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Harmonic Oscillations and Resonances in 3-D Nonlinear Dynamical System

Usama H. Hegazy*, Mousa A. ALshawish

Department of Mathematics, Faculty of Science, Al-Azhar University, Gaza, Palestine
*Corresponding author: uhijazy@yahoo.com, u.hejazy@alazhar.edu.ps

Abstract This paper is concerned with the three dimensional motion of a nonlinear dynamical system. The motion is described by nonlinear partial differential equation, which is converted by Galerkin method to three dimensional ordinary differential equations. The three dimensional differential equations, under the influence of external forces, are solved analytically and numerically by the multiple time scales perturbation technique and the Runge-Kutta fourth order method. Phase plane technique and frequency response equations are used to investigate the stability of the system and the effects of the parameters of the system, respectively.

Keywords: *Galerkin method, resonances, nonlinearities*

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1. Introduction

Problems involving nonlinear differential equations are extremely diverse, and methods of solutions or analysis are problem dependent. Nonlinear systems are interesting for engineers, physicists and mathematicians because most physical systems are nonlinear in nature. The sub-combination internal resonance of a uniform cantilever beam of varying orientation with a tip mass under vertical base excitation is studied. The Euler–Bernoulli theory slender beam was used to derive the governing nonlinear partial differential equation [1]. The dynamic stability of a moving string in three-dimensional vibration is investigated [2]. Three nonlinear integro-differential equations of motion are studied and the analysis is focused on the case of primary resonance of the first in-plane flexural mode when its frequency is approximately twice the frequency of the first out-of-plane flexural-torsional mode [3]. The method of multiple time scales is applied to investigate the response of nonlinear mechanical systems with internal and external resonances. The stability of vibrating systems is investigated by applying both the frequency response equation and the phase plane methods. The numerical solutions are focused on both the effects of the different parameters and the behavior of the system at the considered resonance cases [4,5]. The nonlinear characteristics in the large amplitude three-dimensional free vibrations of inclined sagged elastic cables are investigated [6]. The nonlinear forced vibration of a plate-cavity system is analytically studied. Galerkin method is used to derive coupled nonlinear equations of the system. In order to solve the nonlinear equations of plate-cavity system, multiple scales method is employed. Closed form expressions are obtained for the frequency-amplitude relationship in different resonance conditions [7]. The steady-state periodic response of the forced vibration for an axially moving viscoelastic beam in the supercritical speed range is studied [8]. For this motion, the model is cast in the standard form of continuous gyroscopic systems. Internal Various approximate analytical methods are developed for obtaining solutions for strongly nonlinear differential equations in a complex function. The methods of harmonic balance, Krylov-Bogoliubov and elliptic perturbation are utilized [9]. The problem of suppressing the vibrations of a hinged-hinged flexible beam when subjected to external harmonic and parametric excitations is considered and studied. The multiple scale perturbation method is applied to obtain a first-order approximate solution. The equilibrium curves for various controller parameters are plotted. The stability of the steady state solution is investigated using frequency-response equations. The approximate solution was numerically verified. It is found that all predictions from analytical solutions were in good agreement with the numerical simulations [10].

2. Equations of Motion

The nonlinear partial differential equation governing the flexural deflection $u(x,t)$ of the beam, subject to harmonic axial excitation $p = p_0 - p_1 \cos \Omega t$, is given by [11,12]

$$
\begin{aligned}
& m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + EI\frac{\partial^4 u}{\partial x^4} + (p_0 + p_1 \cos\Omega t)\frac{\partial^2 u}{\partial x^2} \\
& + \frac{3}{2}(p_0 + p_1\cos\Omega t)\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} \\
& + EI\left[\frac{27}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 3\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^4 u}{\partial x^4} + \frac{9}{4}\left(\frac{\partial u}{\partial x}\right)^4\frac{\partial^4 u}{\partial x^4}\right] = 0
\end{aligned} \quad (1)
$$

under the following boundary conditions:

$$u(x)=0 \ and \ \frac{\partial u}{\partial x}=0 \ at x=0, x=L. \tag{2}$$

Equation (1) can be converted to a three dimensional nonlinear ordinary differential equations applying the method of Galerkins and using the following expression

$$u(x,t)=g(t)\sin\left(\frac{\pi x}{L}\right)+h(t)\sin\left(\frac{2\pi x}{L}\right)+k(t)\sin\left(\frac{3\pi x}{L}\right) \tag{3}$$

into equation (1). Then we have

$$\begin{aligned} &g''+\omega_1^2 g+\varepsilon(\alpha g'+\eta_1 h^2 g+\eta_2 g k^2+\eta_3 h_2 k \\ &+\eta_4 g^3+\eta_5 g^2 k)+\varepsilon^2(\eta_6 g k^4+\eta_7 g^2 k^3+\eta_8 h^2 k^3 \\ &+\eta_9 h^2 g^3+\eta_{10} h^4 k+\eta_{11} h^4 g+\eta_{12} g^3 k^2+\eta_{13} g^5 \\ &+\eta_{14} h^2 k^2 g+\eta_{15} g^4 k+\eta_{16} h^2 g^2 k)=\varepsilon F\cos\Omega t, \end{aligned} \tag{4}$$

$$\begin{aligned} &h''+w_2^2 h+\varepsilon(\beta h'+\lambda_1 hgk+\lambda_2 h^3+\lambda_3 hk^2+\lambda_4 hg^2) \\ &+\varepsilon^2(\lambda_5 hg^4+\lambda_6 h^3 k^2+\lambda_7 h^5+\lambda_8 hgk^3+\lambda_9 hk^4 \\ &+\lambda_{10} h^3 g^2+\lambda_{11} gkh^3+\lambda_{12} hg^2 k^2+\lambda_{13} hkg^3)=\varepsilon F\cos\Omega t, \end{aligned} \tag{5}$$

$$\begin{aligned} &k''+w_3^2 k+\varepsilon(\delta k'+\tau_1 h^2 k+\tau_2 g k^2+\tau_3 g h^2 \\ &+\tau_4 g^3+\tau_5 k^3)+\varepsilon^2(\tau_6 k^5+\tau_7 g^4 k+\tau_8 g^2 k^3 \\ &+\tau_9 k^2 g^3+\tau_{10} h^2 g^3+\tau_{11} h^4 g+\tau_{12} g h^2 k^2 \\ &+\tau_{13} k h^2 g^2+\tau_{14} g^5+\tau_{15} h^4 k+\tau_{16} h^2 k^3)=\varepsilon F\cos\Omega t. \end{aligned} \tag{6}$$

3. Perturbation Solution

The method of multiple scales is applied to determine an approximate solution for the differential equations (4-6). Assuming that *g*, *h* and *k* are in the forms

$$\begin{aligned} g(T_0,T_1)&=g_0(T_0,T_1)+\varepsilon g_1(T_0,T_1)+..., \\ h(T_0,T_1)&=h_0(T_0,T_1)+\varepsilon h_1(T_0,T_1)+..., \\ k(T_0,T_1)&=k_0(T_0,T_1)+\varepsilon k_1(T_0,T_1)+..., \end{aligned} \tag{7}$$

Where $T_0=t$, $T_1=\varepsilon T_0=\varepsilon t$.

