

On the Incompressible Fluid Flow over the Prismatic Bodies

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Abstract In the paper the unsteady incompressible fluid flow over the infinite and finite prismatic bodies is studied. Mathematically this problem is modeled as 3D Navier-Stokes equations (NSE) for the fluid velocity components with the appropriate initial-boundary conditions. The study of the fluid flow over the bodies with the sharp edges is the important problem of Aerodynamics and Hydrodynamics. We admit that near the sharp edges the velocity components are non-smooth. By the methods of mathematical physics the bounded novel exact solutions are obtained for the specific pressure. The profile of the velocity is plotted by means of “Maplesoft”.

Keywords: *incompressible fluid flows, Navier-stokes equations, prismatic bodies*

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1. Introduction

We study the incompressible viscous fluid flow over a prism and between two finite or infinite similar prisms. Besides, the fluid flow between two similar octahedrons is considered. This process is described by the 3D Navier-Stokes Equations (NSE) with the appropriate initial-boundary conditions. We admit that solutions can be non-smooth near sharp edges. Our goal is to obtain exact solutions of NSE in case of the specific pressure. There is no doubt how important it is to find exact solutions for the non-linear equations. Besides, they serve as tests for the numerical analysis.

Navier-Stokes equations describe a wide range of physical phenomena and were studied by numerous authors [1-22]. Many open problems are connected with NSE. The existence of smooth solutions in 3D is still the open problem [9].

NSE with the equations of heat and mass transfer plays an important role in the theory of micropolar fluids [23-36]. In some specific cases it is possible to obtain exact solutions of NSE [1-22,30-89].

Exact solutions for micropolar fluids were obtained by Crane (1970) [30]; Ahmadi, (1976) [31]; Miklavc̃ic̃, Wang, (2006), [32]; Hauat, Javed, Sajid, [33]; Fang, and Zhang (2009) [34]; Noor, Kechil, Hashim (2010), [35]; Khan, Qasim, Haq, Al-Mdallal (2017) [36].

Here we consider the exact solutions for incompressible viscous fluids and do not discuss the exact solutions for micropolar and Magnetohydrodynamic fluids (MHD fluids) i.e. the NSE with the Maxwell equation is not considered.

The full survey of all known exact solutions of XIX and XX centuries is given in the book of Drazin, P.G. and Riley, N. [44]. Most of the new results are given in the works [30-36,51,52,63,71,72,81,85-89].

Let us recall some of the cases of incompressible NSE for which exact solutions are known:

- Stokes flow over the sphere with the specific pressure fall (Stokes) (1840) [1-22,44,75]

-fluid flow in pipes of the circular, elliptical, rectangular, triangular cross-sections (Poiseuille (1840), Boussinesq (1868)) [1-22,38,44,60]

- the stationary fluid flow between two parallel moving walls for the constant pressure fall (Couette (1890), Proudman (1914)) [1-22,42,44,62,75]

- Rayleigh (1911) derived the solutions for the impulsive motion of fluid over a flat plate [67]

- the steady fluid flow in a half plane (Riabouchinsky (1926)) [44,68]

- Von Karman (1921) has obtained the solution of NSE for the swirling flow in terms of cylindrical coordinates [44,49]

-the fluid flow in the neighborhood of the stagnation point of a rotating circular cylinder placed in a uniform stream is described in (Hiemens (1911) and Rott (1956)) [44,70,78]

- solutions for the Jeffery-Hamel (1915) fluid flow are given in 2D in terms of an elliptic functions [44,47]

- solutions for the Beltrami flow were obtained by Taylor (1923) [79].

- the exact solution in 2D was obtained by Tsien (1943) for the specific pressure [44,82]

- Landay (1943) and Squire (1951) derived exact solutions for a round submerged jet from a point source into an infinite fluid medium of the same kind [44,56,74]

