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Existence and Stability of Mixed Stochastic Fractional Order Differential Inclusion Equations via Cosine Dynamical System

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Abstract In this paper, we shall consider the existence and stability of stochastic fractional order differential inclusion nonlinear equations in infinite dimensional space by mixed fractional Brownian motion in Hilbert space H.

Keywords: *Neutral mixed stochastic fractional order differential inclusion equations, existence, stability, via cosine dynamical system with fractional derivative as component in nonlinear functions 0<α, β<1*

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1. Introduction

In this article we study the neutral second- order abstract differential inclusion problem

$$d\left[x'(t)-g\left(t,x(t)\right)\right]\in A\,x(t)+F_1\left(t,x(t),D^{\alpha}x(t)\right)dw + F_2\left(t,x(t),D^{\beta}x(t)\right)dw^H. \tag{3.1}$$

$$x(0)=x^0$$

$$x(0)=x',\ 0<\alpha,\ \beta\le 1,\ \frac{1}{2}<H\le 1.$$

Where $A: X\to X$ is a generator of cosine semigroup on a Hilbert space $(X,\|\cdot\|)$, $\{W(t): t\ge 0\}$ and $\{W^H(t): t\ge 0\}$ are K −valued Brownian motion and fractional Brownian motion respectively with afinit trace nuclear covariance operator >0 . F_1, F_2, g satisfy suitable conditions that will be established later on. The random *variable* $x_0\in H$ *satisfies* $E\|x_0\|^2<\infty$.

This problem has been studied in case $0<\alpha,\beta\le 1$, ([1,2,3,16]). Well-posedness has been established using different fixed point theorems and the theory of strongly continuous cosine families in Banach spaces. We refer the reader to [27,28] for a good account on the theory of cosine families.

The theory of integro-differential equations or inclusions has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology and so on. On can see ([7,8,25] and references therein). Several authors have established the existence results of mild solutions for these equations ([4,5,21,24]). In addition, the nonlinear integro-differential equations with resolvent operators serve as an abstract formulation of partial integro-differential equations that arise in many physical phenomena. One can see [15] and references therein. The deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic problems. As the generalization of classic impulsive integro-differential equations or inclusions, impulsive neutral stochastic functional integro-differential equations or inclusions have attracted the researchers great interest. And some works have done on the existence results of mild solutions for these equations (see [17,26] and references therein). To the best of our knowledge, there is no work reported on the existence of mild solutions for the impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions and resolvent operators, and the aim of this paper is to close the gap. In this paper, motivated by the previously mentioned papers, we will study this interesting problem. Sufficient conditions for the existence are given by means of the fixed point theorem for multi-valued mapping due to Dhage [11] and the fractional power of operators. Especially, the known results appeared in [9,10] are generalized to the stochastic settings. An example is provided to illustrate the theory. We refer the reader to Da prato and Zabczyk [12]. throughout the paper $(H,\|.\|_H)$*and* $(K,\|.\|_K)$denote two real separable Hilbert spaces. In case without confusion, we just use $\langle .,.\rangle$ for the inner product and $\|.\|$ for the norm.

Let $(\Omega,\mathcal{F},p;F)$ $(F=\{\mathcal{F}(t)\}_{t\ge 0})$ be complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t):\Omega\to H$ and the collection of random variables $S=\{x(t,w):\Omega\to H\setminus t\in J\}$ is called a stochastic process. Generally, we just write *x(t)* instead of $x(t,w)$ and $x(t): J\to H$ in the space of *S. Let* $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of K. Suppose that $\{w(t): t\ge 0\}$

is a cylindrical K −valued wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$. *So,* actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, w_i(t) e_i$, where $\{w_i\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s): 0 \leq s \leq t\}$ is the σ −algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\Psi \in L(K,H)$ and define $\|\Psi\|_Q^2 = Tr(\Psi Q \Psi^*) = \sum_{n=1}^{\infty} \left\|\sqrt{\lambda_n}\Psi e_n\right\|^2$.

If $\|\Psi\|_Q < \infty$, then Ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$denote the space of all Q-Hilbert-Schmidt operators $\Psi: K \to H$. *The* completion $L_Q(K,H)$ *of L(K,H)* with respect to topology induced by the norm $\|.\|_Q$ where $\|\Psi\|_Q^2 = \langle \Psi, \Psi \rangle$ *is* a Hilbert space with the above norm topology. Let $A: D(A) \to H$ be infinitesimal generator of a compact, analytic resolvent operator $S(t), t \geq 0$. Let $L_2(\Omega, \mathcal{F}_{t,H})$ denote the Hilbert space of all $\mathcal{F}_t$ − measurable square integrable random variables with values in H. Let $L_2^{\mathcal{F}}([0,b], H)$ be the Hilbert space of all square integrable and $\mathcal{F}_t$ − measurable processes with values in H. $\boldsymbol{\beta}$([0,b])= $\{x: [0,b] \to H, x_K \in C(J_K, H)\}$ let $L_2^0([,\Omega,],H)$ denote the family of all $\mathcal{F}_0$ − measurable, β –valued random variables $x(0)$. We use the notations $p_{cl}(H)$ for the family of all subsets of H and denote