The time derivatives are written as

$$\begin{aligned} \frac{d}{dt}&=D_0+\varepsilon D_1+..., \\ \frac{d^2}{dt^2}&=D_0^2+2\varepsilon D_0 D_1+..., \end{aligned} \tag{8}$$

Where

$$D_0=\frac{\partial}{\partial T_0}, D_1=\frac{\partial}{\partial T_1}. \tag{9}$$

Substituting eqs. (7-9) into eqs. (4-6) and equating coefficients of same powers of ε yields:

$o(\varepsilon^0)$:

$$\left(D_0^2+\omega_1^2\right)g_0=0 \tag{10}$$

$$\left(D_0^2+\omega_2^2\right)h_0=0. \tag{11}$$

$$\left(D_0^2+\omega_1^2\right)k_0=0 \tag{12}$$

$o(\varepsilon^1)$:

$$\begin{aligned} &\left(D_0^{\,2}+\omega_1^{\,2}\right)g_1=-2D_0D_1g_0-\alpha D_0 g_0-\eta_1 h_0^{\,2} g_0 \\ &-\eta_2 k_0^{\,2} g_0-\eta_3 h_0^{\,2} k_0-\eta_4 g_0^{\,3}-\eta_5 g_0^{\,2} k_0+F\cos(\Omega t) \end{aligned} \tag{13}$$

$$\begin{aligned} &\left(D_0^{\,2}+\omega_2^{\,2}\right)h_1=-2D_0D_1h_0-\beta D_0 h_0-\lambda_1 h_0 g_0 k_0 \\ &-\lambda_2 h_0^{\,3}-\lambda_3 h_0 k_0^{\,2}-\lambda_4 h_0 g_0^{\,2}+F\cos\Omega t. \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(D_0^{\,2}+\omega_3^{\,2}\right)k_1=-2D_0D_1k_0-\delta D_0 k_0-\tau_1 h_0^{\,2} k_0 \\ &-\tau_2 k_0^{\,2} g_0-\tau_3 h_0^{\,2} g_0-\tau_4 g_0^{\,3}-\tau_5 k_0^{\,3}+F\cos(\Omega t). \end{aligned} \tag{15}$$

The solution of eqs. (10-12) is expressed as

$$g_0(T_0,T_1)=A(T_1)e^{i\omega_1 T_0}+\bar{A}(T_1)e^{-i\omega_1 T_0}. \tag{16}$$

$$h_0(T_0,T_1)=B(T_1)e^{i\omega_2 T_0}+\bar{B}(T_1)e^{-i\omega_2 T_0}. \tag{17}$$

$$k_0(T_0,T_1)=C(T_1)e^{i\omega_3 T_0}+\bar{C}(T_1)e^{-i\omega_3 T_0}. \tag{18}$$

Where A, B, C are complex functions in T_1 Substituting eqs. (16-18) into eqs (13-15),we get

$$\begin{aligned} &\left(D_0^{\,2}+\omega_1^{\,2}\right)g_1=\begin{pmatrix}-2i\omega_1 A'-\alpha i\omega_1 A-2\eta_1 B\bar{B}A \\ -2\eta_2 C\bar{C}A-3\eta_4 A^2\bar{A}\end{pmatrix}e^{i\omega_1 T_0} \\ &+\left(-2\eta_3 CB\bar{B}-2\eta_5 CA\bar{A}\right)e^{i\omega_3 T_0}-\eta_4 A^3 e^{3i\omega_1 T_0} \\ &-\eta_1 B^2 A e^{iT_0(2\omega_2+\omega_1)}-\eta_1\bar{B}^2 A e^{iT_0(-2\omega_2+\omega_1)} \\ &-\eta_2 C^2 A e^{iT_0(2\omega_3+\omega_1)}-\eta_2\bar{C}^2 A e^{iT_0(-2\omega_3+\omega_1)} \\ &-\eta_3 CB^2 e^{iT_0(\omega_3+2\omega_2)}-\eta_3\bar{C}B^2 e^{iT_0(-\omega_3+2\omega_2)} \\ &-\eta_5 CA^2 e^{iT_0(\omega_3+2\omega_1)}-\eta_5\bar{C}A^2 e^{iT_0(-\omega_3+2\omega_1)} \\ &+\frac{1}{2}Fe^{i\Omega T_0}+cc. \end{aligned} \tag{19}$$

$$\begin{aligned} &\left(D_0^{\,2}+\omega_2^{\,2}\right)h_1=(-2i\omega_2 B'-\beta i\omega_2 B-3\lambda_2 B^2\bar{B} \\ &-2\lambda_3 C\bar{C}B-2\lambda_4 BA\bar{A})e^{i\omega_2 T_0}-\lambda_2 B^3 e^{3i\omega_2 T_0} \\ &-\lambda_1 BACe^{iT_0(\omega_2+\omega_1+\omega_3)}-\lambda_1 BA\bar{C}e^{iT_0(\omega_2+\omega_1-\omega_3)} \\ &-\lambda_1\bar{B}A\bar{C}e^{iT_0(-\omega_2+\omega_1-\omega_3)}-\lambda_1\bar{B}ACe^{iT_0(-\omega_2+\omega_1+\omega_3)} \\ &-\lambda_3 C^2 Be^{iT_0(2\omega_3+\omega_2)}-\lambda_3\bar{C}^2 Be^{iT_0(-2\omega_3+\omega_2)} \\ &-\lambda_4 BA^2 e^{iT_0(\omega_2+2\omega_1)}-\lambda_4\bar{B}A^2 e^{iT_0(-\omega_2+2\omega_1)} \\ &+\frac{F}{2}e^{i\Omega T_0}+cc. \end{aligned} \tag{20}$$

$$\begin{aligned} &\left(D_0^{\,2}+\omega_3^{\,2}\right)k_1=\begin{pmatrix}-2i\omega_3 C'-\delta i\omega_3 C \\ -2\tau_1 CB\bar{B}-3\tau_5 C^2\bar{C}\end{pmatrix}e^{i\omega_3 T_0} \\ &+\left(-\tau_2 2AC\bar{C}-2\tau_3 B\bar{B}A-3\tau_4 A^2\bar{A}\right)e^{i\omega_1 T_0} \\ &-\tau_1 B^2 Ce^{iT_0(2\omega_2+\omega_3)}-\tau_1 B^2\bar{C}e^{iT_0(2\omega_2-\omega_3)} \\ &-\tau_2 AC^2 e^{iT_0(2\omega_3+\omega_1)}-\tau_2 A\bar{C}^2 e^{iT_0(-2\omega_3+\omega_1)} \\ &-\tau_3 B^2 Ae^{iT_0(2\omega_2+\omega_1)}-\tau_3\bar{B}^2 Ae^{iT_0(-2\omega_2+\omega_1)} \\ &-\tau_4 A^3 e^{3i\omega_1 T_0}-\tau_5 C^3 e^{3i\omega_3 T_0}+\frac{F}{2}e^{i\Omega T_0}+cc. \end{aligned} \tag{21}$$

Where cc denotes a complex conjugate of the preceding term.