- exact vertical solutions with the Burgers vortex in R^3 (Burgers 1948) are given in [39]
- exact solutions for the flow between homofocal ellipses and non-concentric circles are given in (Berker (1963), [35,44])
- Wang (1990) derived the solutions in terms of stream function for a share flow over convection cells [44,86,87]
- Polyanin (2001) has found exact solutions of incompressible NSE for the specific pressure by means of the generalized separation of variables [61]
- Tsangaris et al. (2006) derived the exact solution of the Navier-Stokes equations for the pulsating Dean flow in a channel with porous walls [81]
- Siddiqui and Iftikhar (2012) have found exact solutions for the unsteady flow of viscous fluid flow between two oscillating cylinders [73]
- Khatiashvili et al. (2013) derived exact solutions for the axi-symmetric Stokes flow over ellipsoidal bodies in the infinite channel [51,52]
- Prosviryakov (2019) obtained exact solutions of NSE in case of quadratic dependence of the velocity on two horizontal coordinates for rotating liquids [63]
- Zubarev and Prosviryakov (2019) have considered 3D unsteady incompressible fluid flow at a constant pressure and obtained polynomial and spatially localized self similar exact solutions [89]
- Sheng Yin Cheng and Falin Chen (2020) obtained exact solutions for the flow impinging obliquely on a moving flat plate [59]
- Khatiashvili (2021) has obtained non-smooth solutions for 3D fluid flow over an octahedron [53]

Here we present vortex solutions obtained in 1937 by Taylor and Green [44,77,79,80]

$$\begin{aligned}
 u &= A \cos ax \sin by \sin cz, \\
 v &= B \sin ax \cos by \sin cz, \\
 w &= C \sin ax \sin by \cos cz,
 \end{aligned}$$

where $A a + B b + C c = 0$, A, a, B, b, C, c are some constants, u, v, w are the velocity components.

In the paper we study the incompressible viscous fluid flow which is described by the dimensionless NSE

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}, \nabla \right) \vec{V} = \vec{F} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \Delta \vec{V}, \quad (\text{NSE})$$

where $\vec{V} (u, v, w)$ is the velocity, $\vec{F} (F_x, F_y, F_z)$ is the body force, p is the pressure, ρ is the density, ν is the kinematic viscosity of the fluid.

NSE is studied in the domain $D_t = D \times [0, T]$; $0 \leq t \leq T; T > 0$; (t is the time), where D is the infinite or finite area of R^3 space with the boundary S .

In our paper NSE is studied with the specific pressure and specific initial-boundary conditions, D is R^3 cut along the infinite or finite rectangular prism or the area between two similar rectangular prisms. Besides we consider the case when D is the area between two similar octahedrons.

Our goal is to obtain a new type of exact non-smooth solutions of NSE.

The results can be applied to aircrafts, submarines, underwater constructions.

2. Statement of the Problem

Let us suppose that the body force has some potential i.e. there exists the function Φ for which $\vec{F} = \text{grad } \Phi$ and rewrite the Navier-Stokes equations (NSE) in terms of velocity components

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \Delta u, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \Delta v, \quad (2)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \Delta w, \quad (3)$$

where $P = p + p_0 - \rho \Phi$ is the dynamical pressure, p_0 is the certain constant.

We consider the system (1), (2), (3) with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (4)$$

The system (1), (2), (3), (4) is a basic set of equations governing the flow of an incompressible fluid. We consider this system in the area D_t with the initial-boundary conditions

$$\begin{aligned}
 u(x, y, z, 0) &= u_0(x, y, z), \quad v(x, y, z, 0) = v_0(x, y, z), \\
 w(x, y, z, 0) &= w_0(x, y, z),
 \end{aligned} \quad (5)$$

$$u|_S = v|_S = w|_S = 0, \quad (6)$$

where $u_0(x, y, z), v_0(x, y, z), w_0(x, y, z)$ are some bounded functions.

We suppose that the pressure P has continuous derivatives of the first order except for some planes and the velocity components (u, v, w) are also non-smooth functions, i.e. we admit that if the body has sharp edges the first order derivatives of the velocity has discontinuities near these edges.

We have to solve the following problem.

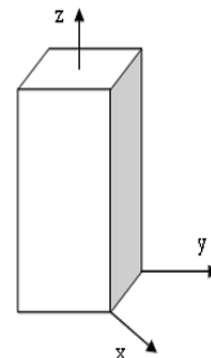


Figure 1. The prism with the axis of symmetry Oz in the Cartesian coordinates system $Oxyz$

Problem 1. In the area D_t to find bounded functions (u, v, w) having continuous derivatives of the first order except for some planes, and satisfying the system (1), (2), (3), (4) with the conditions (5), (6).

