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Solving the Partial Differential Problems Using Differentiation Term by Term Theorem

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Abstract This paper took advantage of the mathematical software Maple as the auxiliary tool to study the partial differential problems of four types of two-variables functions. We can obtain the infinite series forms of any order partial derivatives of these two-variables functions by using differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order partial derivative values. On the other hand, we proposed some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying our answers by using Maple.

Keywords: *partial derivatives, infinite series forms, differentiation term by term theorem, Maple*

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1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics curricula, the study of Laplace equation, wave equation, and other important physical equations are involved the partial differentiation. So the evaluation and numerical calculation of the partial derivatives of multivariable functions are important. On the other hand, calculating the q-th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the q-th order partial derivative of this function, and then taking the point into the q-th order partial derivative. These two procedures will make us face with increasingly complex calculations when calculating higher order partial derivative values (i.e. q is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we studied the partial differential problem of the following four types of two-variables functions

$$f(x,y)=\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^p\times\frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x-\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y}{\cosh 2\alpha x+\cos 2\beta y}\quad(1)$$

$$g(x,y)=\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^p\times\frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y+\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x}{\cosh 2\alpha x+\cos 2\beta y}\quad(2)$$

$$u(x,y)=\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^p\times\frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x+\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y}{\cosh 2\alpha x-\cos 2\beta y}\quad(3)$$

$$w(x,y)=\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^p\times\frac{-\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y+\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x}{\cosh 2\alpha x-\cos 2\beta y}\quad(4)$$

where α, β are real numbers, $\beta \neq 0$, and p is an integer. We can obtain the infinite series forms of any order partial derivatives of these four types of two-variables functions using differentiation term by term theorem; these are the major results of this study (i.e., Theorems 1-4), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. The study of related partial differential problems can refer to [1-13]. In addition, we provided some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations, formulas and some Maple's commands used in this paper.

2.1. Notations

2.1.1. Let $z = a + ib$ be a complex number, where $i=\sqrt{-1}$, a, $b$are real numbers. We denote a the real part of z by Re(z), and b the imaginary part of z by Im(z).

2.1.2. Suppose m, n are non-negative integers. For the two-variables function $f(x, y)$, its n-times partial derivative with respect to x, and m-times partial derivative with respect to y, forms a $m+n$-th order partial derivative, and denoted by $\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y)$.

2.1.3. Suppose r is any real number, m is any positive integer. Define $(r)_m = r(r-1)\cdots(r-m+1)$, and $(r)_0 = 1$.

2.1.4. evalf ( ); the Maple's command of calculating the approximation.

2.1.5. sum ( ); the command of evaluating the summation.

2.1.6. D [1$3, 2$4] (f) (1, 2); the command of finding the 7-th order partial derivative of $f(x, y)$ at (1, 2), $\frac{\partial^7 f}{\partial y^4 \partial x^3}(1,2)$.

2.1.7. Bernoulli (n); the command of evaluating the n-th bernoulli number.

2.1.8. product (8-j, j = 0..6); the command of determining the product $8\times 7\times\cdots\times 2$.

2.2. Formulas

2.2.1. Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta$, where θ is any real number.

2.2.2. De Moivre's Formula

$(\cos\theta + i\sin\theta)^p = \cos p\theta + i\sin p\theta$, where p is any integer, and θ is any real number.

2.2.3. ([14])

$\sinh(a+ib) = \sinh a\cos b + i\cosh a\sin b$, where a, b are real numbers.

2.2.4. ([14])

$\cosh(a+ib) = \cosh a\cos b + i\sinh a\sin b$, where a, b are real numbers.

2.2.5. Taylor Series Expression of Complex Hyperbolic Tangent Function ([15])

$\tanh z = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}z^{2k-1}$, where z is a complex number, $|z| < \frac{\pi}{2}$, and B_k are the k-th Bernoulli number.

2.2.6. Taylor Series Expression of Complex Hyperbolic Cotangent Function ([15])

$\coth z = \frac{1}{z} + \sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{(2k)!}z^{2k-1}$, where z is a complex number, $0 < |z| < \pi$.

In the following, we introduce an important theorem used in this study.