The general solution of eqs. (19-20) can be written in the following form

$$
\begin{aligned}
g_1(T_0,T_1) &= \frac{\left(-2\eta_3 CB\bar{B} - 2\eta_5 CA\bar{A}\right)}{(\omega_1-\omega_3)(\omega_1+\omega_3)} e^{i\omega_3 T_0} \\
&+ \frac{\eta_4 A^3}{8\omega_1{}^2} e^{3i\omega_1 T_0} + \frac{\eta_1 B^2 A}{4\omega_2(\omega_2+\omega_1)} e^{iT_0(2\omega_2+\omega_1)} \\
&- \frac{\eta_1 \bar{B}^2 A}{4\omega_2(\omega_1-\omega_2)} e^{iT_0(-2\omega_2+\omega_1)} \\
&+ \frac{\eta_2 C^2 A}{4\omega_3(\omega_3+\omega_1)} e^{iT_0(2\omega_3+\omega_1)} \\
&- \frac{\eta_2 \bar{C}^2 A}{4\omega_3(\omega_1-\omega_3)} e^{iT_0(-2\omega_3+\omega_1)} \\
&- \frac{\eta_3 CB^2}{(2\omega_2+\omega_3+\omega_1)(2\omega_2+\omega_3-\omega_1)} e^{iT_0(\omega_3+2\omega_2)} \\
&+ \frac{\eta_3 CB^2}{(-2\omega_2+\omega_3-\omega_1)(-2\omega_2+\omega_3+\omega_1)} e^{iT_0(-\omega_3+2\omega_2)} \\
&- \frac{\eta_5 CA^2}{(\omega_3+3\omega_1)(\omega_3+\omega_1)} e^{iT_0(\omega_3+2\omega_1)} \\
&+ \frac{\eta_5 \bar{C}A^2}{(\omega_3-3\omega_1)(\omega_3-\omega_1)} e^{iT_0(-\omega_3+2\omega_1)} \\
&+ \frac{F}{2(\omega_1-\Omega)(\omega_1+\Omega)} e^{i\Omega T_0} + cc.
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
h_1(T_0,T_1) &= \frac{-\lambda_2 B^3}{-8\omega_2{}^2} e^{3i\omega_2 T_0} \\
&+ \frac{\lambda_1 BAC}{(\omega_3+\omega_1)(\omega_3+\omega_1+2\omega_2)} e^{iT_0(\omega_2+\omega_1+\omega_3)} \\
&+ \frac{\lambda_1 BA\bar{C}}{(-\omega_1+\omega_3)(\omega_3-\omega_1-2\omega_2)} e^{iT_0(\omega_2+\omega_1-\omega_3)} \\
&+ \frac{\lambda_1 \bar{B}A\bar{C}}{(-\omega_1+\omega_3)(\omega_3-\omega_1+2\omega_2)} e^{iT_0(-\omega_2+\omega_1-\omega_3)} \\
&+ \frac{\lambda_1 \bar{B}AC}{(\omega_3+\omega_1)(\omega_3+\omega_1-2\omega_2)} e^{iT_0(-\omega_2+\omega_1+\omega_3)} \\
&+ \frac{\lambda_3 C^2 B}{4\omega_3(\omega_2+\omega_3)} e^{iT_0(2\omega_3+\omega_2)} \\
&+ \frac{\lambda_3 \bar{C}^2 B}{4\omega_3(-\omega_2+\omega_3)} e^{iT_0(-2\omega_3+\omega_2)} \\
&+ \frac{\lambda_4 BA^2}{4\omega_1(\omega_2+\omega_1)} e^{iT_0(\omega_2+2\omega_1)} \\
&+ \frac{\lambda_4 \bar{B}A^2}{4\omega_1(\omega_2-\omega_1)} e^{iT_0(-\omega_2+2\omega_1)} \\
&- \frac{F}{2(\omega_2-\Omega)(\omega_2+\Omega)} e^{i\Omega T_0} + cc.
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
k_1(T_0,T_1) &= \frac{\left(-\tau_2 2AC\bar{C} - 2\tau_3 B\bar{B}A - 3\tau_4 A^2\bar{A}\right)}{(\omega_3-\omega_1)(\omega_3-\omega_1)} e^{i\omega_1 T_0} \\
&+ \frac{\tau_1 B^2 C}{4\omega_2(\omega_2+\omega_3)} e^{iT_0(2\omega_2+\omega_3)} \\
&+ \frac{\tau_1 B^2 \bar{C}}{4\omega_2(\omega_2-\omega_3)} e^{iT_0(2\omega_2-\omega_3)} \\
&+ \frac{\tau_2 AC^2}{(\omega_1+3\omega_3)(\omega_1+\omega_3)} e^{iT_0(2\omega_3+\omega_1)} \\
&+ \frac{\tau_2 A\bar{C}^2}{(\omega_1-\omega_3)(\omega_1-3\omega_3)} e^{iT_0(-2\omega_3+\omega_1)} \\
&+ \frac{\tau_3 B^2 A}{(2\omega_2+\omega_1+\omega_3)(2\omega_2+\omega_1-\omega_3)} e^{iT_0(2\omega_2+\omega_1)} \\
&+ \frac{\tau_3 \bar{B}^2 A}{(-2\omega_2+\omega_1-\omega_3)(-2\omega_2+\omega_1+\omega_3)} e^{iT_0(-2\omega_3+\omega_1)} \\
&+ \frac{\tau_4 A^3}{(3\omega_1-\omega_3)(3\omega_1+\omega_3)} e^{3i\omega_1 T_0} \\
&+ \frac{\tau_5 C^3}{8\omega_3{}^2} e^{3i\omega_3 T_0} + \frac{F}{2(\omega_3-\Omega)(\omega_3+\Omega)} e^{i\Omega T_0} + cc.
\end{aligned} \tag{24}
$$

From eqs. (22-24) the following resonance cases are extracted:

- Internal Resonance:
  1. $\omega_1 = \omega_3$,
  2. $\omega_1 = \omega_2 = \omega_3$,
- External Resonance:
  - a. Primary resonance
    1. $\Omega = \omega_1$,
    2. $\Omega = \omega_2$,
    3. $\Omega = \omega_3$,
  - b. Simultaneous resonance
    1. $\Omega = \omega_1 = \omega_2 = \omega_3$.

4. Stability Analysis

We shall consider the resonance case $\Omega \approx \omega_2$ when $\omega_1 \approx \omega_2 \approx \omega_3$. Using the detuning parameter σ, the resonance case are expressed as

$$\Omega = \omega_2 + \varepsilon\sigma_1, \omega_1 = \omega_2 + \varepsilon\sigma_2, \omega_3 = \omega_2 + \varepsilon\sigma_3. \tag{25}$$

Substituting eq. (25) into eqs. (19-21) and eliminating terms that produce secular term then performing some algebraic manipulations, we obtain the following modulation equations:

$$
\begin{aligned}
&\begin{pmatrix} -2i\omega_1 A' - \alpha i\omega_1 A - 2\eta_1 B\bar{B}A \\ -2\eta_2 C\bar{C}A - 3\eta_4 A^2\bar{A} \end{pmatrix} e^{i\omega_1 T_0} \\
&+\left(-2\eta_3 CB\bar{B} - 2\eta_5 CA\bar{A}\right) e^{i\omega_3 T_0} - \eta_3 \bar{C}B^2 e^{iT_0(-\omega_3+2\omega_2)} \\
&-\eta_5 \bar{C}A^2 e^{iT_0(-\omega_3+2\omega_1)} + \frac{1}{2} F e^{i\Omega} = 0.
\end{aligned} \tag{26}
$$

$$\begin{pmatrix}-2i\omega_2 B' - i\beta\omega_2 B - 3\lambda_2 B^2\bar{B}\\ -2\lambda_3 C\bar{C}B - 2\lambda_4 BA\bar{A}\end{pmatrix} e^{i\omega_2 T_0}$$
$$-\lambda_1 BA\bar{C}e^{iT_0(\omega_2+\omega_1-\omega_3)} - \lambda_1\bar{B}ACe^{iT_0(-\omega_2+\omega_1+\omega_3)} \quad (27)$$
$$-\lambda_4\bar{B}A^2e^{iT_0(-\omega_2+2\omega_1)} + \frac{1}{2}Fe^{i\Omega} = 0.$$

$$\left(-2i\omega_3 C' - \delta i\omega_3 C - 2\tau_1 CB\bar{B} - 3\tau_5 C^2\bar{C}\right)e^{i\omega_3 T_0}$$
$$+\left(-2\tau_2 AC\bar{C} - 2\tau_3 B\bar{B}A - 3\tau_4 A^2\bar{A}\right)e^{i\omega_1 T_0} \quad (28)$$
$$-\tau_1 B^2\bar{C}e^{iT_0(2\omega_2-\omega_3)} + \frac{1}{2}Fe^{i\Omega T_0} = 0.$$