3. Solutions of Problem 1. In Case of the Single Prism

Here we consider two cases

1). We suppose, $D = R^3 \setminus D_1$ (D_1 is the infinite prism with the surface $\alpha|x| + \alpha|y| = B, B \geq 0, B$ is some given constant--Figure 1). In case of $B = 0$, D_1 is the axis $Oz: \{x = 0, y = 0\}$.

In 2013 we obtained a new class of solitary waves with the modulus [54,55]

$$r = R \sin \exp(-\alpha|x| - \beta|y| - \gamma|z| - D) + \sum_k R_k \exp(-\alpha_k|x| - \beta_k|y| - \gamma_k|z| - 2D),$$

where $R, A_0, \alpha, \beta, \gamma, \alpha_k, \beta_k, \gamma_k, R_k, D, (D > 0); k = 1, 2, \dots, n;$ are some parameters, $\exp(-5D)$ is negligible and $\alpha^2 + \beta^2 + \gamma^2 = A_0$.

According to this in the present paper we combine the functions $\exp(-\alpha|x| - \alpha|y| + B), \operatorname{sgn} x, \operatorname{sgn} y$ and in the area $D_t = D \times [0, T]$ consider the following representations

$$u_1 = \operatorname{sgn} x (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \quad (7)$$

$$v_1 = -\operatorname{sgn} y (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \quad (8)$$

$$w_1 = R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t), \quad (9)$$

R_1, α, γ, B are some real constants, $\alpha > 0, t$ is a time $0 \leq t \leq T$.

The functions (7), (8), (9) satisfy the initial-boundary conditions

$$u_1(x, y, z, 0) = \operatorname{sgn} x R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1), \quad (10)$$

$$v_1(x, y, z, 0) = -\operatorname{sgn} y R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1), \quad (11)$$

$$w_1(x, y, z, 0) = R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1), \quad (12)$$

$$u_1|_S = v_1|_S = w_1|_S = 0. \quad (13)$$

If $\gamma = 2\alpha^2\nu$,

$$u_0(x, y, z) = u_1(x, y, z, 0),$$

$$v_0(x, y, z) = v_1(x, y, z, 0),$$

$$w_0(x, y, z) = w_1(x, y, z, 0)$$

and the dynamical pressure is represented as

$$P = p_0 + R_1 \rho \gamma (|x| - |y| + z) \exp(\gamma t) + A_1(t), \quad (14)$$

$A_1(t)$ is the continuous function of time, then the functions u_1, v_1, w_1 given by formulas (7), (8), (9) are the solutions of Problem 1.

The velocity modulus will be given by the formula

$$\left| \vec{V} \right| = \sqrt{3} R_1 |\exp(-\alpha|x| - \alpha|y| + B + \gamma t) - \exp(\gamma t)|. \quad (15)$$

The vortex $\vec{\Omega}(\Omega_1, \Omega_2, \Omega_3)$ in the flow will be defined by the formula [1-22]

$$\begin{aligned} \Omega_1 &= \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \\ \Omega_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \\ \Omega_3 &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \end{aligned} \quad (16)$$

According to (7), (8), (9), (16) the components of the vortex in our case will be given by the formulas

$$\Omega_1 = -\alpha R_1 \operatorname{sgn} y \exp(-\alpha|x| - \alpha|y| + B + \gamma t),$$

$$\Omega_2 = \alpha R_1 \operatorname{sgn} x \exp(-\alpha|x| - \alpha|y| + B + \gamma t),$$

$$\Omega_3 = 2\alpha R_1 \operatorname{sgn} y \operatorname{sgn} x \exp(-\alpha|x| - \alpha|y| + B + \gamma t).$$