2.3. Differentiation Term by Term Theorem ([16])

For all non-negative integers k, if the functions $g_k: (a, b) \to R$ satisfy the following three conditions : (i) there exists a point $x_0 \in (a,b)$ such that $\sum_{k=0}^{\infty} g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a, b), (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a, b). Then $\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent and differentiable on (a, b). Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

Before deriving the major results in this study, we need three lemmas.

2.4. Lemma 1

Suppose a, b are real numbers, $b > 0$, and p is an integer. Then

$$(a+ib)^p$$

$$= \left(\sqrt{a^2+b^2}\right)^p\left[\cos\left(p\cot^{-1}\frac{a}{b}\right) + i\sin\left(p\cot^{-1}\frac{a}{b}\right)\right] \tag{5}$$

$$= \left(\sqrt{a^2+b^2}\right)^p \exp\left(ip\cot^{-1}\frac{a}{b}\right) \tag{6}$$

Proof $(a+ib)^p$

$$= \left[\sqrt{a^2+b^2} \left(\frac{a}{\sqrt{a^2+b^2}} + i \frac{b}{\sqrt{a^2+b^2}} \right) \right]^p$$

$$= \left(\sqrt{a^2+b^2} \right)^p (\cos\theta + i\sin\theta)^p$$

(Where $\theta = \cot^{-1}\frac{a}{b}$)

$$= \left(\sqrt{a^2+b^2} \right)^p \left[\cos\left(p\cot^{-1}\frac{a}{b} \right) + i\sin\left(p\cot^{-1}\frac{a}{b} \right) \right]$$

(By DeMoivre's formula)

$$= \left(\sqrt{a^2+b^2} \right)^p \exp\left(ip\cot^{-1}\frac{a}{b} \right)$$

(By Euler's formula)

2.5. Lemma 2

Suppose a, b are real numbers with $\cosh 2a + \cos 2b \neq 0$. Then

$$\tanh(a+ib) = \frac{\sinh 2a + i\sin 2b}{\cosh 2a + \cos 2b} \quad (7)$$

Proof $\tanh(a+ib)$

$$= \frac{\sinh(a+ib)}{\cosh(a+ib)}$$

$$= \frac{\sinh a\cos b + i\cosh a\sin b}{\cosh a\cos b + i\sinh a\sin b}$$

(By Formulas 2.2.3 and 2.2.4)

$$= \frac{(\sinh a\cos b + i\cosh a\sin b)(\cosh a\cos b - i\sinh a\sin b)}{\cosh^2 a\cos^2 b + \sinh^2 a\sin^2 b}$$

$$= \frac{\sinh a\cosh b + i\sin b\cos b}{\sinh^2 a + \cos^2 b}$$

$$= \frac{\sinh 2a + i\sin 2b}{\cosh 2a + \cos 2b}$$

2.6. Lemma 3

Suppose a, b are real numbers with $\cosh 2a - \cos 2b \neq 0$. Then

$$\coth(a+ib) = \frac{\sinh 2a - i\sin 2b}{\cosh 2a - \cos 2b} \quad (8)$$

Proof $\coth(a+ib)$

$$= \frac{\cosh(a+ib)}{\sinh(a+ib)}$$

$$= \frac{\cosh a\cos b + i\sinh a\sin b}{\sinh a\cos b + i\cosh a\sin b}$$

$$= \frac{(\cosh a\cos b + i\sinh a\sin b)(\sinh a\cos b - i\cosh a\sin b)}{\sinh^2 a\cos^2 b + \cosh^2 a\sin^2 b}$$

$$= \frac{\sinh a\cosh a - i\sin b\cos b}{\cosh^2 a - \cos^2 b}$$

$$= \frac{\sinh 2a - i\sin 2b}{\cosh 2a - \cos 2b}$$

Next, we determine the infinite series forms of any order partial derivatives of the two-variables function (1).