Letting

$$A = \frac{1}{2}a_1e^{i\theta_1},\ B = \frac{1}{2}a_2e^{i\theta_2},\ C = \frac{1}{2}a_3e^{i\theta_3}.$$

where $a_1, a_2, a_3, \theta_1, \theta_2, \theta_3$ are functions of T_1. Separating real and imaginary parts gives the following six equations governing the amplitude and phase modulations

$$a_1\nu_5' = a_1(\sigma_1 - \sigma_2) - \frac{1}{4\omega_1}\eta_1 a_1 a_2^2 - \frac{1}{4\omega_1}\eta_2 a_1 a_3^2$$
$$-\frac{3}{8\omega_1}\eta_4 a_1^3 - \frac{1}{4\omega_1}\eta_3 a_2^2 a_3\cos\nu_1 -$$
$$\frac{1}{2\omega_1}\eta_5 a_1^2 a_3\cos\nu_2 - \frac{1}{8\omega_1}\eta_3 a_2^2 a_3\cos\nu_3 \quad (29)$$
$$-\frac{1}{8\omega_1}\eta_5 a_1^2 a_3\cos\nu_4 + \frac{1}{2\omega_1}F\cos\nu_5$$

$$a_1' = -\frac{1}{2}\alpha a_1 - \frac{1}{4\omega_1}\eta_3 a_2^2 a_3\sin\nu_1$$
$$-\frac{1}{2\omega_1}\eta_5 a_1^2 a_3\sin\nu_2 - \frac{1}{8\omega_1}\eta_3 a_2^2 a_3\sin\nu_3 \quad (30)$$
$$-\frac{1}{8\omega_1}\eta_5 a_1^2 a_3\sin\nu_4 + \frac{1}{2\omega_1}F\sin\nu_5$$

$$a_2\left(\frac{\nu_4'}{5} - \frac{\nu_3'}{10} - \frac{\nu_1'}{10}\right) = \frac{a_2}{5}\sigma_3 - \frac{3}{16\omega_2}\lambda_5 a_1^4 a_2$$
$$-\frac{3}{16\omega_2}\lambda_6 a_3^2 a_2^3 - \frac{5}{16\omega_2}\lambda_7 a_2^5 - \frac{3}{16\omega_2}\lambda_9 a_3^4 a_2$$
$$-\frac{3}{16\omega_2}\lambda_{10}a_1 a_2^4 - \frac{1}{4\omega_2}\lambda_{12}a_1^2 a_3^2 a_2 \quad (31)$$
$$-\frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3^3\cos\nu_2 - \frac{1}{32\omega_2}\lambda_{11}a_1 a_3 a_2^3\cos\nu_3$$
$$-\frac{1}{32\omega_2}\lambda_6 a_2^3 a_3^2\cos\nu_4 - \frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3^3\cos\nu_5,$$

$$a_2' = -\frac{1}{2}\beta a_2 - \frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3^3\sin\nu_2$$
$$-\frac{1}{32\omega_2}\lambda_{11}a_1 a_3 a_2^3\sin\nu_3 - \frac{1}{32\omega_2}\lambda_6 a_2^3 a_3^2\sin\nu_4 \quad (32)$$
$$-\frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3^3\sin\nu_5,$$

$$a_3\nu_{12}' = a_3(\sigma_1 - \sigma_3) - \frac{1}{4\omega_3}\tau_1 a_3 a_2^2 - \frac{3}{8\omega_3}\tau_5 a_3^3$$
$$+\left(-\frac{1}{4\omega_3}\tau_2 a_1 a_3^2 - \frac{1}{4\omega_3}\tau_3 a_1 a_2^2 - \frac{3}{8\omega_3}\tau_4 a_1^3\right)\cos\nu_{10} \quad (33)$$
$$-\frac{1}{8\omega_3}\tau_1 a_2^2 a_3\cos\nu_{11} + \frac{1}{2\omega_3}F\cos\nu_{12},$$

$$a_3' = -\frac{1}{2}\delta a_3 + \begin{pmatrix}-\frac{1}{4\omega_3}\tau_2 a_1 a_3^2 - \frac{1}{4\omega_3}\tau_3 a_1 a_2^2\\ -\frac{3}{8\omega_3}\tau_4 a_1^3\end{pmatrix}\sin\nu_{10} \quad (34)$$
$$-\frac{1}{8\omega_3}\tau_1 a_2^2 a_3\sin\nu_{11} + \frac{1}{2\omega_3}F\sin\nu_{12}.$$

where

$$\nu_1 = -\theta_1 + \sigma_3 T_1 - \sigma_2 T_1,$$
$$\nu_2 = \theta_3 - \theta_1 + \sigma_3 T_1 - \sigma_2 T_1,$$
$$\nu_3 = 2\theta_2 - \theta_3 - \theta_1 - \sigma_3 T_1 - \sigma_2 T_1,$$
$$\nu_4 = -\nu_2, \nu_5 = -\theta_1 + \sigma_1 T_1 - \sigma_2 T_1,$$
$$\nu_6 = \theta_2 - \theta_3 - \theta_1 - \sigma_3 T_1,$$
$$\nu_7 = \theta_1 - \sigma_1 T_1 + \sigma_2 T_1,$$
$$\nu_8 = \theta_1 - \theta_3 - \theta_2 + \sigma_1 T_1,$$
$$\nu_9 = \theta_1 - \theta_2 - \sigma_3,$$
$$\nu_{10} = \theta_1 - \theta_3 + \sigma_2 T_1 - \sigma_3 T,$$
$$\nu_{11} = 2\theta_2 - 2\theta_3 - 2\sigma_3 T,$$
$$\nu_{12} = -\theta_3 + \sigma_1 T_1 - \sigma_3 T_1.$$

The steady-state solutions of eqs. (29-34) are obtained by setting $a_1' = a_2' = a_3' = \nu_5' = \nu_8' = \nu_{12}' = 0$. into eqs. (29-34). This results in the following nonlinear algebraic equations, which are called the frequency response equations:

$$\Lambda_1 a_1^6 + \Lambda_2 a_1^4 + \Lambda_3 a_1^2 + \Lambda_4 = 0. \quad (35)$$

$$\Lambda_5 a_2^6 + \Lambda_6 a_2^4 + \Lambda_7 a_2^2 + \Lambda_8 a_2 + \Lambda_9 = 0. \quad (36)$$

$$\Lambda_{10}a_3^6 + \Lambda_{11}a_3^4 + \Lambda_{12}a_3^3 + \Lambda_{13}a_3^2 + \Lambda_{14}a_3 + \Lambda_{15} = 0. \quad (37)$$

The coefficients $\Lambda_i, i = (1,2,...,15)$, are given in Appendix.

5. Numerical Results and Discussions

In this section, the Runge-Kutta fourth order method is applied to determine the numerical time series solutions (*t*, *g*), (*t*, *h*), and (*t*, *k*)and the phase planes (*g*, *v*), (*h*, *v*), (*k*, *v*), respectively, for the three modes of the nonlinear system (4-6). Moreover, the fixed points of the model is obtained by solving the frequency response equations (35-37) numerically.

5.1. Time-response Solution

A non-resonant time response and the phase plane of the three modes of vibration of the system is shown in Figure 1. In Figure 2, different resonance cases are investigated and an approximate percentage of increase, if exists, in maximum steady-state amplitude compared to that in the non-resonant case is indicated.