2) Now let us consider the case, when the prism D_1 moves along the axis Oz vertically with the speed $B_0(t)$, where $B_0(t)$ is some continuous function of time. In this case if the pressure is of the form

$$P = p_0 + R_1 \rho \gamma (|x| - |y|) \exp(\gamma t) - B_0'(t) z + A_1(t),$$

and $\gamma = 2\alpha^2\nu$, the solutions of the system (1), (2), (3), (4) will be given by

$$u_0 = \operatorname{sgn} x (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \quad (17)$$

$$v_0 = -\operatorname{sgn} y (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \quad (18)$$

$$w_0 = B_0(t). \quad (19)$$

The functions (17), (18), (19), satisfy the following initial-boundary conditions

$$u_0(x, y, z, 0) = \operatorname{sgn} x R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1),$$

$$v_0(x, y, z, 0) = -\operatorname{sgn} y R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1),$$

$$w_0(x, y, z, 0) = B_0(0),$$

$$u_0|_S = v_0|_S = 0,$$

$$w_0|_S = B_0(t).$$

Below (Figure 2 and Figure 3) the profile of the velocity given by the formula (15) is plotted for the data $\nu = 1; R_1 = 1; B = 1; \alpha = 1; t = 0$ and $\nu = 1; R_1 = 1; B = 1; \alpha = 1; t = 1$.

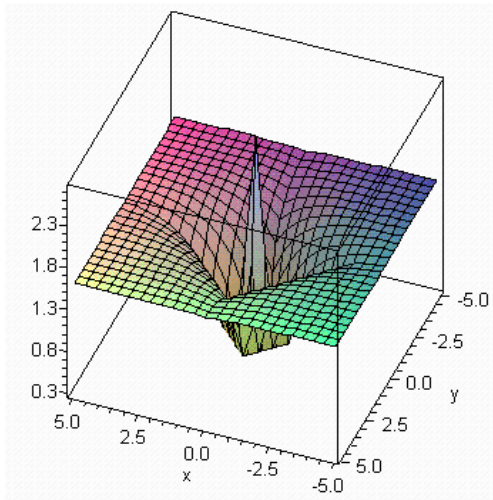


Figure 2. The profile of the velocity given by (15) in case of $v = 1; R_1=1; B=1; \alpha=1; t=0$

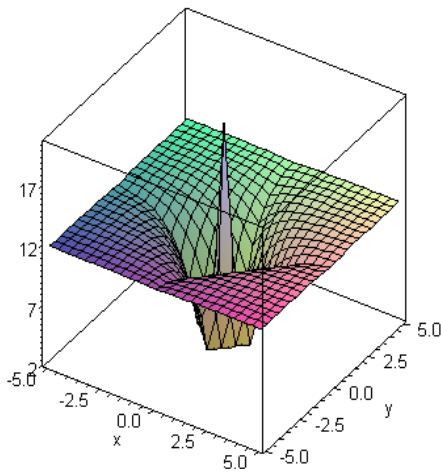


Figure 3. The profile of the velocity given by (15) in case of $v = 1; R_1=1; B=1; \alpha=1; t=1$

Remark 1. The functions (7), (8), (9) are also the solutions of NSE for the fluid flow inside the infinite prism.

Remark 2. In the author’s previous paper exact solutions for the fluid flow over the infinite prism was obtained in case of $\gamma = 2\alpha^2\nu$ and the pressure [53]

$$P = p_0 + R_1 \rho \gamma (|x| - |y|) \exp(\gamma t). \tag{20}$$

Those solutions are given by the formulas

$$\begin{aligned} u_1 &= \operatorname{sgn} x (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \\ v_1 &= -\operatorname{sgn} y (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \\ w_1 &= 0, \end{aligned}$$

and satisfy the initial-boundary conditions

$$\begin{aligned} u_1(x, y, z, 0) &= \operatorname{sgn} x R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1), \\ v_1(x, y, z, 0) &= -\operatorname{sgn} y R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1), \\ w_1(x, y, z, 0) &= 0. \end{aligned}$$

We note that in the paper [53] there was a misprint in the pressure representation - the pressure should be given by the formula (20).

Remark 3. Analogously, according to the previous results, we can obtain solutions of NSE in 2D for the fluid flow over the rectangle D_1^0 of xOy plane with the boundary $S_1^0 : \alpha|x| + \alpha|y| = B, B \geq 0;$

