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Review on Different Special Functions in Fractional Calculus

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Abstract In this paper, we reviewed the specific tasks involved in solving different mathematical equations or fractional integral. Since solutions for different component of modeling models from various fields such as physics, engineering, chemistry, biological etc. involve special work as part of their solutions, here, we review and include the type and general- ization of such specialized functions.

Keywords: *special functions, fractional integral calculus, Mittag-Leffler function, Fox H-function, hyper-Bessel function*

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1. Introduction

The discipline of fractional calculus, which deals with the assignment of fractional values and their derivatives, was first introduced by Leibniz as a mathematical curiosity but has since grown into an important area of study. Incorporating partial calculations into everyday mathematics was made possible by the work of famous mathematicians like Riemann, Liouville, Grunwald, Euler, Lagrange, Caputo, and others who built the groundwork for current theory.

This area of study is growing right now. The domains of visco-elasticity, fluid flow, rheology, etc., have also benefited from the emergence of new ideas and concepts. The roots of fractional calculus can be traced all the way back to the early days of calculus. The foundation of fractional calculus is the debate over whether or not real numbers can be extended to complex numbers, and whether or not the meaning of integer factorials can be extended to complex number factorials. A large number of academics, however, remain uneducated in these areas. They often ask: What is a fractional derivative? Is this a new field or an old field? Is there a use of fractional applications? What are these outputs?

$\frac{d^n p}{dx^n}$, p a function of x, can always be written algebraically. Indeed, for $n \le m$

$$\frac{d^n p}{dx^n} = m(m-1)(m-2).....(m-n+1)x^{m-n},$$

$$= \frac{m!}{(m-n)!}x^{m-n}, \quad (1)$$

$$= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}. \quad (2)$$

He put forward that it could be possible to interpolate if the order n of the derivative is a fraction. Fourier in 1822 started with the integral representation of $f(x)$ given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} cos(px - pz)dp, \quad (3)$$

and made the following generalization:

$$\frac{d^\mu f(x)}{dx^\mu} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} p^\mu cos(px - pz + \frac{\mu\pi}{2})dp, \quad (4)$$

adding that μ could take any arbitrary value whether positive or negative. In his definition, one can see that the existence of the fractional derivative or integral depends on the convergence of the improper integrals. Ross [1,2,3] attributes the first application of fractional calculus to Abel. In 1823, Abel solved the integral equation

$$\frac{1}{\Gamma(\mu)}\int_0^x \frac{\phi(t)}{(x-t)^{1-\mu}}dt = f(x) \qquad 0<\mu<1, \tag{5}$$

which arises in connection with the tautochrone problem: A bead on a frictionless wire starts from rest at some point and slides down under the influence of gravity. What should the shape of the wire be so that the amount of time it makes the bead to descend to its lowest point is independent of its starting point? Abel obtained the solution

$$\phi(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{\mu}}dt. \tag{6}$$

As far as Butzer and Westphal are concerned, Abel's use of fractional calculus was limited to demonstrating that the solution could be written as a fraction, rather than actually solving the issue. The ideas of Abel, on the other hand, were instrumental in spreading awareness of fractional calculus. Many people consider Liouville to be the man who developed the modern theory of fractional calculus. Between 1832 and 1855, he authored a number of academic works on the subject. Liouville derived his first concept of partial derivatives from the procedure for differentiating an exponential function:

$$\frac{d^m e^{ax}}{dx^m} = a^m e^{ax}. \tag{7}$$

He considered functions which can be written as the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \tag{8}$$

and defined the derivative as

$$\frac{d^{\mu} f(x)}{dx^{\mu}} = \sum_{n=0}^{\infty} c_n a_m^{\mu} e^{a_n x} \tag{9}$$

This definition is clearly too restrictive as it depends on the convergence of the series. He also derived the formula

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^{\mu}\Gamma(\mu)}\int_0^{\infty} \phi(x+\alpha)\alpha^{\mu-1}d\mu \tag{10}$$
$$-\infty < x < \infty, \ \mathbb{R}(\mu) > 0,$$

If we let $\tau = x+\alpha$, then

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{(-1)^{\mu}\Gamma(\mu)}\int_x^{\infty} \phi(\tau)(\tau-x)^{\mu-1}d\tau. \tag{11}$$

This formula is what is now known as the Liouville type of fractional integration with the factor $(-1)^{\mu}$ being excluded. He applied these formulae to solve various problems in electrodynamics, mechanics and geometry.