2.7. Theorem 1

Suppose α, β are real numbers, $\beta \neq 0$, p is an integer, and m, n are non-negative integers. If the domain of the two-variables function

$$f(x,y) = \left(\sqrt{\alpha^2x^2+\beta^2y^2} \right)^p \times \frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y} \right)\sinh 2\alpha x - \sin\left(p\cot^{-1}\frac{\alpha x}{\beta y} \right)\sin 2\beta y}{\cosh 2\alpha x + \cos 2\beta y}$$

is $\left\{ (x,y)\in R^2 \,\middle|\, \beta y > 0, \alpha^2x^2+\beta^2y^2 < \frac{\pi^2}{4} \right\}$. Then the m+n-th order partial derivative of $f(x, y)$,

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y) = \alpha^n\beta^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot \left(\sqrt{\alpha^2x^2+\beta^2y^2} \right)^{2k+p-1-m-n} \times \cos\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2} \right] \quad (9)$$

Proof Let $z = \alpha x + i\beta y$, and $\beta y > 0$, $\alpha^2x^2+\beta^2y^2 < \frac{\pi^2}{4}$, then

$$z^p \tanh z$$

$$= (\alpha x + i\beta y)^p \tanh(\alpha x + i\beta y)$$

$$= \left(\sqrt{\alpha^2x^2+\beta^2y^2} \right)^p \left[\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y} \right) + i\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y} \right) \right] \times \frac{\sinh 2\alpha x + i\sin 2\beta y}{\cosh 2\alpha x + \cos 2\beta y} \quad (10)$$

(By (5) and (7))

Therefore,

$$f(x,y)$$

$$= \mathrm{Re}(z^p \tanh z)$$

$$= \mathrm{Re}\left[(\alpha x + i\beta y)^p \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} (\alpha x + i\beta y)^{2k-1} \right]$$

(By Formula 2.2.5)

$$= \mathrm{Re}\left[\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} (\alpha x + i\beta y)^{2k+p-1} \right] \quad (11)$$

Using differentiation term by term theorem, differentiating n-times with respect to x, and m-times with respect to y on both sides of (11), we obtain

$$\frac{\partial^{m+n} f}{\partial y^m \partial x^n}(x,y)$$

$$= \alpha^n \beta^m \operatorname{Re}\left[i^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot (\alpha x + i\beta y)^{2k+p-1-m-n} \right]$$

$$= \alpha^n \beta^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot \operatorname{Re}[i^m (\alpha x + i\beta y)^{2k+p-1-m-n}]$$

$$= \alpha^n \beta^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^{2k+p-1-m-n} \times \cos\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2}\right]$$

(By (6))

Next, we find the infinite series forms of any order partial derivatives of the two-variables function (2).

2.8. Theorem 2

If the assumptions are the same as Theorem 1. Suppose the domain of the two-variables function

$$g(x,y) = \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^p \times \frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y + \sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x}{\cosh 2\alpha x + \cos 2\beta y}$$

is $\left\{(x,y)\in R^2 \middle| \beta y > 0, \alpha^2 x^2 + \beta^2 y^2 < \frac{\pi^2}{4}\right\}$. Then

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y) = \alpha^n \beta^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^{2k+p-1-m-n} \times \sin\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2}\right] \quad (12)$$

Proof $g(x,y)$

$$= \operatorname{Im}(z^p \tanh z)$$

(By (10))

$$= \operatorname{Im}\left[\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}(\alpha x + i\beta y)^{2k+p-1}\right] \quad (13)$$

Thus, by differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y) = \alpha^n \beta^m \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!} \cdot \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^{2k+p-1-m-n} \times \sin\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2}\right]$$

In the following, we determine the infinite series forms of any order partial derivatives of the two-variables function (3).

2.9. Theorem 3

If the assumptions are the same as Theorem 1. Suppose the domain of the two-variables function

$$u(x,y) = \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^p \times \frac{\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x + \sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y}{\cosh 2\alpha x - \cos 2\beta y}$$

is $\left\{(x,y)\in R^2 \middle| \beta y > 0, \alpha^2 x^2 + \beta^2 y^2 < \pi^2\right\}$. Then

$$\frac{\partial^{m+n} u}{\partial y^m \partial x^n}(x,y) = \alpha^n \beta^m \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^{p-1-m-n} (p-1)_{m+n} \cdot \cos\left[(p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2}\right] + \alpha^n \beta^m \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}(2k+p-1)_{m+n}}{(2k)!}\left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^{2k+p-1-m-n} \times \cos\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y} + \frac{m\pi}{2}\right] \quad (14)$$

Proof Let $z = \alpha x + i\beta y$, then

$$z^p \coth z$$

$$= (\alpha x + i\beta y)^p \coth(\alpha x + i\beta y)$$

$$= \left(\sqrt{\alpha^2 x^2 + \beta^2 y^2}\right)^p \left[\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right) + i\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\right] \times \frac{\sinh 2\alpha x - i\sin 2\beta y}{\cosh 2\alpha x - \cos 2\beta y} \quad (15)$$