$$p_{cl}(H) = \{Y \in p(H): Y \text{ is closed}\},$$

$$p_{cv}(H) = \{Y \in p(H): Y \text{ is convex}\},$$

$$p_{bd}(H) = \{Y \in p(H): Y \text{ is bounded}\},$$

$$p_{cp}(H) = \{Y \in p(H): Y \text{ is compact}\}.$$

(closed) for all $x \in H$. $\Gamma(x)$ is bounded on bounded sets if $\Gamma(B) = \cup_{x \in B} \Gamma(x)$ is bounded in H, for any bounded set B of H, that is, $\sup_{x \in B} \sup\{\|y\| \in \Gamma(x)\} < \infty$. Γ is called upper semi continuous (u .s .c. for short) on H, if for any $x \in H$,the set $\Gamma(x)$ is a nonempty, closed subset of H, and if for each open set B of H containing $\Gamma(x)$, there exists an open neighborhood N of x such that $\Gamma(N) \subseteq B$. Γ is said to be completely continuous if $\Gamma(B)$ is relatively compact, for every bounded subset $B \subseteq H$. If the multi –valued map Γ is completely continuous with nonempty compact values, then Γ is u .s .c. if and only if Γ has a closed graph, i.e., $x_n \to x, y_n \to y, y_n \in \Gamma(x_n)$ imply $y \in \Gamma(x)$. Γ has a fixed point if there is $x \in H$ such that $x \in \Gamma(x)$. A multi-valued map $\Gamma: J \to p_{cl}$ is said to be measurable if for each $x \in H$, the mean –square distance between x and $\Gamma(t)$ is measurable.

2. Preliminaries

In this section we present some notation. Assumptions and results needed our proofs later.

Definition (3-1) [18]

The multi-valued map $F: J \times H \to p_{bd,cl,cv}(H)$ is said to be L^2-Caratheodory if

i) $t \mapsto F(t,v)$is measurable for each $v \in H$;

ii) $v \mapsto F(t,v)$ is u. s. c. for almost all $t \in J$;

iii) for each $q > 0$, there exists $h_1 \in L^1(J, R_+)$

such that $\|F(t,v)\|^2 = \sup_{f \in F(t,v)}$, $E\|f\|^2 \leq h_q(t)$, for all $\|v\|_\beta^2 \leq q$ and for a. e. $t \in J$.

Lemma (3-1) [19]

If $\varphi: [0, b] \to L_2^0(Y,X)$ satisfies $\int_0^T \|\varphi(s)\|_{L_2^0}^2 ds < \infty$ then the above sum in

$$\int_0^t \Phi(s)\, dW_{(s)}^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\,\Phi(s) e_n dB_{n(s)}^H$$

$$= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n}\,(K_H^*(\Phi e_n)(s)\, dB_{n(s)}^H$$

is well defined as an X-valued random variable and we have

$$E\left\|\int_0^t \varphi(s)\, dW_{(S)}^H\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\varphi(s)\|_{L_2^0}^2 ds. \quad (3.2)$$

Definition (3-2) [6]

A semigroup $T(t)$, $0 \leq t < \infty$ of bounded linear operators on a Banach space X is a C_0 −semigroup of bounded linear operators if: $\lim_{t\downarrow 0} T(t)x = x$, for every $x \in X$.

Definition (3-3), [14]

If $\{C(t): t \in R\}$ is a strongly continuous cosine family in X,

i. $\{S(t): t \in R\}$, associated to the given strongly continuous cosine family, is defined by $S(t)x = \int_0^t C(s)x\, ds\ x \in X, t \in R$.

ii. The infinitesimal generator $A: X \to X$ of a cosine family $\{C(t): t \in R\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t) x_{t=0}, x \in D(A),$$

Where $D(A) = \{x \in X : C(t)\, x \in C^2(R,X)\}$.

Lemma(3-2), [28]

Let $C(t)$, $(\text{resp}, S(t))$, $t \in R$, be a strongly continuous cosine (resp, sin) family on X, then there exist constants $M \geq 1$ and $w \geq 0$ such that $\|C(t)\| \leq Me^{|t|}$, for all $t \in R$, $\|S(t_1) - S(t_2)\| \leq M\left|\int_{t_1}^{t_2} e^{w|s|} ds\right|$, for all $t_1, t_2 \in R$.