It is also worth noting that in both the Fourier's and Liouville's interpretations, the fractional derivatives take the form of an integral. Grunwald in 1867 and Letnikov in 1868 introduced what is now known as the Grunwald-Letnikov fractional derivative. Their idea was to start with the ordinary derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \tag{12}$$

and apply this recursively to obtain higher-order derivatives. For example, the

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h},$$

$$f''(x) = \lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2},$$

In general, we have

$$f^n(x) = \lim_{h\to 0}\frac{1}{h^n}\sum_{0\le m\le n}(-1)^m \binom{n}{m} f(x+(m-n)h), \tag{13}$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}. \tag{14}$$

By allowing n to be any real number, the Grunwald-Letnikov derivative is obtained:

$$D^{\alpha} f(x) = \lim_{h\to 0} h^{-\alpha}\sum_{m=0}^{\frac{x-\alpha}{h}}(-1)^m \binom{\alpha}{m} f(x-mh). \tag{15}$$

Riemann's theory of fractional calculus was developed during his time as a student, although it was not published until after his death in 1876. While looking for a generalisation of a Taylor's series expansion, Riemann came up with the following definition of fractional integration:

$$\frac{d^{-\mu} f(x)}{dx^{-\mu}} = \frac{1}{\Gamma(\mu)}\int_c^x (x-t)^{\mu-1} f(t)dt + \psi(x). \tag{16}$$

He felt the need to add the complementary function $\psi(x)$ to deal with the ambiguity of the lower limit of integration c, which only created even confusion as to what is meant by it. The detail study about the fractional development of differential and integral calculus can be found in [6,7,8,9] and since the finding the analytically solution for the fractional order is not that simple so numerical approach to find the approximate solution is one of the alternate and simple method, several researcher work on numerical methods [10-15].

2. Special Function in Fractional Calculus

Improper integrals and series are examples of special functions; depending on their significance, these functions have well-established names and notations in various areas of mathematics and the natural and social sciences. Among the integer-ordered special functions are the Bessel and all cylindrical functions; the Gauss and Kummer functions; the confluent and generalised hypergeometric functions; the classical orthogonal polynomials; the incomplete Gamma- and Beta-functions and Error functions; the Airy and Whittaker functions; etc., all of which are useful in the application of differential and integral equations. However, in recent years, there has been a surge in interest and widespread use of many mathematical and fractional order systems (i.e., random order), such as better models for a wide range of physical phenomena, engineering, automation, biology, chemistry, earth science, economics, etc. The theory of differentiation and integration of random order (i.e. Fractional Calculus) and fractional order (or many orders) and essential figures now necessitate its introduction accompanied with significant development, making the so-called Special Functions of Fractional Calculus a vital tool. The detail discussion of these special functions can be found in [16-21], in addition these special functions are now available in mathematical software's like Matlab, mathematica, Mapple etc. We will start the brief review with the first Gamma and Beta function then restrict the functions for the fractional calculus.

2.1. Gamma and Beta Function

L. Euler first in the eighteenth century introduced

$$n! = \int_0^\infty e^{-t^n} dt, n = 0,1,2,3\ldots$$

After that in 1729 he generalized factorial n, where n is any positive integer to gamma function, where x is any positive real number $\Re^+ \cdot$ The detail growth of the Gamma function can be observed in [22,23,24,25,26]. This result was extended by L. C. Andrews [27] in 1985 to certain negative numbers and even to complex numbers.

$$\Gamma(x) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re(x > 0) \tag{17}$$

After this, Legendre, Whittsker and Watson in (1990) defined another special function beta, which is a two parameters composition of gamma function.

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \tag{18}$$

$x \geq 1, y \geq 1$ for proper integral and $x > 0, y > 0$ and $x < 1, y < 1$ either or both for convergent improper integral.