Let $D_0 = R^2 \setminus D_1^0$ be R^2 cut along the rectangle D_1^0 . In D_0 we consider 2D NSE with the equation of continuity for the velocity components (u, v)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \Delta u, \tag{21}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \Delta v, \tag{22}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{23}$$

with the initial-boundary conditions

$$u(x, y, 0) = u_0(x, y), v(x, y, 0) = v_0(x, y), \tag{24}$$

$$u \Big|_{S_1^0} = v \Big|_{S_1^0} = 0, \tag{25}$$

where $u_0(x, y), v_0(x, y)$ are some functions. It is obvious that if $\gamma = 2\alpha^2\nu$ and the pressure is given by the formula (14), than the functions

$$u_1^0 = \operatorname{sgn} x (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \tag{26}$$

$$v_1^0 = -\operatorname{sgn} y (R_1 \exp(-\alpha|x| - \alpha|y| + B + \gamma t) - R_1 \exp(\gamma t)), \tag{27}$$

are the solutions of the system (21), (22), (23) with the initial-boundary conditions

$$u_1^0(x, y, 0) = \operatorname{sgn} x R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1),$$

$$v_1^0(x, y, 0) = -\operatorname{sgn} y R_1 (\exp(-\alpha|x| - \alpha|y| + B) - 1),$$

$$u_1^0 \Big|_{S_1^0} = v_1^0 \Big|_{S_1^0} = 0.$$

According to (26) and (27) we can derive the stream function $\psi(x, y, t)$ by the formulas [1-22]

$$u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x}. \tag{28}$$

The stream function will be given by the formula

$$\begin{aligned} \psi(x, y, t) &= \frac{R_1}{\alpha} \operatorname{sgn} x \operatorname{sgn} y \exp(-\alpha|x| - \alpha|y| + B + \gamma t) \\ &\quad - y \operatorname{sgn} x R_1 \exp(\gamma t) - x \operatorname{sgn} y R_1 \exp(\gamma t) \\ &\quad + \operatorname{sgn} x \operatorname{sgn} y B_1(t), \end{aligned} \tag{29}$$

where $B_1(t)$ is some function of time. Below (Figure 4 and Figure 5) the profile of the stream lines (the border of the gray area) for the stream function given by the formula (29) are constructed by means of “Maple” for the data

$$v = 1; R_1 = 1; B = 1; B_1 = 5; \alpha = 1; t = 1.$$

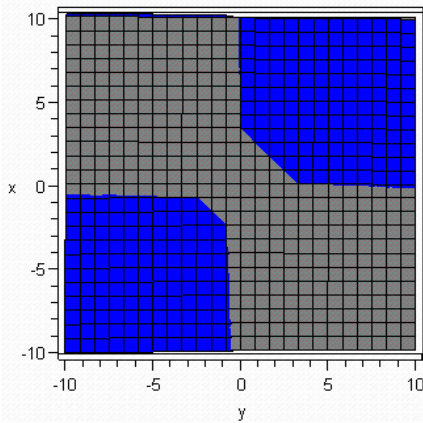


Figure 4. The stream lines in case of $\operatorname{sgn} x \operatorname{sgn} y \geq 0$; $v=1$; $R_1=1$; $B_1=5$; $\alpha=1$; $t=1$

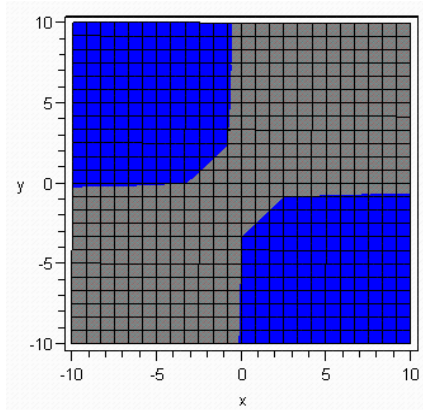


Figure 5. The stream lines in case of $\operatorname{sgn} x \operatorname{sgn} y < 0$; $v=1$; $R_1=1$; $B_1=5$; $\alpha=1$; $t=1$

4. Solutions of Problem 1 for the Fluid Flow Between two Infinite Similar Prisms

1). Now let us consider the fluid flow between two infinite similar rectangular prisms, i.e. in the area $D_2 : B \leq \alpha|x| + \alpha|y| \leq B + \pi, B \geq 0$. In this case we also consider the basic set of equations (1), (2), (3), (4) with the initial-boundary conditions (5), (6).