(By (5) and (8))

Hence,

$$u(x,y)$$

$$= \operatorname{Re}(z^p \coth z) \quad (16)$$

$$= \operatorname{Re}\left[(\alpha x + i\beta y)^{p-1} + \sum_{k=1}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}(\alpha x + i\beta y)^{2k+p-1}\right]$$

(By Formula 2.2.6)

Also, by differentiation term by term theorem, we have

$$\frac{\partial^{m+n}u}{\partial y^m \partial x^n}(x,y)$$
$$=\alpha^n\beta^m\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{p-1-m-n}(p-1)_{m+n}\cdot$$
$$\cos\left[(p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]+$$
$$\alpha^n\beta^m\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+p-1)_{m+n}}{(2k)!}\cdot$$
$$\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{2k+p-1-m-n}\times$$
$$\cos\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]$$

Finally, we obtain the infinite series forms of any order partial derivatives of the two-variables function (4).

2.10. Theorem 4

Let the assumptions be the same as Theorem 1. If the domain of the two-variables function

$$w(x,y)=\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^p\times$$
$$\frac{-\cos\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sin 2\beta y+\sin\left(p\cot^{-1}\frac{\alpha x}{\beta y}\right)\sinh 2\alpha x}{\cosh 2\alpha x-\cos 2\beta y}$$

is $\left\{(x,y)\in R^2 \middle| \beta y>0, \alpha^2x^2+\beta^2y^2<\pi^2\right\}$. Then

$$\frac{\partial^{m+n}w}{\partial y^m \partial x^n}(x,y)$$
$$=\alpha^n\beta^m\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{p-1-m-n}(p-1)_{m+n}\cdot$$
$$\sin\left[(p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]+$$
$$\alpha^n\beta^m\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+p-1)_{m+n}}{(2k)!}\cdot \tag{17}$$
$$\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{2k+p-1-m-n}\times$$
$$\sin\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]$$

Proof By (15), we have

$$w(x,y)$$
$$=\operatorname{Im}(z^p\coth z) \tag{18}$$
$$=\operatorname{Im}\left[(\alpha x+i\beta y)^{p-1}+\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{(2k)!}(\alpha x+i\beta y)^{2k+p-1}\right]$$

Using differentiation term by term theorem, we obtain

$$\frac{\partial^{m+n}w}{\partial y^m \partial x^n}(x,y)$$
$$=\alpha^n\beta^m\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{p-1-m-n}(p-1)_{m+n}\cdot$$
$$\sin\left[(p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]+$$
$$\alpha^n\beta^m\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+p-1)_{m+n}}{(2k)!}\cdot$$
$$\left(\sqrt{\alpha^2x^2+\beta^2y^2}\right)^{2k+p-1-m-n}\times$$
$$\sin\left[(2k+p-1-m-n)\cot^{-1}\frac{\alpha x}{\beta y}+\frac{m\pi}{2}\right]$$

3. Examples

In the following, for the partial differential problem of the four types of two-variables functions discussed in this study, we proposed four examples and use Theorems 1-4 to obtain the infinite series forms of any order partial derivatives of these functions, and evaluate some of their higher order partial derivative values. On the other hand, we employed Maple to calculate the approximations of these higher order partial derivative values and their solutions for verifying our answers.

3.1. Example 1

Suppose the domain of the two-variables function

$$f(x,y)=\left(\sqrt{25x^2+4y^2}\right)^5\times$$
$$\frac{\cos\left(5\cot^{-1}\frac{5x}{2y}\right)\sinh 10x-\sin\left(5\cot^{-1}\frac{5x}{2y}\right)\sin 4y}{\cosh 10x+\cos 4y} \tag{19}$$

is $\left\{(x,y)\in R^2 \middle| y>0, 25x^2+4y^2<\frac{\pi^2}{4}\right\}$. By (9), we obtain any $m+n$-th order partial derivative of $f(x,y)$,

$$\frac{\partial^{m+n}f}{\partial y^m \partial x^n}(x,y)$$
$$=5^n2^m\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}(2k+p-1)_{m+n}}{(2k)!}\cdot \tag{20}$$
$$\left(\sqrt{25x^2+4y^2}\right)^{2k+p-1-m-n}\times$$
$$\cos\left[(2k+p-1-m-n)\cot^{-1}\frac{5x}{2y}+\frac{m\pi}{2}\right]$$