Using a regularization factor $e^{-\left(\frac{p}{t}\right)}$, Chaudhry and Zubair in 1994 [24,25] have defined the following extension of gamma function as

$$\Gamma_p(x) = \int_0^\infty e^{\left(-t-\frac{p}{E}\right)} t^{x-1} dt, \quad \Re(p > 0) \tag{19}$$

which removes the singularity coming from $t = 0$ limit and setting $p = 0$ reduces this extension to original gamma function.

Chaudhry, et al. introduced the following extension of Eulers beta function in 1997 [28] as

$$B_{\mathrm{p}}(x, y) = \int_0^1 t^{x-1} (1-t)^{\square-1} e^{\left(-\frac{\mathrm{R}}{\mathrm{II-}}\right)} dt, \quad \Re(p > 0) \tag{20}$$

2.2. Mittag Leffler Function of One Parameter

Definition 2.2.1. *(Mittag-Leffler Function) [31] The Mittag - Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as,*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{21}$$

where $z, \alpha \in C, Re(\alpha) > 0.$

If we put $\alpha = 1,$ then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{22}$$

2.3. Mittag Leffler Function of Two Parameter

Definition 2.3.1. *(Mittag-Leffler Function for two parameters) The generalization of $E_\alpha(z)$ was studied by Wiman (1905) [36], Agarwal [29] and Humbert and Agarwal [32] defined the function as,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k \tag{23}$$

where $z, \alpha, \beta \in C, Re(\alpha) > 0, Re(\beta) > 0.$

2.4. Mittag Leffler Function of Three Parameter

In 1971, The more generalized function is introduced by Prabhakar [37] as

$$E_{\alpha,\beta}^{\gamma}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}. \tag{24}$$

where $z, \alpha, \beta, \gamma \in C, Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0,$ where $\gamma \neq 0, \gamma)_k = \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+k-1)$ is the Pochhammer symbol [34], and

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

2.5. Mittag Leffler Function of Four Parameter

In 2007, Shulka and Prajapati [34] introduced the function which is defined as,

$$E_{\alpha,\beta}^{\gamma,q}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k!\Gamma(\alpha k + \beta)}. \tag{25}$$

where $z, \alpha, \beta, \gamma \in C, min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0,$ and $q \in (0,1) \cup N$

2.6. Mittag Leffler Function of Five Parameter

In 2012, further generalization of Mittag - Leffler function was defined by Salim [35] and Chauhan [30] as,

$$E_{\alpha,\beta}^{\gamma,\delta,q}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_{(qk)} \Gamma(\alpha k + \beta)}. \tag{26}$$

where $z, \alpha, \beta, \gamma \in C, min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0,$ and $q \in (0,1) \cup N$

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)} \ \ and \ \ (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$$

denote the generalized Pochhammer symbol [34].

2.7. Generalized Mittag Leffler Function

Definition 2.7.1. *[33] The generalization of Mittag – Leffler function denoted by* $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x)$ *and defined by*

$$Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) = Q^{\gamma,q,r}_{\alpha,\beta,\delta}(a_1,a_2,...,a_r,b_1,b_2,...,b_r,x) = \sum_{s=0}^{\infty} \frac{\Pi^r_{n=1}\beta(b_n,s)(\gamma)_{qs}}{\Pi^r_{n=1}\beta(a_n,s)(\delta)_{qs}\Gamma(\alpha s+\beta)} x^s, \quad (27)$$

where $x,\alpha,\beta,\gamma,\delta,a_i,b_i \in C$,

$min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, *and* $q \in (0,1) \cup N$,

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)} \ \ and \ \ (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$$