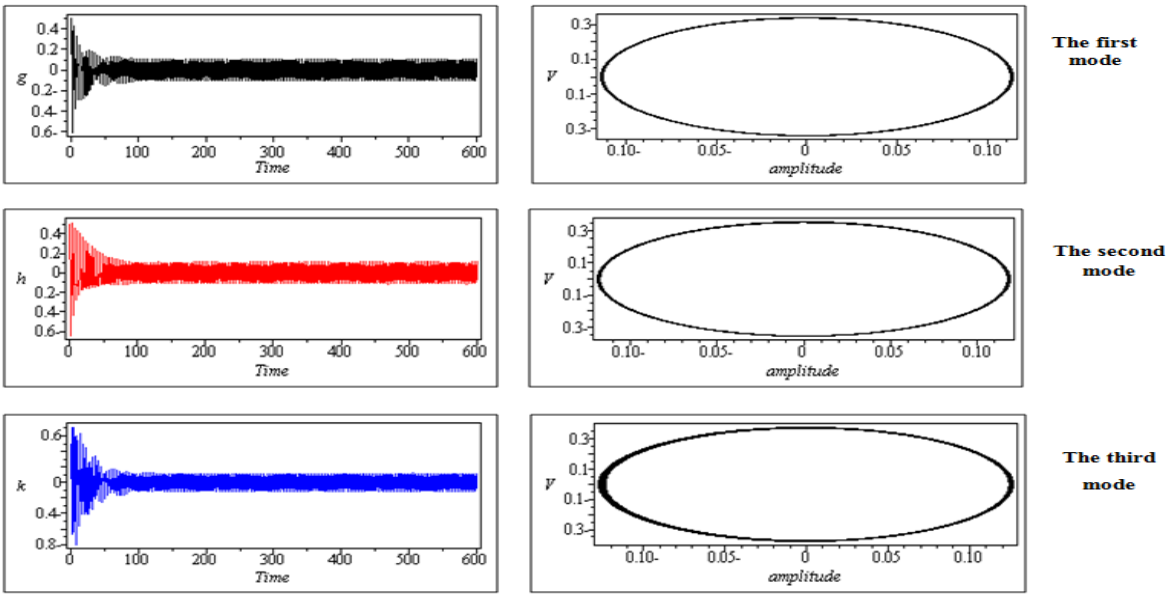

Figure 1. Non-resonant time solution of the 3-D model to external excitation

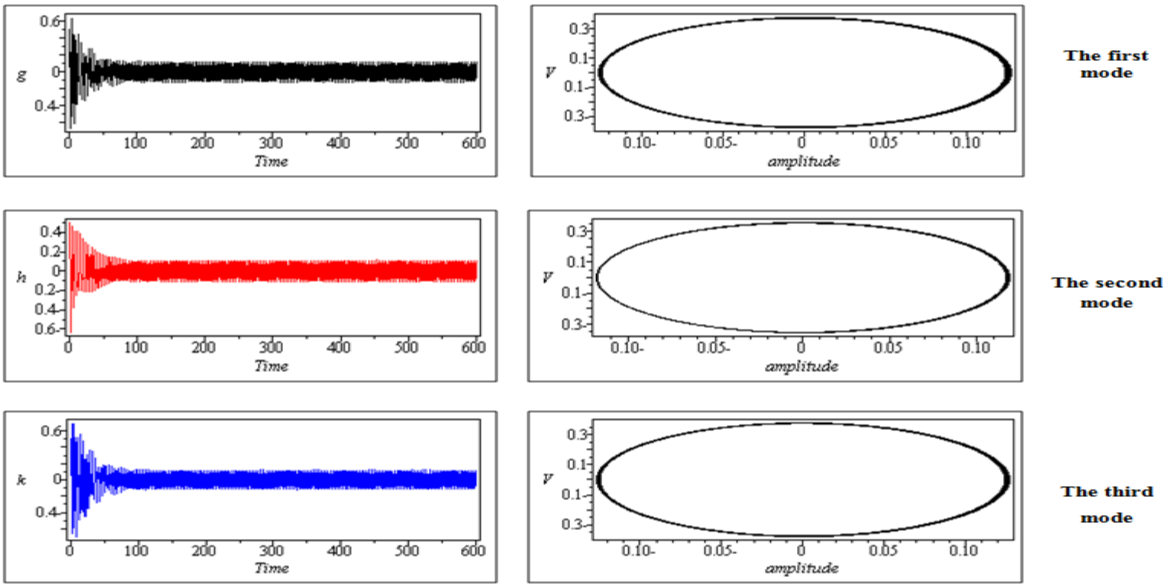

Figure 2(a). The internal resonance condition $\omega_1 = \omega_3 = 2.6$

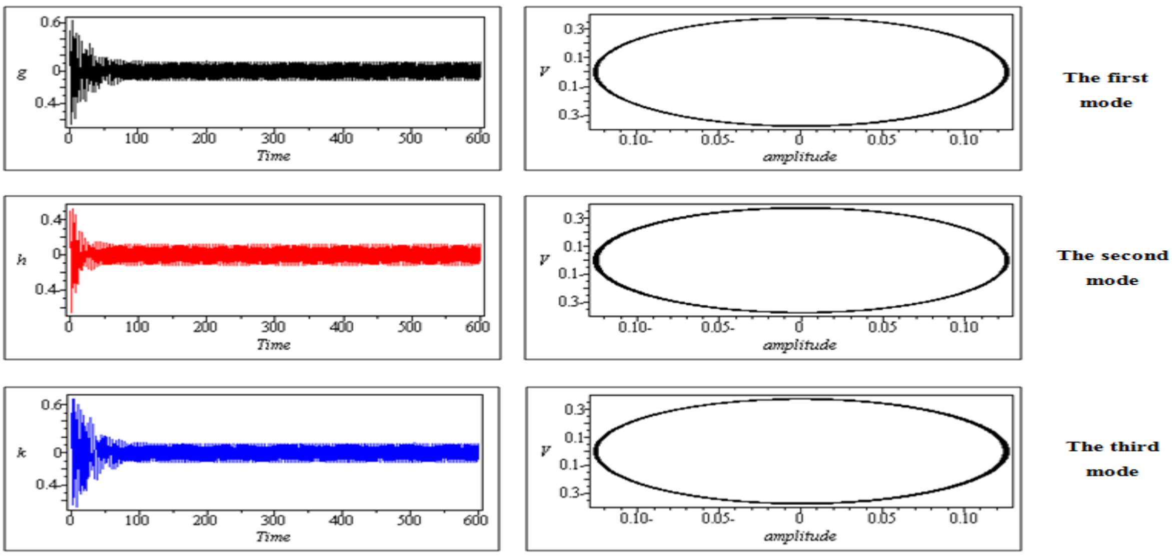

Figure 2(b). The internal resonance condition $\omega_1 = \omega_2 = \omega_3 = 2.6$