Therefore, we can evaluate the 7-th order partial derivative value of $f(x,y)$ at $\left(\frac{1}{5},\frac{1}{2}\right)$,

$$\frac{\partial^7 f}{\partial y^4 \partial x^3}\left(\frac{1}{5},\frac{1}{2}\right)$$
$$= 5^3 2^4 \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+4)_7}{(2k)!}(\sqrt{2})^{2k-3}\cos\frac{(2k-3)\pi}{4} \tag{21}$$

Next, we use Maple to verify the correctness of (21).

```
>f:=(x,y)->(sqrt(25*x^2+4*y^2))^5*(cos(5*arccot((5*x)/(2*y)))*sinh(10*x)-sin(5*arccot((5*x)/(2*y)))*sin(4*y))/(cosh(10*x)+cos(4*y));
>evalf(D[1$3,2$4](f)(1/5,1/2),14);
```

$$\mathbf{-2.6908842324474 \cdot 10^7}$$

```
>evalf(5^3*2^4*sum(2^(2*k)*(2^(2*k)-1)*bernoulli(2*k)*product(2*k+4-j,j=0..6)/(2*k)!*(sqrt(2))^(2*k-3)*cos((2*k-3)*Pi/4),k=1..infinity),14);
```

$$\mathbf{-2.6908842324474 \cdot 10^7}$$

3.2. Example 2

If the domain of the two-variables function

$$g(x,y) = \left(\sqrt{x^2+y^2}\right)^7 \times \frac{\cos\left(7\cot^{-1}\frac{x}{y}\right)\sin 2y + \sin\left(7\cot^{-1}\frac{x}{y}\right)\sinh 2x}{\cosh 2x + \cos 2y} \tag{22}$$

is $\left\{(x,y)\in R^2 \middle| y>0, x^2+y^2<\frac{\pi^2}{4}\right\}$. By (12), we obtain

$$\frac{\partial^{m+n} g}{\partial y^m \partial x^n}(x,y) = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+6)_{m+n}}{(2k)!}\left(\sqrt{x^2+y^2}\right)^{2k+6-m-n} \times \sin\left[(2k+6-m-n)\cot^{-1}\frac{x}{y}+\frac{m\pi}{2}\right] \tag{23}$$

Therefore,

$$\frac{\partial^9 g}{\partial y^3 \partial x^6}\left(\frac{2}{3},\frac{1}{3}\right) = -\sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}(2k+6)_9}{(2k)!}\left(\sqrt{\frac{5}{9}}\right)^{2k-3} \cdot \cos[(2k-3)\cot^{-1} 2] \tag{24}$$

```
>g:=(x,y)->(sqrt(x^2+y^2))^7*(cos(7*arccot(x/y))*sin(2*y)+sin(7*arccot(x/y))*sinh(2*x))/(cosh(2*x)+cos(2*y));
>evalf(D[1$6,2$3](g)(2/3,1/3),18);
```

$$\mathbf{-2.68926222712736 \cdot 10^5}$$

```
>evalf(-sum(2^(2*k)*(2^(2*k)-1)*bernoulli(2*k)*product(2*k+6-j,j=0..8)/(2*k)!*(sqrt(5/9))^(2*k-3)*cos((2*k-3)*arccot(2)),k=1..infinity),18);
```

$$\mathbf{-2.68926222712726 \cdot 10^5}$$

3.3. Example 3

Let the domain of the two-variables function

$$u(x,y) = \left(\sqrt{3x^2+4y^2}\right)^9 \times \frac{\cos\left(9\cot^{-1}\frac{\sqrt{3}x}{2y}\right)\sinh 2\sqrt{3}x + \sin\left(9\cot^{-1}\frac{\sqrt{3}x}{2y}\right)\sin 4y}{\cosh 2\sqrt{3}x - \cos 4y} \tag{25}$$

be $\left\{(x,y)\in R^2 \middle| y>0, 3x^2+4y^2<\pi^2\right\}$. By (14), we have

$$\frac{\partial^{m+n} u}{\partial y^m \partial x^n}(x,y) = \sqrt{3}^n 2^m \left(\sqrt{3x^2+4y^2}\right)^{8-m-n}(8)_{m+n} \cdot \cos\left[(8-m-n)\cot^{-1}\frac{\sqrt{3}x}{2y}+\frac{m\pi}{2}\right] + \sqrt{3}^n 2^m \sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+8)_{m+n}}{(2k)!}\left(\sqrt{3x^2+4y^2}\right)^{2k+8-m-n} \times \cos\left[(2k+8-m-n)\cot^{-1}\frac{\sqrt{3}x}{2y}+\frac{m\pi}{2}\right] \tag{26}$$

