GARCH Models. Revista de Economia del Rosario, Vol. 10, No. 2, December 2007.

[5] Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica: Journal of the econometric society, 987-1007.

[6] Tim Bollerslev (1986). Generalized Autoregressive Conditional Heteroskedasticity, Journal of Econometrics. Journal of Econometrics Volume 31, Issue 3, April 1986.

[7] Daniel B. Nelson (1991). Conditional Heteroskedasticity in asset returns: A new approach, Econometrica, Vol. 59, No. 2 (Mar., 1991), pp. 347-370.

[8] Bollerslev, T., Engle, R. F., & Wooldridge, J. M. (1988). A capital asset pricing model with time-varying covariances. Journal of political Economy, 96(1), 116-131.

[9] Engle, R. F., & Kroner, K. F. (1995). Multivariate Simultaneous Generalized Arch. Econometric Theory, 11(1), 122 – 150. http://www.jstor.org/stable/3532933

[10] Caporin, M., & McAleer, M. (2014). Robust ranking of multivariate GARCH models by problem dimension. Computational Statistics & Data Analysis, 76, 172-185.

[11] Luc Bauwens, Bastien Laurent, Jeroen V. K. Rombouts (2006). Multivariate GARCH models: A Survey. Journal of Applied Econometrics, Volume21, Issue1, January/February 2006, Pages79-109.

[12] Roy Van Der Weide (2002). GO-GARCH: A Multivariate Generalized Orthogonal GARCH Model. Econ. 17: 549564 (2002).

[13] Tim Bollerslev (1990). Modelling the Coherence in Short-Run Nominal Exchange Rates: A Multivariate Generalized Arch Model. The Review of Economics and Statistics, MIT Press, vol. 72(3), pages 498-505, August.

[14] Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. Journal of business & economic statistics, 339-350.

[15] Tse, Y. K., & Tsui, A. K. C. (2002). A multivariate generalized autoregressive conditional heteroscedasticity model with time-varying correlations. Journal of Business & Economic Statistics, 20(3), 351-362.

[16] French, Kenneth R., G. William Schwert, and Robert F. Stambaugh. "Expected stock returns and volatility." Journal of financial Economics 19.1 (1987): 3-29.

[17] Robert F. Engle and Victor K. Ng (1993). Measuring and Testing the Impact of News on Volatility. The Journal of Finance, Vol. 48, No. 5 (Dec., 1993).

[18] Gouriéroux, C. (1997). ARCH models and financial applications. Springer Science & Business Media.

[19] Jose A. Lopez (1998) Methods for Evaluating Value-at-Risk Estimates. Economic Policy Review, Vol. 4, No. 3, October 1998.

[20] Christoffersen, P. (2008). Evaluating Interval Forecasts. International Economic Review. Vol. 39, No. 4.

[21] Perry Sadorsky (2006). Modeling and Forecasting Petroleum Futures Volatility. Energy Economics, 2006, vol. 28, issue 4, 467-488.

[22] Kupiec P. (1995). Techniques for Verifying the Accuracy of Risk Management Models. The Journal of DerivativesWinter 1995,3(2)73-84.

[23] Tsay, R. S. (2014). An introduction to analysis of financial data with R. John Wiley & Sons.

[24] Yuliya Plyakha et al. (2012). Why Does an Equal-Weighted Portfolio Outperform Value- and Price-Weighted Portfolios? SSRN Electronic Journal. October 2012.

[25] Manuela Braione, Nicolas K. Scholtes (2016). Forecasting Value-at-Risk under Different Distributional Assumptions. Econometrics 2016.

[26] Alain-Philippe Fortin, Jean-Guy Simonato and Georges Dionne (2018). Forecasting Expected Shortfall: Should we use a Multivariate Model for Stock Market Factors? Working Papers18-4, HEC Montreal, Canada Research Chair in Risk Management.

[27] Rui Luo et al. (2018). A Neural Stochastic Volatility Model. arXiv:1712.00504v2 [cs.LG].

[28] Chen, D. Q., Mao, S. Q., & Niu, X. F. (2023). Tests and classification methods in adaptive designs with applications. Journal of Applied Statistics, 50(6), 1334-1357.

[29] Li, Y., Mao, S., Yuan, Y., Wang, Z., Kang, Y., & Yao, Y. (2023). Beyond Tides and Time: Machine Learning's Triumph in Water Quality Forecasting. American Journal of Applied Mathematics and Statistics, 11(3), 89-97.

[30] Mao, S., Yuan, Y., Li, Y., Wang, Z., Yao, Y., & Kang, Y. (2023). Room Occupancy Prediction: Exploring the Power of Machine Learning and Temporal Insights. arXiv preprint arXiv:2312.14426.

[31] Mao, S. (2022). Time Series and Machine Learning Models for Financial Markets Forecast. The Florida State University.