2.8. Fox *H*-Function

The Fox *H*-function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral,

$$H^{m,n}_{PA}\left[z \,|\, \begin{matrix} (a_j,A_j)^p_1 \\ (b_h,B_k)^q_1 \end{matrix} \right] = \frac{1}{2\pi i}\int_L H^{m,n}_{p,q}(s) z^{-s} ds, z \neq 0, \quad (28)$$

where L is a suitable contour (of three possible types in $C: L_{-\infty}, L_{\infty}$, $(\gamma - i\infty, \gamma + i\infty)$), the orders (m,n,p,q) are non-negative integers such that $0 \le m \le q, 0 \le n \le p$, the parameters $A_j > 0, B_k > 0$ are positive, and a_j, b_k, $j = 1,\dots,p; k = 1\dots,q$, can be arbitrary complex such that $A_j(b_k + l) \neq B_k(a_j - l' - 1), l, l' = 0,1,2,\dots; \quad j = 1,\dots,n;$ $k = 1,\dots,m$, and the integrand has the form

Define also

$$\rho = \prod_{j=1}^{p} A_j^{-A_j} \prod_{k=1}^{q} B_k^{B_k}; \quad \Delta = \sum_{k=1}^{1} B_k - \sum_{j=1}^{p} A_j$$

$$\mu = \sum_{k=1}^{q} b_k - \sum_{j=1}^{p} a_j + \frac{p-q}{2}; \quad (29)$$

$$a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{k=1}^{m} B_k - \sum_{k=m+1}^{q} B_{k*}$$

Then it is an analytic function of *z* in circle domains $|z| < \rho$ (or sectors of them, or in the whole C), depending on the parameters and the contours [16-21].

2.9. Meijer's *G*-function

When all $A_f = B_k = 1, \quad j = 1,\dots,p; \quad k = 1,\dots,q$, in H-function,

$$H^{m,n}_{PA}\left[z \,|\, \begin{matrix} (a_j,A_j)^p_1 \\ (b_h,B_k)^q_1 \end{matrix} \right] == \frac{1}{2\pi i}\int_L H^{m,n}_{p,q}(s) z^{-s} ds \quad z \neq 0, \quad (30)$$

the FOX *H*-function reduces to the simpler Meijer's *G*-finction, denoted by $G^{mn}_{m}\left[z \left[\begin{matrix} (a_j)^p_1 \\ (b_k)^q_1 \end{matrix} \right] \right]$; [16-21].

2.10. Fox- Wright Function (F-W GHF)

The Wright generalized hypergeometric function $\Psi_Q(z)$, called also the Fox-Wright function (F-W ghf), is defined as

$$_p\Psi_q\left[\begin{matrix} (a_1,A_1),\dots,(a_p,A_p) \\ (b_1,B_1),\dots,(b_q,B_q) \end{matrix} \,|\, z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+kA_1)\cdots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\cdots\Gamma(b_q+kB_q)} \frac{z^k}{k!} = H^{1,p}_{p,q+1}\left[z \,|\, \begin{matrix} (1-a_1,A_1),\dots,(1-a_p,A_p) \\ (0,1),(1-b_1,B_1),\dots,(1-b_q,B_q) \end{matrix} \right]. \quad (31)$$

For $A_1 = \cdots = A_p = 1, B_1 = \cdots = B_q = 1$, the functions reduce to a $_pF_q$-function and a G-function, respectively:

$$_p\Psi_q\left[\begin{matrix} (a_1,1),\dots,(a_p,1) \\ (b_1,1),\dots,(b_q,1) \end{matrix} \,|\, z \right] = c\,{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k} \frac{z^k}{k!} = G^{1,p}_{p,q+1}\left[-z \left| \begin{matrix} 1-a_1,\dots,1-a_p \\ 0,1-b_1,\dots,1-b_q \end{matrix} \right. \right]$$

$$c = \left[\prod_{j=1}^{p} \Gamma(a_j) / \prod_{k=1}^{q} \Gamma(b_k) \right], \quad (a)_k := \Gamma(a+k)/\Gamma(a).$$

Among the special functions this is the simplest $_p\Psi_q(z)$ -function with $p = 0, q = 1$, called the Wright function when denoted as $\varphi(\alpha,\beta;z)$ or the Bessel Maitland (Wright-Bessel) function when denoted as J^{μ}_{ν}, as a fractional index analogue of the Bessel function J_{ν}.