Thus,

$$\frac{\partial^7 u}{\partial y^4 \partial x^3}\left(\frac{1}{2},\frac{3}{4}\right) = 2903040 + 144\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+8)_7 3^k}{(2k)!}\cos\frac{(2k+1)\pi}{3} \tag{27}$$

```
>u:=(x,y)->(sqrt(3*x^2+4*y^2))^9*(cos(9*arccot(sqrt(3)*x/(2*y)))*sinh(2*sqrt(3)*x)+sin(9*arccot(sqrt(3)*x/(2*y)))*sin(4*y))/(cosh(2*sqrt(3)*x)-cos(4*y));
>evalf(D[1$3,2$4](u)(1/2,3/4),24);
```

$$\mathbf{3.19133553043736659056376 \cdot 10^7}$$

```
>evalf(2903040+144*sum(2^(2*k)*bernoulli(2*k)*product(2*k+8-j,j=0..6)*3^k/(2*k)!*cos((2*k+1)*Pi/3),k=1..infinity),24);
```

$$\mathbf{3.19133553043736659056323 \cdot 10^7}$$

3.4. Example 4

Suppose the domain of the two-variables function

$$w(x,y) = \left(\sqrt{9x^2+y^2}\right)^{11} \times \frac{-\cos\left(11\cot^{-1}\frac{3x}{y}\right)\sin 2y + \sin\left(11\cot^{-1}\frac{3x}{y}\right)\sinh 6x}{\cosh 6x - \cos 2y} \tag{28}$$

is $\left\{(x,y)\in R^2 \middle| y>0, 9x^2+y^2<\pi^2\right\}$. By (17), we obtain

$$\frac{\partial^{m+n}w}{\partial y^m \partial x^n}(x,y)$$
$$=3^n\left(\sqrt{9x^2+y^2}\right)^{10-m-n}(10)_{m+n}\cdot$$
$$\sin\left[(10-m-n)\cot^{-1}\frac{3x}{y}+\frac{m\pi}{2}\right]+ \quad (29)$$
$$3^n\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+10)_{m+n}}{(2k)!}\left(\sqrt{9x^2+y^2}\right)^{2k+10-m-n}\times$$
$$\sin\left[(2k+10-m-n)\cot^{-1}\frac{3x}{y}+\frac{m\pi}{2}\right]$$

Hence,

$$\frac{\partial^8 w}{\partial y^5 \partial x^3}\left(\frac{2}{3},\frac{4}{3}\right)$$
$$=78\cdot 10!\cdot\cos\left(2\cot^{-1}\frac{3}{2}\right)+$$
$$27\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}(2k+10)_8}{(2k)!}\left(\frac{52}{9}\right)^{k+1}\cos\left[(2k+2)\cot^{-1}\frac{3}{2}\right] \quad (30)$$

```
>w:=(x,y)->(sqrt(9*x^2+y^2))^11*(-cos(11*arccot(3*x/y))*sin(2*y)+sin(11*arccot(3*x/y))*sinh(6*x))/(cosh(6*x)-cos(2*y));
>evalf(D[1$3,2$5](w)(2/3,4/3),22);
```

$$-5.154590255812792862\cdot 10^8$$

```
>evalf(78*10!*cos(2*arccot(3/2))+27*sum(2^(2*k)*bernoulli(2*k)*product(2*k+10-j,j=0..7)*(52/9)^(k+1)/(2*k)!*cos((2*k+2)*arccot(3/2)),k=1..infinity),22);
```

$$-5.154590255812792852\cdot 10^8$$

4. Conclusion

In this article, we provided a new technique to evaluate any order partial derivatives of four types of two-variables functions. We will use this technique to solve another partial differential problems. On the other hand, the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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