Theorem (3-1) **[11]**

Let $B(0,r)$ and $B[0,r]$ denote respectively the open and closed balls in a Hilbert space H centered at the origin and of radius r and let $\Phi_1: H \to P_{bd,cl,cv}(H)$ and $\Phi_2: B[0,r] \to P_{bd,cl,cv}(H)$, two multi-valued operators satisfying

(i) Φ_1 is a contraction, and

(ii) Φ_2 is u.s.c. and completely continuous.

Then, either

(1) the operator inclusion $x \in \Phi_1 x + \Phi_2 x$ has a solution, or

(2) there exists an $x \in H$ with $\|x\| = r$ such that $\lambda x \in f\Phi_1 x + \Phi_2 x\ for\ some\ \lambda > 1$.

3. Main Result of the Existence and Stability

The following lemma and definition are begging to explain the main results.

To investigate the existence of the mixed-stochastic mild solution to the system (3.1), and for the operators A we make the following assumption:

1. A is the infinitesimal generator of a cosine semi group $S(t)$, $C(t)$, $t \geq 0$ in the Hilbert space H and there exist constants $N^{\wedge}, M^{\wedge}$ and M_1 such that $\|S(t)\|^2 \leq N^{\wedge}, \|C(t)\|^2 \leq M^{\wedge}, t \in J$ on $I = [0,T]$.
2. There exist constant M_2 such that $g : J \times H \to H$, satisfies the following Lipchitz condition, that is, for any $s,t \in J, x,y \in H$ such that $\|g(s,x) - g(t,y)\|^2 \leq M_2[|s-t| + \|x-y\|^2]$.
3. The multi-valued maps $F_{1,2}: J \times H \to P_{bd,cl,cv}(L(k,H))$ is an $L^2 -$ Caratheodory function satisfies the following condition :-
   i. $F_{1,2}(t,.,.): J \times H \times H \to H$ is uniformly semi continuous for a. e. $t \in J$, for every $(.,y_1,y_2) \in J \times H \times H \to H$ the function $F_{1,2}(.,y_1,y_2)$ is strongly measurable.
   ii. $F_{1,2}N_{y_1,y_2} = \{ \sigma_{1,i} \in L^2(L(k,H)) : \sigma_{2,i}(t) \in F1,2t,y1,y2 \text{for } t \in J$ nonempty.
   iii. There exists a nonnegative continuous function $K_{F_{1,2}}(t)$ and a continuous non decreasing positive function $\Omega_{F_{1,2}}$ such that

$$F_{1,2}(t,y_1,y_2) \leq K_{F_{1,2}}(t)\Omega_{F_{1,2}}(y_1+y_2).$$

4. $L_0 = (4b M^{\wedge 2} M_2) < 1$.
5. The function $\sigma: [0,T] \to L_2^0(y\,;x)$ satisfies from $\int_0^t \|\sigma(s)\|_{L_2}^2 ds < \infty$ and there exists $C_1 > 0$ such that $Sup\|\sigma(s)(t)\|_{L_2}^2 \leq C_1$.
6. For each $r > 0$,the set $F_2(I \times B_r(0,X^n))$ is relatively compact in.

Lemma (3.3):

Let $\{C(t)\}_{t\geq 0}$ be a cosine semigroup and the $H-$valued function $V(s) = C(t-s)x(s) + S(t-s)[x'(s) - g(s,x(s))]$, then (3.1) has a mixed–stochastic mild solution

$$\begin{aligned} x(t) &= C(t)x(0) - S(t)\left[x'(0) - g(0,x(0))\right] \\ &+ \int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ &+ \int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \\ &+ \int_0^t C(t-s)g(s,x(s))ds. \end{aligned}$$

Proof:

$$V(s) = C(t-s)x(s) + S(t-s)[x'(s) - g(s,x(s))]$$

different both sides for s and use properties in lemma (3.1), we get

$$\begin{aligned} \frac{dV(s)}{dS} &= C(t-s)x'(s) - AS(t-s)x(s) + S(t-s) \\ &\frac{d}{ds}\left[x'(s) - g(s,x(s))\right] - C(t-s)\left[x'(s) - g(s,x(s))\right] \\ &= C(t-s)x'(s) - AS(t-s)x(s) + S(t-s) \\ &\begin{bmatrix} A\,x(s) + F_1\left(s,x(s),D^{\alpha}x(s)\right)dw \\ +F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H \end{bmatrix} \\ &- C(t-s)x'(s) + C(t-s)g(s,x(s)) \\ &= S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw \\ &+ S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H \\ &+ C(t-s)g(s,x(s)) \end{aligned}$$

$$\begin{aligned} V(t) - V(0) &= \int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ &+ \int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \\ &+ \int_0^t C(t-s)g(s,x(s))ds \end{aligned}$$

$$\begin{aligned} x(t) &= C(t)x(0) - S(t)\left[x'(0) - g(0,x(0))\right] \\ &+ \int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ &+ \int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \\ &+ \int_0^t C(t-s)g(s,x(s))ds. \end{aligned} \tag{3.3}$$