The function $\sin(-\alpha|x| - \alpha|y| + B)$ is the non-smooth solution of the Helmholtz equation [90]

$$\Delta U + 2\alpha^2 U = 0.$$

The functions

$$u_2 = R_2 \operatorname{sgn} x \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \quad (30)$$

$$v_2 = -R_2 \operatorname{sgn} y \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \quad (31)$$

$$w_2 = R_2 \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B). \quad (32)$$

satisfy the equations (1), (2), (3), (4) and the initial-boundary conditions

$$u_2(x, y, z, 0) = R_2 \operatorname{sgn} x \sin(-\alpha|x| - \alpha|y| + B), \quad (33)$$

$$v_2(x, y, z, 0) = -R_2 \operatorname{sgn} y \sin(-\alpha|x| - \alpha|y| + B), \quad (34)$$

$$w_2(x, y, z, 0) = R_2 \sin(-\alpha|x| - \alpha|y| + B), \quad (35)$$

$$u_2|_{S_2} = v_2|_{S_2} = w_2|_{S_2} = 0, \quad (36)$$

where S_2 is the boundary of D_2 , R_2 is some constant.

If $\gamma = -2\alpha^2 v$ and the dynamical pressure is represented as

$$P = A_1(t), \quad (37)$$

where $A_1(t)$ is the continuous function of time, then the functions u_2, v_2, w_2 given by the formulas (30), (31), (32) are solutions of Problem 1.

The velocity modulus will be given by

$$\left| \vec{V} \right| = \sqrt{3} R_2 \left| \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B) \right|. \quad (38)$$

Below (Figure 6 and Figure 7) the profile of the velocity given by the formula (38) is constructed for the data $v = 1; R_1 = 1; B = 1; \alpha = 1; t = 0$ and $v = 1; R_1 = 1; B = 1; \alpha = 1; t = 1$.

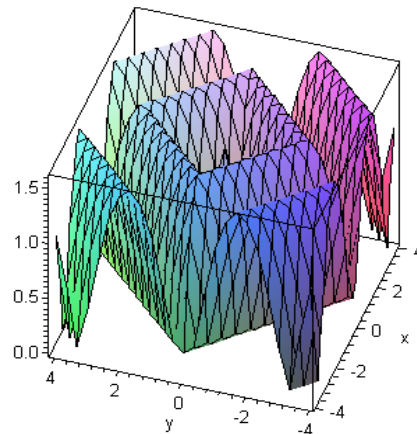


Figure 6. The profile of the velocity given by (38) in case of $v=1$; $R_1=1$; $B=1$; $\alpha=1$; $t=0$

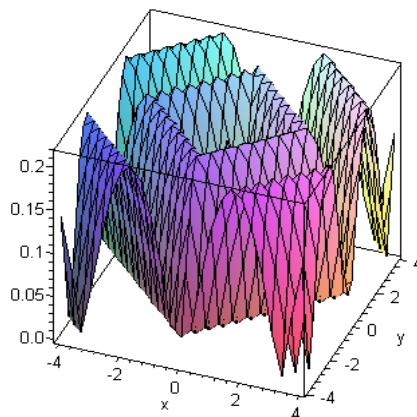


Figure 7. The profile of the velocity given by (38) in case of $v=1$; $R_1=1$; $B=1$; $\alpha=1$; $t=1$

Remark 4. The functions

$$\begin{aligned}
 u_2 &= R_2 \operatorname{sgn} x \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \\
 v_2 &= -R_2 \operatorname{sgn} y \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \\
 w_2 &= 0, \\
 u_2(x, y, z, 0) &= R_2 \operatorname{sgn} x \sin(-\alpha|x| - \alpha|y| + B), \\
 v_2(x, y, z, 0) &= -R_2 \operatorname{sgn} y \sin(-\alpha|x| - \alpha|y| + B), \\
 w_2(x, y, z, 0) &= 0,
 \end{aligned}$$

Are solutions of Problem 1. in the area D_2 with the initial-boundary conditions (33), (34), (35), (36) if $\gamma = -2\alpha^2\nu$ and the pressure P is given by the formula (37).