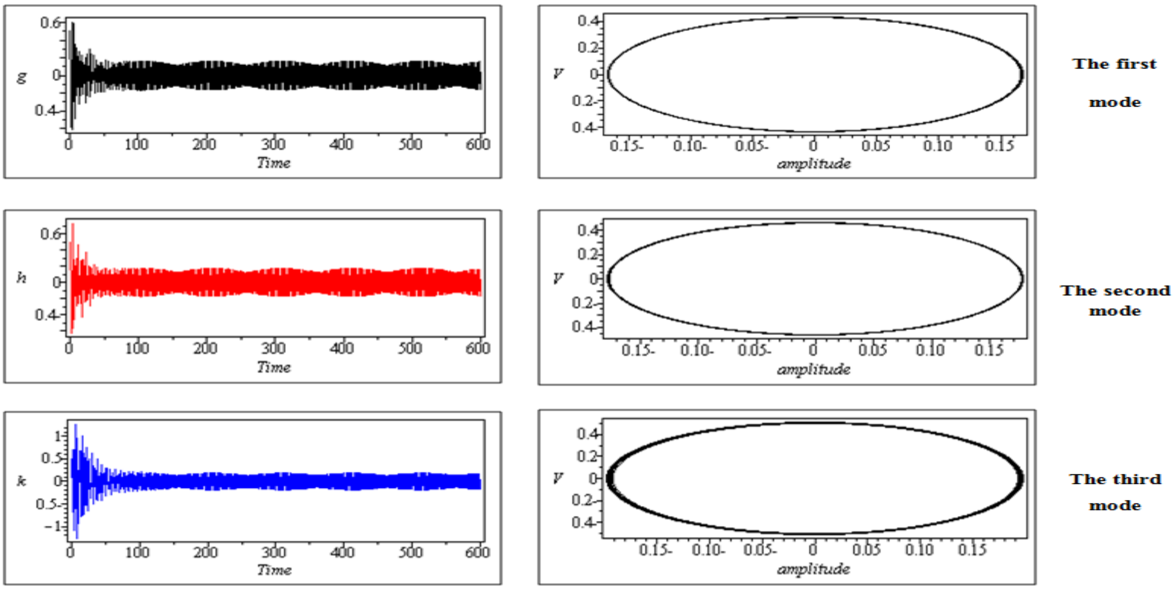

Figure 2(c). The primary resonance condition $\Omega = \omega_3 = 2.6$

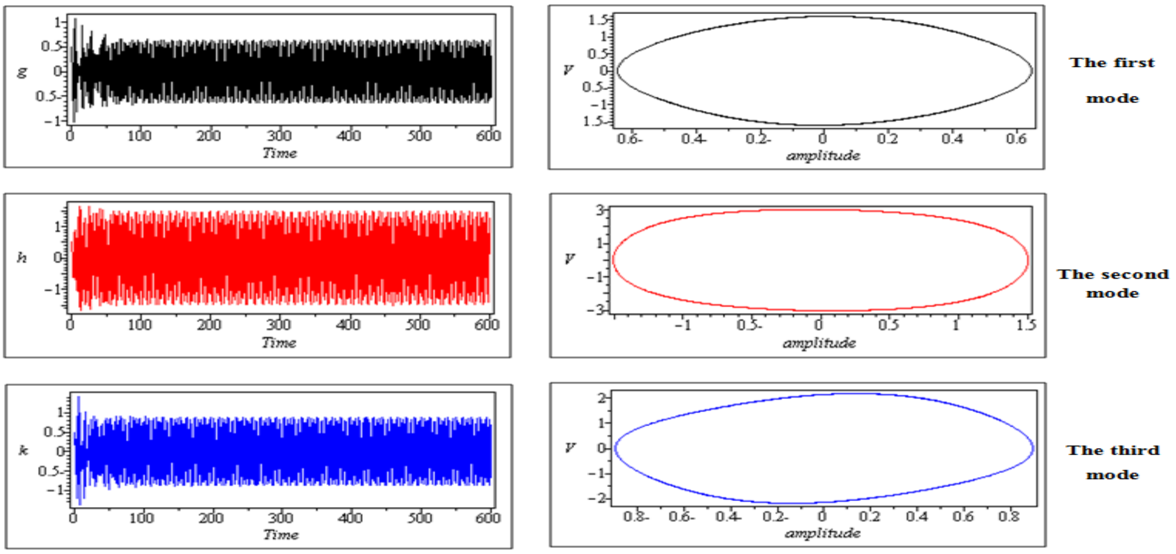

Figure 2(d). The Simultaneous resonance condition $\Omega = \omega_1 = \omega_2 = \omega_3 = 2.2$

(a) Internal resonance cases

$$(\omega_1 = \omega_3), (150\%, \text{None}, \text{None}), \text{ Figure 2(a)}$$

$$(\omega_1 = \omega_2 = \omega_3), (150\%, \text{None}, \text{None}), \text{Figure 2(b)}$$

(b) External resonance cases

(1) primary resonance:

$$(\Omega = \omega_3), (150\%, 150\%, 160\%), \text{ Figure 2(c).}$$

(2) Simultaneous resonances

$$(\Omega = \omega_1 = \omega_2 = \omega_3), (250\%, 250\%, 160\%), \text{ Figure 2(d).}$$

5.2. Theoretical Frequency Response Solution

The numerical results are presented graphically in Figs. (3-5) as the amplitudes a_1, a_2, a_3 against the detuning parameters $\sigma_1, \sigma_2, \sigma_3$ for different values of other parameters. Each curve in these figures consists of two branches. Considering Figure 3(a) as basic case to compare with, it can be seen from Figure 3(b), (c) that the steady-state amplitude a_1 decreases as each of α, η_3, ω_1 are increased but in Figure 3(e) the steady- state amplitude a_1 increases as each of F increases .Whereas the frequency response curves in Figure 3(h) are shifted to the right as σ_2 increases.

Considering Figure 4(a) as basic case to compare with, it can be seen from Figure 4(c) that the steady-state amplitude a_2 increases as each of F are increasing. But in Figures 4(b),(c), (d) and(i), the steady- state amplitude a_2 decreases as each of $a_1, a_3, \beta, \omega_2$ are increased. In Figure 4(h) the curves are shifted to the right as σ_1 increases. Whereas, the frequency response curve are bent to right as λ_3 varies from negative to positive values, showing hardening nonlinearity effect, Figure 4(f), (g).

Considering Figure 5(a) as basic case to compare with, it can be seen From Figures 5(b),(c) that the steady-state amplitude a_3 increases as each of a_1, τ_2, F are increased. The nonlinearity effect of τ_5 is shown in Figure 5(f), (g), whereas the curves are being shifted in Figure 5(d).

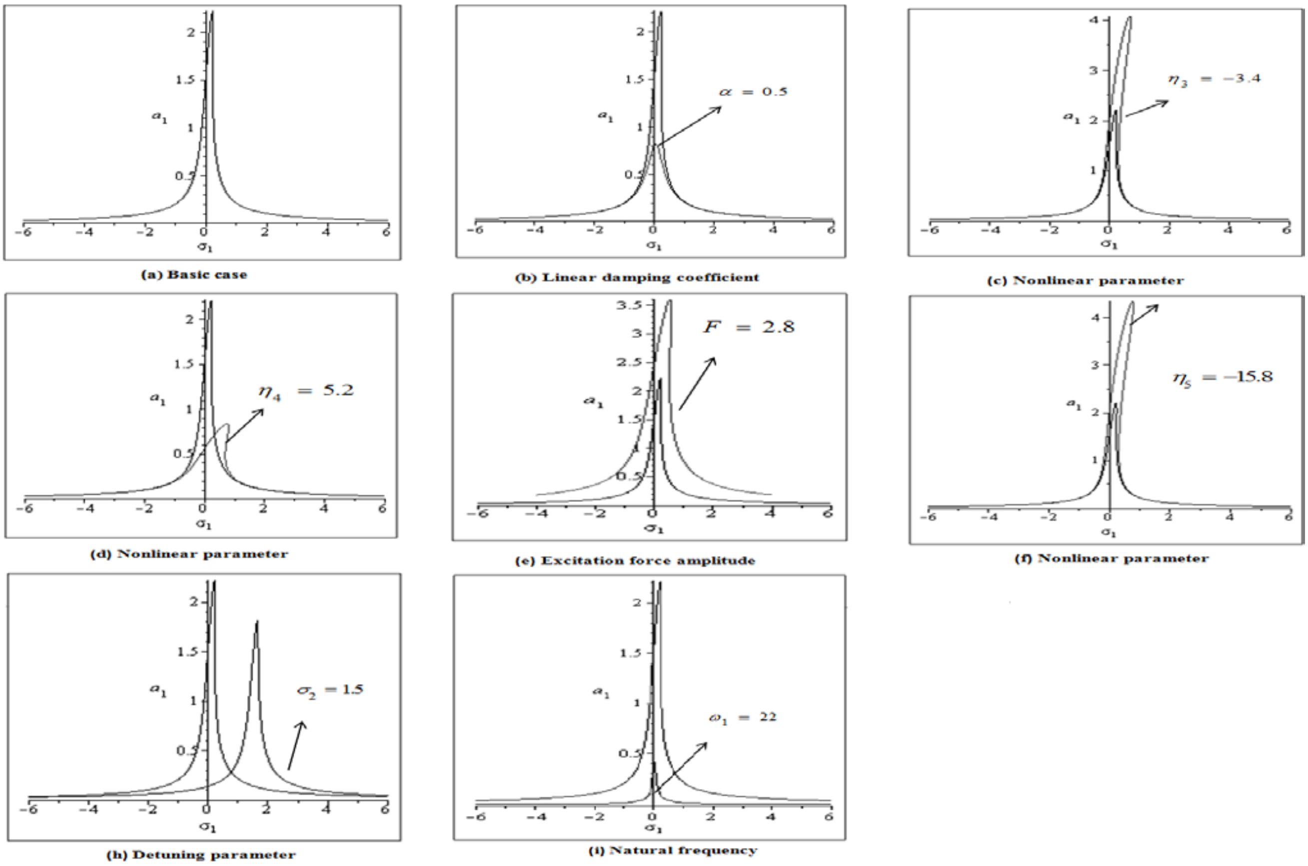

Figure 3. Frequency response curves of the first mode of the system at resonance $\eta_1 = 0.5, \eta_2 = 0.6, \eta_3 = 0.4, \eta_4 = 0.2, \eta_5 = 0.8$, $a_2 = 0.03$, $a_3 = 0.05, \sigma_2 = 0.02, \alpha = 0.08, F = 0.8, \omega_1 = 1.9$