Definition (3.4):

A bounded function $x(t): R \to X$ is called mixed–stochastic mild solution of the fractional inclusion system (3.1) if for any $t \in J$

$$\begin{aligned} x(t) &= C(t)x(0) - S(t)\left[x'(0) - g(0,x(0))\right] \\ &+ \int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ &+ \int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \\ &+ \int_0^t C(t-s)g(s,x(s))ds. \end{aligned}$$

4. Existence of the Fractional Stochastic Integro-Differential Inclusion Equations via Cosine Dynamical System

In this section, the existence of the mixed–stochastic mild solution to the stochastic fractional order inclusion problem formulation (3.1) has been discussed.

Theorem (3.2):

Assume the Hypotheses (1-6) are hold with

$$\int_0^t H(s)ds < \frac{\left(1 - 12M^{\wedge 2}L_4\right)}{C_3}\int_{c_2}^{\infty}\frac{ds}{\Omega_{F_1}(s) + \Omega_{F_2}(s)},$$

$$\left(1 - 12M^{\wedge 2}L_4\right) > 0.$$

For the mixed stochastic fractional order system (3.1) with initial conditions

$$x(0) = x^0,\, x(0) = x',\, 0 < \alpha, \beta \leq 1, \frac{1}{2} < H \leq 1.$$

Then has a mixed-stochastic mild solution $x \in \beta$.

Proof:

Let the operator $\emptyset: B \to P(B)$ defined by

$$\emptyset(x)=\left\{\begin{array}{l} x\in B, x(t)=C(t)x(0)+S(t)\left[x'(0)-g(0,x(0))\right] \\ +\int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ +\int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \\ +\int_0^t C(t-s)g(s,x(s))ds \end{array}\right\}.$$

It is clear that the fixed point of $\emptyset$ are mild solutions of the system (3.1). Let

$$\emptyset_1(x) = \left\{\begin{array}{l} x\in B: x(t)=C(t)x(0)+S(t)\left[x'(0)-g(0,x(0))\right] \\ +\int_0^t C(t-s)g(s,x(s))ds \end{array}\right\} \quad (3.4)$$

$$\emptyset_2(x) = \left\{\begin{array}{l} x\in B: x(t)=\int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s) \\ +\int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s) \end{array}\right\}. \quad (3.5)$$

We prove that the operators $\emptyset_1$ and $\emptyset_2$ are satisfy all the condition for theorem (3.1).

Let $B_l = \{x \in \beta, \mathrm{E}\|x\|^2 \le l\}$.

Step(1):- Now to prove that $\emptyset_1$is a contraction.

Let x_1, x_2 from assuming that

$$\emptyset_1(x)=\left\{\begin{array}{l} C(t)x(0)+S(t)\left[x'(0)-g(0,x(0))\right] \\ +\int_0^t C(t-s)g(s,x(s))ds \end{array}\right\}.$$

By using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & E\left\|\emptyset_1(x_1)(t)-\emptyset_1(x_2)(t)\right\|^2 \\ & \le 4\|C(t)\|^2 E\left\|x_1(0)-x_2(0)\right\|^2 \\ & +4\|S(t)\|^2 E\left\|x_1'(0)-x_2'(0)\right\|^2 \\ & +4\|S(t)\|^2 E\left\|g(0,x_1(0))-g(0,x_2(0))\right\|^2 \\ & +4\int_0^t \|C(t-s)\|^2 E\left\|g(s,x_1(s))-g(s,x_2(s))\right\|^2 ds. \end{aligned}$$

From the conditions (1),(2) and taking supremum over $t \in [0,T]$for both sides, we get

$$\begin{aligned} & \le 4\mathrm{M}^{\wedge 2}\,\mathrm{Sup}_{t\in J} E\left\|x_1(0)-x_2(0)\right\|^2 \\ & +4N^{\wedge 2}\,Sup_{t\in J} E\left\|x_1'(0)-x_2'(0)\right\|^2 \\ & +4N^{\wedge 2} M_2 \mathrm{Sup}_{t\in J}\left\|x_1(t)-x_2(t)\right\|^2 \\ & +4\int_0^t M^{\wedge 2} M_2\left[|s-s|+\left\|x_1(t)-x_2(t)\right\|^2\right]ds. \end{aligned}$$