Remark 5. According to the previous results we can obtain the solutions of NSE in 2D for the fluid flow between two rectangles

$$D_2^0 : B \leq \alpha|x| + \alpha|y| \leq B + \pi, B \geq 0.$$

The functions

$$u_2^0 = R_2 \operatorname{sgn} x \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \quad (39)$$

$$v_2^0 = -R_2 \operatorname{sgn} y \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B), \quad (40)$$

satisfy the equations (21), (22), (23) and the initial-boundary conditions

$$u_2^0(x, y, 0) = R_2 \operatorname{sgn} x \sin(-\alpha|x| - \alpha|y| + B),$$

$$v_2^0(x, y, 0) = -R_2 \operatorname{sgn} y \sin(-\alpha|x| - \alpha|y| + B),$$

$$u_2^0 \Big|_{S_2^0} = v_2^0 \Big|_{S_2^0} = 0,$$

where S_2^0 is the boundary of D_2^0 . If $\gamma = -2\alpha^2\nu$, $u_0(x, y, z) = u_2^0(x, y, 0)$, $v_0(x, y) = v_2^0(x, y, 0)$, and the pressure is given by the formula $P = A_1(t)$, then the functions (39), (40), are the solutions of Problem 1 in 2D case.

Based on the formulas (28), (39), (40) the stream function is of the form

$$\begin{aligned}
 \psi(x, y, t) &= \frac{R_2}{\alpha} \operatorname{sgn} x \operatorname{sgn} y \exp(\gamma t) \\
 &\times \cos(-\alpha|x| - \alpha|y| + B) + \operatorname{sgn} x \operatorname{sgn} y B_1(t),
 \end{aligned} \quad (41)$$

where $B_1(t)$ is some function of time.

Below Figure 8 and Figure 9 the stream lines (the border of the gray area) for the stream function given by the formula (41) are plotted for the data $\nu = 1; R_2 = 1; B = 1; B_1 = 0.2; \alpha = 1; t = 0$ and $\nu = 1; R_2 = 1; B = 1; B_1 = 1; \alpha = 1; t = 1$

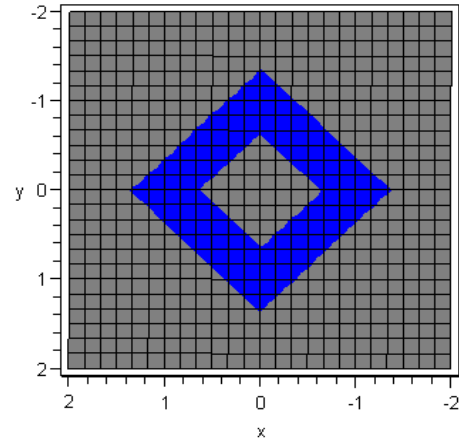


Figure 8. The stream line for the stream function given by (41) in case of $\nu=1; R_2=1; B_1=0.2; \alpha=1; t=0$

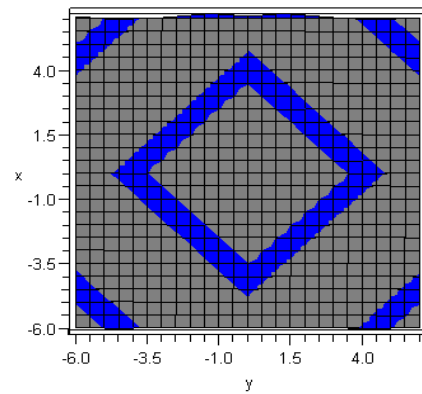


Figure 9. The stream line for the stream function given by (41) in case of $\nu=1; R_2=1; B_1=0.2; \alpha=1; t=1$

5. The Fluid Flow Between two Finite Similar Prisms and Similar Octahedrons

1) Let us consider the fluid flow between two finite similar rectangular prisms, i.e. in the area $D_1^* : B_1 \leq \alpha|x| + \alpha|y| \leq B_1 + \pi; 0 \leq z \leq \pi / \beta_1$, where $B_1 > 0$ and $\beta_1 > 0$ are some given constants. If $\gamma = -(2\alpha^2 + \beta^2)\nu$ and the dynamical pressure is given by the formula (28) the solutions of the system (1), (2), (3), (4) will be given by

$$u_3 = R_3 \operatorname{sgn} x \sin \beta z \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B_1), \quad (42)$$

$$v_3 = -R_3 \operatorname{sgn} y \sin \beta z \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| + B_1), \quad (43)$$

$$w_3 = 0, \quad (44)$$

where

$$u_0(x, y, z) = R_3 \operatorname{sgn} x \sin \beta z \sin(-\alpha|x| - \alpha|y| + B_1),$$

$$v_0(x, y, z) = -R_3 \operatorname{sgn} y \sin \beta z \sin(-\alpha|x| - \alpha|y| + B_1),$$

$$w_0(x, y, z) = 0,$$

$$u_3 \Big|_{S_1^*} = v_3 \Big|_{S_1^*} = w_3 \Big|_{S_1^*} = 0,$$

S_1^* is the boundary of D_1^* , R_3 is some given constant.