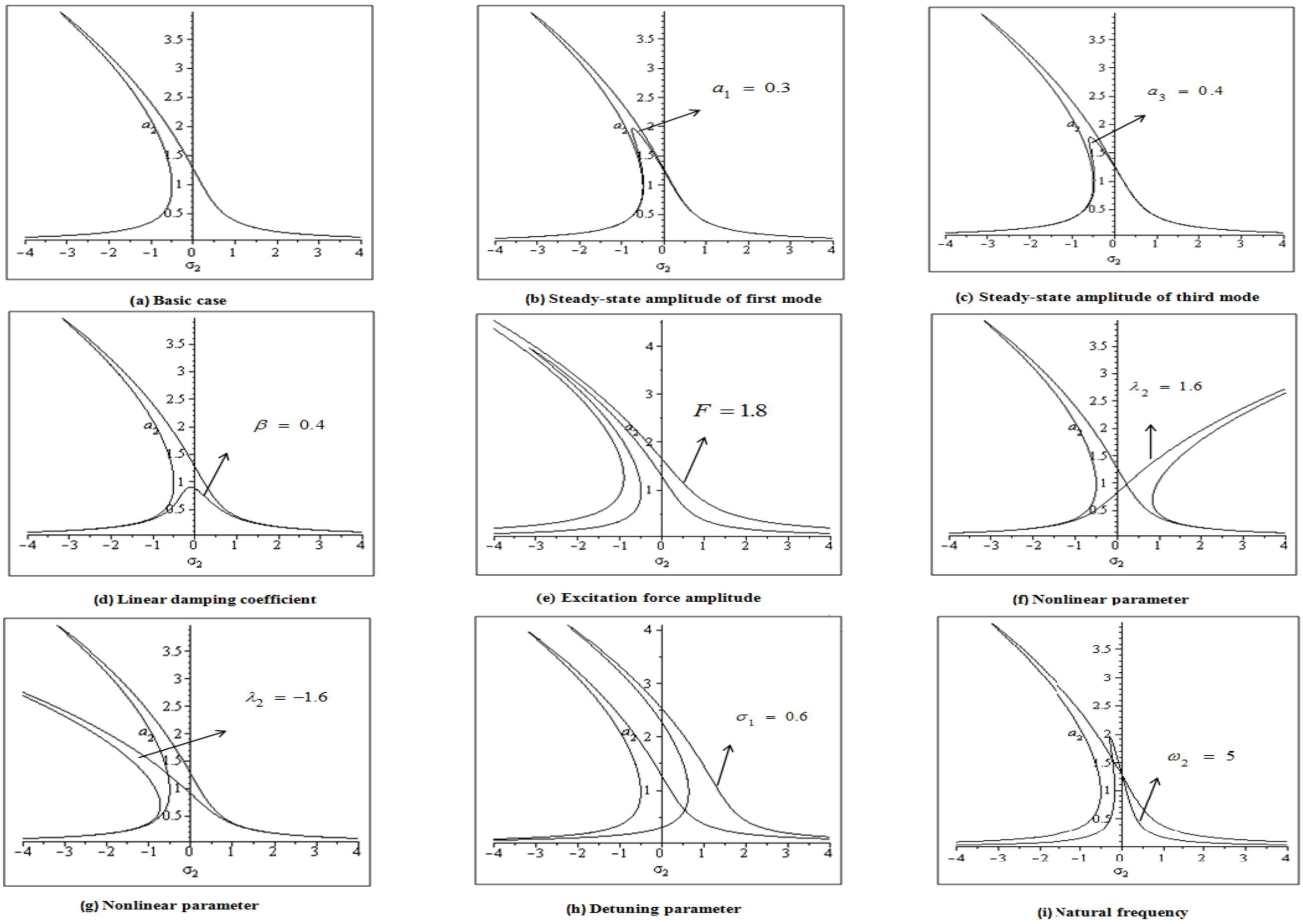

Figure 4. Frequency response curves of the second mode of the system at resonance $a_3 = 0.01, \sigma_1 = 0.03, \beta = 0.08, F = 0.8, \omega_2 = 2.2$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6, \lambda_3 = 0.4, \lambda_4 = 0.8, a_1 = 0.04$

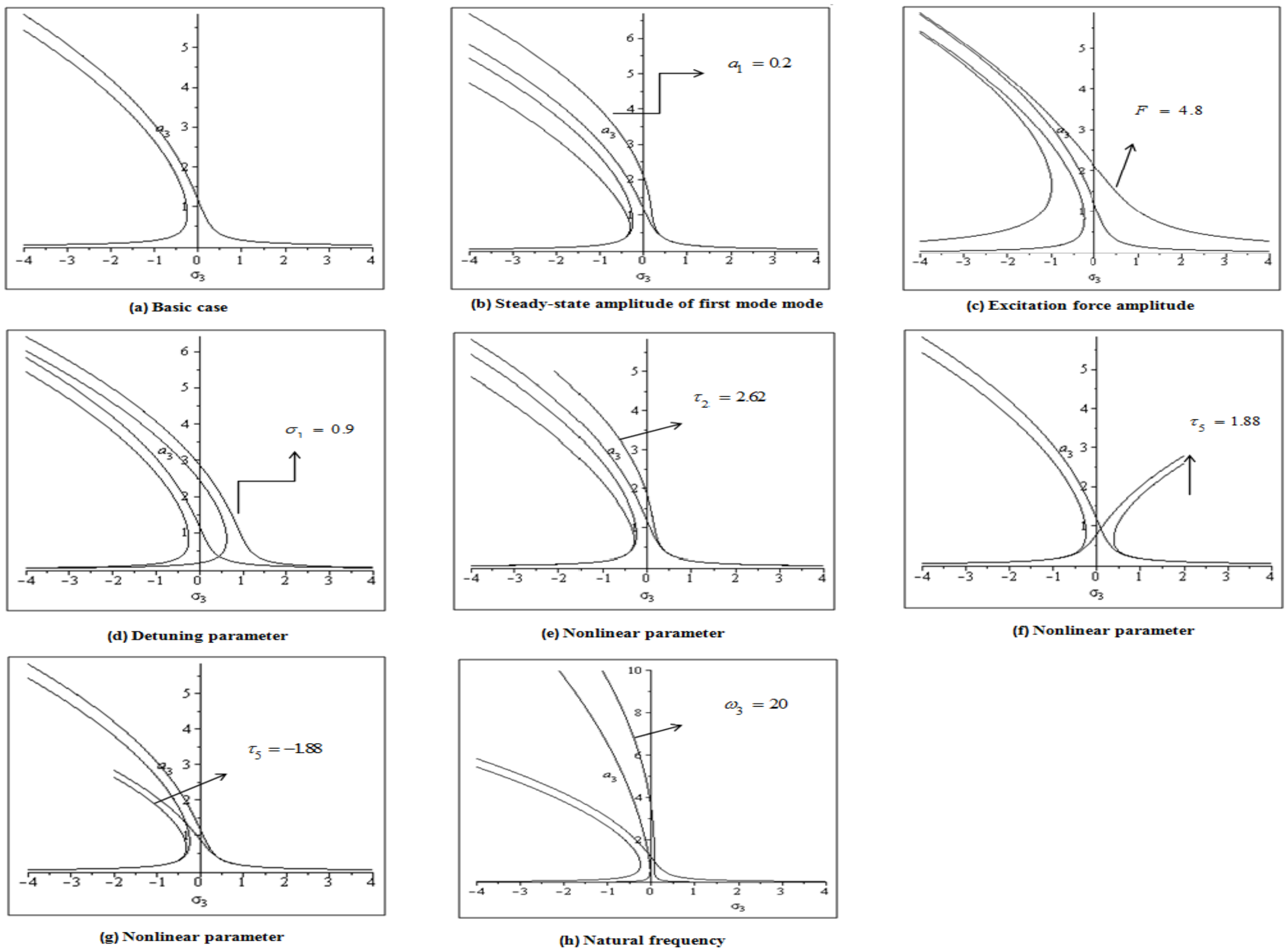

Figure 5. Frequency response curves of the third mode of the system at resonance $a_1 = 0.04, a_2 = 0.03, \sigma_1 = 0.03, \delta = 0.08, F = 0.8, \omega_3 = 2.6$, $\tau_1 = 0.55, \tau_2 = 0.62, \tau_3 = 0.44, \tau_4 = 0.22, \tau_5 = 0.88$

6. Conclusions

We have studied the analytic and numerical solutions of three dimensional nonlinear differential equations that describe the oscillations of abeam subjected to external forces. The multiple scales method and Runge-Kutta fourth order numerical method are utilized to investigate the system behavior and its stability .All possible resonance cases were be extracted and effect of different parameters on system behavior at resonant condition were studied. We may conclude the following:

(1) The steady-state amplitude of the first mode increases as each of the external force amplitude F and nonlinear coefficients η_3, η_5 are increased.