By initial condition $x(0) = x^0$, $x'(0) = x^0$, $x'(0) = x'$, we get

$$\begin{aligned} & \le 4N^{\wedge 2} M_2 \mathrm{Sup}_{t\in J} E\left\|x_1(t)-x_2(t)\right\|^2 \\ & +4bM^{\wedge 2} M_2 \mathrm{Sup}_{t\in J} E\left\|x_1(t)-x_2(t)\right\|^2 \\ & \le (4N^{\wedge 2} M_2+4bM^{\wedge 2} M_2)\mathrm{Sup}E\left\|x_1(t)-x_2(t)\right\|^2. \end{aligned}$$

Let

$$\begin{aligned} & L_0 = 4(N^{\wedge 2} M_2 + bM^{\wedge 2} M_2) \\ & \le L_0 \mathrm{Sup}_{t\in J} E\left\|x_1(t)-x_2(t)\right\|^2 \\ & E\left\|\emptyset_1(x_1)-\emptyset_1(x_2)\right\|^2 \le L_0 E\left\|x_1-x_2\right\|^2. \end{aligned}$$

Step (2):- Now to prove that $\emptyset_2(x)$ is convex for each $x \in \beta$. Let $u_{1,1}, u_{1,2} \in \emptyset_2(x)$, then, there exists $\sigma_{1,1}, \sigma_{1,2} \in F_{N,x}$ such that $\sigma_{1,1}, \sigma_{1,2}, \sigma_{2,1}, \sigma_{2,2} \in N_{F_{1,2}}$ such that

$$\begin{aligned} u_{1,2}(t) &= \int_0^t S(t-s)\sigma_{1,1}(s)dw(s) \\ &\quad +\int_0^t S(t-s)\sigma_{1,2}(s)dw^H(s) \end{aligned} \quad (3.6)$$

$$\begin{aligned} u_{2,1}(t) &= \int_0^t S(t-s)\sigma_{2,1}(s)dw(s) \\ &\quad +\int_0^t S(t-s)\sigma_{2,2}(s)dw^H(s). \end{aligned} \quad (3.7)$$

Let $\lambda \in [0,1]$, then

$$\begin{aligned} u_{1,1}(t) &= \int_0^t S(t-s)\lambda\sigma_{1,1}(s)dw(s) \\ &\quad +\int_0^t \mathrm{S}(\mathrm{t}-\mathrm{s})\lambda\sigma_{1,2}(s)dw^H(s) \end{aligned}$$

$$\begin{aligned} (1-\lambda)u_{1,2}(t) &= (1-\lambda)\int_0^t S(t-s)\sigma_{2,1}(s)dw(s) \\ &\quad +(1-\lambda)\int_0^t S(t-s)\sigma_{2,2}(s)dw^H(s). \end{aligned}$$

For each $t \in J$, We have

$$\begin{aligned} & \left(\lambda u_{1,1}(t)+(1-\lambda)u_{1,2}(t)\right) \\ & =\int_0^t S(t-s)\left(\lambda\sigma_{1,1}(s)+(1-\lambda)\sigma_{2,1}(s)\right)dw(s) \\ & +\int_0^t S(t-s)\left(\lambda\sigma_{1,2}(s)+(1-\lambda)\sigma_{2,2}(s)\right)dw^H(s). \end{aligned} \quad (3.8)$$

From the condition $(3-i, ii)$ and since $N_{F_{1,2,x}}$ is convex then we have that

$$\lambda u_{1,1}(t)+(1-\lambda)u_{1,2}(t)\in\varnothing_2(x).$$

Remark (3.1) [27]:

We need the following inequalities for complete the proof of existence theorem.

$D^{\eta}S(t)x = \mathrm{I}^{1-\eta}\, C(t)x, 0<\eta<1$, thus

$$\left|I^{1-\eta}C(t)x\right|\le\frac{1}{\Gamma(1-\eta)}\int_0^t\left|S(t-s)x\right|ds$$

$$\le\frac{t^{1-\eta}}{\Gamma(2-\eta)}\sup_{0\le t\le T}\left|C(t)x\right|\le\frac{t^{1-\eta}\tilde{M}}{\Gamma(2-\eta)}.$$

Step (3):- Now to prove that $\emptyset_2$ maps bounded sets into bounded set in β. Indeed, it is enough to show that there exists appositive constant $\wedge$ such that For each $u\in\emptyset_2(x)$. Then there exists $\sigma_1\in N_{F_{1,2}}$ for each $t\in J$ such that

$$u(t)=\left\{\begin{matrix}\int_0^t S(t-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s)\\+\int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)\end{matrix}\right\}.$$

By using remark (3.1),

$$E\left\|u(t)\right\|^2\le 2E\left\|\int_0^t S(t-s)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\right\|^2$$

$$+2E\left\|\int_0^t S(t-s)F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\right\|^2.$$