2.11. Mittag Leffler as Fox *H*-Function

Using the two functions discussed above we can write the Mittag Leffler of two parameter as Fox H-function as

$$E_{\alpha,j}(z) = {}_1\Psi_1\left[\begin{matrix} (1,1) \\ (\beta,\alpha) \end{matrix} \,|\, z \right] = H^{1,1}_{1,2}\left[-z \left| \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right. \right]$$

$$E_{\left(\frac{1}{\beta_1}\right),(\mu_1)}(z) = E^{(m)}_{\left(\frac{1}{e_6}\right),(\mu_k)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\cdots\Gamma(\mu_m + k/\rho_n)}$$

$$=1\Psi_{\mathrm{m}}\left[\begin{array}{c}(1,1)\\ \left(\mu_i,\frac{1}{\rho_t}\right)_1^m \mid z\end{array}\right]=H_{1,m+1}^{1,1}\left[-z|(0,1),\left(1-\mu_i,\frac{1}{A_1}\right)_1^m\right].$$

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(ak+)}\frac{z^k}{k!}$$
$$=0_1\Psi_1[(\beta,\alpha)|\ z]=E_{(a,1),(\beta,1)}^{(2)}(z).$$

2.12. Wright Bessel Function

The Wright function is known as the Wright Bessel function, or misnamed as the Bessel-Maitland function, $v\in\mathbb{R},\ \mu>-1$:

$$J_v(z)=\phi(\mu,v+1;-z)=\sum_{k-0}^{\infty}\frac{(-z)^k}{\Gamma(v+k\mu+1)^k|}$$
$$=0\Psi_1\left[\begin{array}{c}-\\(v+1,\mu)|-z\end{array}\right]=H_{0,2}^{1,0}\left[z\Big|\begin{array}{c}-\\(0,1),(-v,\mu)\end{array}\right]\quad(32)$$
$$=E_{[1/w,1],(v+1,1)}^{(2)}(-z).$$

2.13. Wright-Lonmel Function

The Wright-Lonmel function defined due to Pathak 1966-1967; $v,\lambda\in\mathbb{R},\mu>0$:

$$J_{v,\lambda}^{\mu}(z)=(z/2)^{k+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{kk}}{\Gamma(k+k\mu+\lambda+1)\Gamma(\lambda+k+1)}\quad(33)$$
$$=(z/2)^{v+2\lambda}E_{(1/\mu,1),(v+\lambda+1,\lambda+1)}^{(2)}\left(-(z/2)^2\right)$$

including, for $\mu=1$, the Lommel and thus, also the Struve functions $J_{v,\lambda}^{1}(z)=$ const $S_{2\lambda+v-1,y}(z)$

2.14. Hyper-Bessel Function

A hyper-Bessel function, in Delerue's sense [38]

$$J_{\gamma_i,\cdots,\gamma_{m-1}}^{(m-1)}(z)$$
$$=\left(\frac{z}{m}\right)^{\sum_{i-1}^{m-1}\gamma_i}E_{(1,1,\cdots,1)1(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1,1)}^{(m)}\left(-\left(\frac{z}{m}\right)^m\right)$$
$$=\left[\prod_{i=1}^{m-1}\Gamma\left(\gamma_i+1\right)\right]^{-1}\left(\frac{z}{m}\right)^{\sum_{i-1}^{m-1}\gamma_i}$$
$$\times F_{m-1}\left(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1;-\left(\frac{z}{m}\right)^m\right)$$

As a further extension of both multi-M-L functions

$$E_\rho\left(\left(\alpha_j,\beta_j\right)_{1,m};z\right)=\sum_{k-0}^{\infty}\frac{(\rho)_k}{\Pi_{-1}^{m}\Gamma\left(\alpha_jk+\beta_j\right)}\frac{z^k}{k!}\quad(34)$$
$$=\frac{1}{\rho_1}\Psi_m[(\rho,1)(\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m)|\ z$$

3. Conclusion

A large number of new functions have been developed to accommodate the fractional order handling of differentiation and integration, and there has been additional generalisation of derivatives and integrals. The most commonly used definitions for the special functions of fractional calculus were just reviewed.

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Conflict of Interest

There is no conflict of interest.

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