2) Now let us consider the fluid flow between two similar octahedrons, i.e. in the area

$$D_2^* : B_2 \leq \alpha|x| + \alpha|y| + \alpha|z| \leq B_2 + \pi,$$

where $B_2 > 0$ is some given constant.

If $\gamma = -3\alpha^2\nu$ and the dynamical pressure is given by the formula (37) the solutions of Problem 1 will be given by

$$u_4 = R_4 \operatorname{sgn} x \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| - \alpha|z| + B_2), \quad (45)$$

$$v_4 = -R_4 \operatorname{sgn} y \exp(\gamma t) \times \sin(-\alpha|x| - \alpha|y| - \alpha|z| + B_2), \quad (46)$$

$$w_4 = 0, \quad (47)$$

where

$$u_0(x, y, z) = R_4 \operatorname{sgn} x \sin(-\alpha|x| - \alpha|y| - \alpha|z| + B_2),$$

$$v_0(x, y, z) = -R_4 \operatorname{sgn} y \sin(-\alpha|x| - \alpha|y| - \alpha|z| + B_2),$$

$$w_0(x, y, z) = 0,$$

$$u_4 \Big|_{S_2^*} = v_4 \Big|_{S_2^*} = w_4 \Big|_{S_2^*} = 0,$$

S_2^* is the boundary of D_2^* , R_4 is some given constant.

Remark 6. The exact solutions of NSE for the incompressible fluid flow over the single octahedron

$$S^{**} : \alpha|x| + \alpha|y| + \alpha|z| = B^*, \quad B^* \geq 0,$$

was obtained by the author in [53] for the pressure which is represented by the formula (14). Those solutions are of the form

$$u = \operatorname{sgn} x (R_1 \exp(-\alpha|x| - \alpha|y| - \alpha|z| + B^* + \gamma t) - R_1 \exp(\gamma t)),$$

$$v = -\operatorname{sgn} y (R_1 \exp(-\alpha|x| - \alpha|y| - \alpha|z| + B^* + \gamma t) - R_1 \exp(\gamma t)),$$

$$w_1 = 0,$$

$$u_0(x, y, z, 0) = \operatorname{sgn} x R_1 (\exp(-\alpha|x| - \alpha|y| - \alpha|z| + B^*) - 1),$$

$$v_0(x, y, z, 0) = -\operatorname{sgn} y R_1 (\exp(-\alpha|x| - \alpha|y| - \alpha|z| + B^*) - 1),$$

$$w_0(x, y, z, 0) = 0; \quad u \Big|_{S^{**}} = v \Big|_{S^{**}} = w \Big|_{S^{**}} = 0.$$

Remark 7. Repeating octahedrons and tetrahedrons structures were constructed by Buckminster Fuller as a tensegrity structures (the strongest structures which resist cantilever stresses) [91].

6. Discussions

In this paper we do not discuss the uniqueness of the solution of Problem 1. It is still an open question.

In the future, we plan to obtain exact solutions for the incompressible fluid flows over the bodies with the more complicated configuration in R^2 and R^3 .

7. Conclusion

Hence, we obtain the new type of non-smooth exact solutions of 3D NSE for the incompressible fluid flow in R^3 over the infinite rectangular prism, between two infinite rectangular prisms, between two finite rectangular prisms and between two similar octahedrons:

1. If the pressure is represented by the formula (14), then the components of the velocity of the fluid flow over the infinite prism are given by the formulas (7), (8), (9).
2. If the pressure is represented by the formula (37), then the components of the velocity of the incompressible fluid flow between two infinite similar prisms are given by the formulas (30), (31), (32) and between two finite similar prisms are given by the formulas (42), (43), (44).

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