From conditions (1) and $(3-iii)$, we get

$$\le 2N^{\wedge^2}E\left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\left(\begin{matrix}\mathrm{Sup}_{0\le\tau\le s}\left\|x(s)\right\|\\+\frac{s^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha)}\mathrm{Sup}_{0\le\tau\le s}\left\|y(\tau)\right\|\end{matrix}\right)dw(s)\right)^2$$

$$+2N^{\wedge^2}E\left(\int_0^t K_{F_2}(\tau)\Omega_{F_2}\left(\begin{matrix}\mathrm{Sup}_{0\le\tau\le s}\left\|x(s)\right\|\\+\frac{s^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\\mathrm{Sup}_{0\le\tau\le s}\left\|y(\tau)\right\|dw^H(s)\end{matrix}\right)\right)^2.$$

By using Lemmas (1-8) and (1-14), we get

$$\le 2N^{\wedge^2}E\left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\left(\begin{matrix}\mathrm{Sup}_{0\le\tau\le s}\left\|x(s)\right\|\\+\frac{s^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)}\mathrm{Sup}_{0\le\tau\le s}\left\|y(\tau)\right\|\end{matrix}\right)\right)^2ds$$

$$+2N^{\wedge^2}Ht^{2H-1}\left(\int_0^t K_{F_2}(\tau)\Omega_{F_2}\left(\begin{matrix}\mathrm{Sup}_{0\le\tau\le s}\left\|x(s)\right\|\\+\frac{s^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\\mathrm{Sup}_{0\le\tau\le s}\left\|y(\tau)\right\|\end{matrix}\right)\right)^2ds.$$

Let

$$\theta^{\wedge}(t)=\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}\mathrm{Sup}_{0\le\tau\le s}\left(\left\|x(s)\right\|+\left\|y(s)\right\|\right)$$

were $\eta=\max\{\eta_1,\eta_2\}$, $\theta^{\wedge}=\max\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\}$. Now

$$\begin{aligned}\theta^{\wedge}(t)&\le\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}\\&\left\{2N^{\wedge^2}\left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\left(\theta^{\wedge}(s)\right)ds\right)\right\}\\&+\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}\\&\left\{2N^{\wedge^2}\left(2Ht^{2H-1}\int_0^t\left(K_{F_2}(\tau)\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right)\right)ds\right\}\qquad(3.9)\\&\le\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}\\&\left\{\int_0^t H(s)\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right\}\\&\le C_3\int_0^t H(s)\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds.\end{aligned}$$

If we put,

$$\mu(t)=C_3\int_0^t H(s)\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds.$$

Let us denote by $\theta^{\wedge}(t)$ the right hand side of above then $\mu(0)=C_2$ and

$$\theta^{\wedge}(t)\le\mu(t)$$

$$\frac{d\mu(t)}{dt}=\frac{d}{dt}\left[C_3\int_0^t H(s)\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right]$$

$$=C_3H(t)\left[\Omega_{F_1}\left(\theta^{\wedge}(t)\right)+\Omega_{F_2}\left(\theta^{\wedge}(t)\right)\right].$$

Thus

$$\begin{aligned}\mu'(t)&=C_3H(t)\left[\Omega_{F_1}\left(\theta^{\wedge}(t)\right)+\Omega_{F_2}\left(\theta^{\wedge}(t)\right)\right]\\&\le C_3H(t)\left[\Omega_{F_1}\left(\mu(t)\right)+\Omega_{F_2}\left(\mu(t)\right)\right],\end{aligned}$$

therefore

$$\frac{\mu'(t)}{\Omega_{F_1}\left(\mu(t)\right)+\Omega_{F_2}\left(\mu(t)\right)}\le C_3H(t).$$

By taking the integral in both sides, we get

$$\int_0^t\frac{\mu'(t)}{\Omega_{F_1}\left(\mu^{\wedge}(t)\right)+\Omega_{F_2}\left(\mu^{\wedge}(t)\right)}ds\le C_3\int_0^t H(s)ds.$$

Let $\mu(t)=\tau,\mu(0)=C_2, 0<s<t$.
So,

$$\int_{c_2}^{\mu(t)}\frac{ds}{\Omega_{F_1}(s)+\Omega_{F_2}(s)}\le C_3\int_0^t H(s)ds.$$

From theorem (3.1), we get

$$\int_{c_2}^{\mu(t)} \frac{ds}{\Omega_{F_1}(s)+\Omega_{F_2}(s)} \le C_3 \int_0^t H(s)ds$$

$$\le C_3 \frac{1}{C_3} \int_{c_2}^{\infty} \frac{ds}{\Omega_{F_1}(s)+\Omega_{F_2}(s)}$$

$$\int_{c_2}^{\mu(t)} ds \le C_3 \int_0^t \left(\Omega_{F_1}(s)+\Omega_{F_2}(s)\right) H(s)ds < \int_{c_2}^{\mu(t)} ds$$

$$\mu(t)-C_2 \le C_3 \int_0^t \left(\Omega_{F_1}(s)+\Omega_{F_2}(s)\right) H(s)ds < \infty.$$

Hence $\theta^{\wedge}(t)$ is bounded and then $(\|x(t)\| + \|y(\tau)\|)$ is bounded then the $\emptyset_2$ maps bounded sets in to bounded sets in β.