(2) The steady-state amplitude of the first mode decreases as each of the linear damping coefficient α and the nonlinear coefficient η_4 and the natural frequency ω_1 are increased.

(3) The steady-state amplitude of the second and third mode increase as the external force amplitude F increases.

(4) The steady-state amplitude of the second mode decreases as each of the linear damping coefficient β and nonlinear coefficient λ_2 and the second mode amplitude a_2 and the natural frequency ω_2 are increased.

(5) The steady-state amplitude of the third mode increases as each of the external force amplitude F and the first mode amplitude a_1 are increased.

Nomenclature

$\omega_1, \omega_2, \omega_3$	Natural frequencies of the system
ε	Small dimensionless perturbation parameter
α, β, δ	Linear damping coefficients
$\eta_i, \lambda_i, \tau_i$	Nonlinear parameters
F	Excitation force amplitude
Ω	Excitation frequency
D_0, D_1	Differential operators
r	Radius of gyration of cross-section area
m	Mass per unit length of beam
E	Young's modulus
I	Moment of inertia
L	Length of beam
t	Time
p_0, p_1	Real coefficients
T_0	Fast time scale
T_1	Slow time scale
$t^*, g^*, h^*, k^*, \Omega^*$	Non- dimensional quantities
$g_i, h_i, k_i (i = 0,1)$	Perturbation variables expansion

$\theta_1, \theta_2, \theta_3$	Phase angles of the polar forms
$A(T_1), B(T_1), C(T_1)$	Complex valued quantities
cc	Complex conjugate for preceding terms at the same equation
$\sigma_i, i=1,2,3$	Detuning parameter
a_1, a_2, a_3	Steady-state amplitudes

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Appendix

The coefficients presented in (35), are as follows :

$$\Lambda_1 = \frac{10}{64\omega_1^2}\eta_4^2, \Lambda_2 = \frac{3}{4\omega_1}\eta_4\sigma_2 + \frac{3}{16\omega_1^2}\eta_1\eta_4 a_2^2 + \frac{3}{16\omega_1^2}\eta_2\eta_4 a_3^2 - \frac{3}{4\omega_1}\eta_4\sigma_1 - \frac{1}{4\omega_1^2}\eta_2^2 a_3^2 - \frac{1}{64\omega_1^2}\eta_5^2 a_3^2 - \frac{1}{8\omega_1^2}\eta_2\eta_5 a_3^2,$$

$$\Lambda_3 = \frac{1}{16\omega_1^2}\eta_1^2 a_2^4 + \frac{1}{16\omega_1^2}\eta_2^2 a_3^4 + \frac{1}{8\omega_1^2}\eta_1\eta_2 a_2^2 a_3^2 + \frac{1}{2\omega_1}\eta_1\sigma_2 a_2^2 - \frac{1}{2\omega_1}\eta_1\sigma_1 a_2^2 - \frac{1}{2\omega_1}\eta_1\sigma_1 a_3^2 + \frac{1}{2\omega_1}\eta_2\sigma_2 a_3^2$$

$$-2\sigma_1\sigma_2 + \sigma_1^2 + \sigma_2^2 + \frac{1}{4}\alpha^2 + \frac{1}{16\omega_1^2}\eta_3\eta_5 a_2^2 a_3^2 + \frac{1}{4\omega_1^2}\eta_3^2 a_2^2 a_3^2 + \frac{1}{2\omega_1^2}\eta_3 a_3 F + \frac{1}{8\omega_1^2}\eta_5 a_3 F,$$

$$\Lambda_4 = -\frac{1}{16\omega_1^2}\eta_3^2 a_2^4 a_3^2 - \frac{3}{2\omega_1^2}\eta_3 a_3 a_2^2 F - \frac{1}{4\omega_1^2}F^2.$$

The coefficients presented in (36), are as follows:

$$\Lambda_5 = \frac{9}{64\omega_2^2}\lambda_2^2, \Lambda_6 = -\frac{3}{8\omega_2}\lambda_2\sigma_3 + \frac{3}{16\omega_2^2}\lambda_2\lambda_3 a_3^2 - \frac{3}{8\omega_2}\lambda_2\sigma_2 + \frac{3}{16\omega_2^2}\lambda_2\lambda_4 a_1^2,$$

$$\Lambda_7 = \frac{1}{8\omega_2^2}\lambda_3\lambda_4 a_1^2 a_3^2 - \frac{1}{4\omega_2}\sigma_3\lambda_3 a_3^2 + \frac{1}{2}\sigma_2\sigma_3 + \frac{1}{16\omega_2^2}\lambda_3^2 a_3^4 + \frac{3}{64\omega_2^2}\lambda_4^2 a_1^4 - \frac{1}{4\omega_2}\sigma_3\lambda_4 a_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{4}\sigma_3^2$$

$$-\frac{1}{4\omega_2}\sigma_2\lambda_3 a_3^2 - \frac{1}{4\omega_2}\sigma_2\lambda_4 a_1^2 + \frac{1}{4}\beta^2 - \frac{1}{16\omega_2^2}\lambda_1^2 a_1^2 a_3^2 - \frac{1}{16\omega_2^2}\lambda_1^2 a_1^2 a_3,$$

$$\Lambda_8 = \frac{2}{8\omega_2^2}\lambda_1 a_1 a_3 F + \frac{1}{8\omega_2^2}\lambda_4 a_1^2 F, \Lambda_9 = -\frac{1}{4\omega_2^2}F^2.$$

The coefficients presented in (37), are as follows :

$$\Lambda_{10} = \frac{9}{64\omega_3^2}\tau_5^2, \Lambda_{11} = \frac{3}{4\omega_3}\sigma_3\tau_5 - \frac{3}{4\omega_3}\sigma_1\tau_5 + \frac{3}{16\omega_3^2}\tau_1\tau_5 a_2^2 - 4\tau_2^2 a_1^2 - \frac{1}{16\omega_3^2}\tau_2^2 a_1^2,$$

$$\Lambda_{12} = -\frac{1}{16\omega_3^2}\tau_1\tau_2 a_1 a_2^2, \Lambda_{13} = \sigma_1^2 - 2\sigma_1\sigma_3 - \frac{1}{2\omega_3}\sigma_1\tau_1 a_2^2 + \frac{3}{16\omega_3^2}\tau_1^2 a_2^4 + \sigma_3^2 + \frac{1}{2\omega_3}\sigma_3\tau_1 a_2^2$$

$$+\frac{1}{4\omega_3^2}\tau_2 a_1 F - \frac{3}{16\omega_3^2}\tau_2\tau_4 a_1^4 - \frac{1}{8\omega_3^2}\tau_2\tau_3 a_1^2 a_2^2,$$

$$\Lambda_{14} = \frac{1}{8\omega_3^2}\tau_1 a_2^2 F - \frac{3}{32\omega_3^2}\tau_1\tau_4 a_1^3 a_2^2 - \frac{1}{16\omega_3^2}\tau_1\tau_3 a_1 a_2^4,$$

$$\Lambda_{15} = \frac{1}{4\omega_3^2}\tau_3 a_1 a_2^2 F + \frac{6}{16\omega_3^2}\tau_4 a_1^3 F - \frac{1}{4\omega_3^2}F^2 - \frac{9}{64\omega_3^2}\tau_4^2 a_1^6 - \frac{3}{16\omega_3^2}\tau_3\tau_4 a_1^4 a_2^2 - \frac{1}{16\omega_3^2}\tau_3^2 a_1^2 a_2^4.$$