Step (4):- $\emptyset_2$ maps bounded set into equicontinuous sets of β, Let $\tau \in (0,b) \ni \ 0 < \tau_1 < \tau_2 \le b$ therefore each $x \in b\ \beta_I$ and $u \in \emptyset_2\ x\ \sigma \in N_{F_{1,2}}$ such that, for each, $t \in J$, we have

$$E\left\|\varnothing_1(x)(\tau_2)-\varnothing_1(x)(\tau_1)\right\|^2$$

$$\le 6E\left\|\int_0^{\tau_1-\varepsilon} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2} S(\tau_2-s)F_1\left(s,x(s),D^{\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_0^{\tau_1-\varepsilon} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2} S(\tau_2-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)\right\|^2.$$

From this inequality and using remark (3.1), we have

$$\le 6E\left\|\int_0^{\tau_1-\varepsilon} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2} S(\tau_2-s)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6E\left\|\int_0^{\tau_1-\varepsilon} \left[S(\tau_2-s)-S(\tau_1-s)\right]F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1-\varepsilon}^{\tau_1} \begin{Bmatrix}\left[S(\tau_2-s)-S(\tau_1-s)\right] \\ F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)\end{Bmatrix} dw^H(s)\right\|^2$$

$$+6E\left\|\int_{\tau_1}^{\tau_2} S(\tau_2-s)F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\right\|^2.$$

From condition $(3)-(iii)$ and Lemma (1-8) and (1-14)

$$\le 6\int_0^{\tau_1-\varepsilon} \left\|\left[S(\tau_2-s)-S(\tau_1-s)\right]\right\|^2$$

$$E\left(K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{S^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|dw(s)\end{pmatrix}\right)^2$$

$$+6\int_{\tau_1-\varepsilon}^{\tau_1} \left\|\left[S(\tau_2-s)-S(\tau_1-s)\right]\right\|^2$$

$$E\left(K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{S^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|dw(s)\end{pmatrix}\right)^2$$

$$+6\int_{\tau_1}^{\tau_2} \left\|S(\tau_2-s)\right\|^2 E\left(K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{S^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|dw(s)\end{pmatrix}\right)^2$$

$$+6\int_0^{\tau_1-\varepsilon} \left\|\left[S(\tau_2-s)-S(\tau_1-s)\right]\right\|^2$$

$$E\left(K_{F_2}(\tau)\Omega_{F_2}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{S^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|dw^H(s)\end{pmatrix}\right)^2$$

$$+6\int_{\tau_1-\varepsilon}^{\tau_1} \left\|\left[S(\tau_2-s)-S(\tau_1-s)\right]\right\|^2$$

$$E\left(K_{F_2}(\tau)\Omega_{F_2}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{S^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|dw^H(s)\end{pmatrix}\right)^2$$

$$+6\int_{\tau_1}^{\tau_2}\left\|S(\tau_2-s)\right\|^2$$

$$E\left(K_{F_2}(\tau)\Omega_{F_2}\left(\begin{array}{l}\operatorname{Sup}_{0\le\tau\le s}\|x(s)\|\\ +\dfrac{s^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\ \operatorname{Sup}_{0\le\tau\le s}\|y(\tau)\|dw^H(s)\end{array}\right)\right)^2$$

The right-hand side of the above inequality tends to zero as $\tau_2\to\tau_1$, with ε sufficiently small, then

$$\le\int_{\tau_1}^{\tau_2}\left\|S(\tau_2-s)\right\|^2$$

$$E\left(K_{F_1}(\tau)\Omega_{F_1}\left(\begin{array}{l}\operatorname{Sup}_{0\le\tau\le s}\|x(s)\|\\ +\dfrac{s^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)}\operatorname{Sup}_{0\le\tau\le s}\|y(\tau)\|dw(s)\end{array}\right)\right)^2$$

$$+6\int_{\tau_1}^{\tau_2}\left\|S(\tau_2-s)\right\|^2$$

$$E\left(K_{F_2}(\tau)\Omega_{F_2}\left(\begin{array}{l}\operatorname{Sup}_{0\le\tau\le s}\|x(s)\|\\ +\dfrac{s^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\ \operatorname{Sup}_{0\le\tau\le s}\|y(\tau)\|dw^H(s)\end{array}\right)\right)^2.$$

From the last inequality, we have

$$\le 6N^{\wedge 2}T\,r(Q)\int_{\tau_1}^{\tau_2}\int_0^t\left[\begin{array}{l}\left(K_{F_1}(s)\right)^2\\ \left(\Omega_{F_1}\left(r+\dfrac{s^{\eta_1-\alpha}r}{\Gamma(\eta_1-\alpha+1)}\right)\right)^2\end{array}\right]ds$$

$$+6N^{\wedge 2}\int_{\tau_1}^{\tau_2}2Ht^{2H-1}\int_0^t\left[\begin{array}{l}\left(K_{F_2}(s)\right)^2\\ \left(\Omega_{F_2}\left(r+\dfrac{s^{\eta_2-\beta}r}{\Gamma(\eta_2-\beta+1)}\right)\right)^2\end{array}\right]ds \qquad (3.10)$$

$$\le 6N^{\wedge 2}T\,r(Q)(\tau_2-\tau_1)$$

$$\int_0^t\left(K_{F_1}(s)\right)^2\left(\Omega_{F_1}\left(r+\frac{s^{\eta_1-\alpha}r}{\Gamma(\eta_1-\alpha+1)}\right)\right)^2ds$$

$$+6N^{\wedge 2}(\tau_2-\tau_1)2Ht^{2H-1}$$

$$\int_0^t\left(K_{F_2}(s)\right)^2\left(\Omega_{F_2}\left(r+\frac{s^{\eta_2-\beta}r}{\Gamma(\eta_2-\beta+1)}\right)\right)^2ds.$$

Since $S(t)$ is sine semi group continuous in the uniform operator topology, the set $\{\emptyset_2(x): x\in B_l\}$ is equicontinuous.

Step (5):- Now to prove $(\emptyset_2\,B_L)(t)$ is relativity compact in H for each $t\in J$. Where $(\emptyset_2\,B_L)(t)=\{u(t): u\in\emptyset_2\,B_L\}$, $t\in J$ the set $(\emptyset_2\,B_L)(t)$ is relatively compact in β for $t=0$.

Let $0<t\le b$ and $0<\in<t$, for $x\in B_l$ and $u\in\emptyset_2(x)$, there exists $\sigma\in N_{F,x}$ such that

$$\begin{aligned}u(t)=&\int_0^{t-\in}S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\&+\int_{t-\in}^{t}S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\&+\int_0^{t-\in}S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\\&+\int_{t-\in}^{t}S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s).\end{aligned}\qquad(3.11)$$

$$\begin{aligned}u_\in(t)=&\int_0^{t-\in}S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\&+\int_0^{t}S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s).\end{aligned}\qquad(3.12)$$

For each $0<\in<t$

$$E\left\|u(t)-u_\in(t)\right\|^2$$

$$\le 2E\left\|\begin{array}{l}\int_0^{t-\in}S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\ +\int_{t-\in}^{t}S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\ +\int_0^{t-\in}S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\\ +\int_{t-\in}^{t}S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\\ -\int_0^{t-\in}\left(S(t-\in)-(s-\in)\right)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\\ -\int_0^{t-\in}\left(S(t-\in)-(s-\in)\right)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\end{array}\right\|^2.$$

From definition (1-20) and remark (3.1) for the sine simegroup continuous, we have

$$E\left\|u(t)-u_\in(t)\right\|^2$$

$$\le 2E\left\|\begin{array}{l}\int_0^{t-\in}S(t-s)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\\ +\int_{t-\in}^{t}S(t-s)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\\ +\int_0^{t-\in}S(t-s)F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\\ +\int_{t-\in}^{t}S(t-s)F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\\ -\int_0^{t-\in}S\left((t-\in)-(s-\in)\right)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\\ +\int_0^{t-\in}S\left((t-\in)-(s-\in)\right)F_2\left(s,x(s),I^{\eta_2-\beta}x(s)\right)dw^H(s)\end{array}\right\|^2$$

$$\le \int_{t-\epsilon}^{t} E\left(K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix} \mathrm{Sup}_{0\le\tau\le s}\|x(s)\| \\ +\dfrac{s^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\| \end{pmatrix}\right)^2 dw(s)$$

$$+2N^{\wedge 2}\int_{t-\epsilon}^{t} E\left(K_{F_2}(\tau)\Omega_{F_2}\begin{pmatrix} \mathrm{Sup}_{0\le\tau\le s}(x(s)] \\ +\dfrac{s^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)} \\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\| \end{pmatrix}\right)^2 dw^H(s)$$

$$\le 2N^{\wedge 2} T\ r(Q)\int_{t-\epsilon}^{t}\left(K_{F_1}(\tau)\right)^2\left(\Omega_{F_1}(r+\frac{S^{\eta_1-\gamma}r}{\Gamma(\eta_1-\gamma+1)})\right)^2 ds$$

$$+2N^{\wedge 2}\int_{t-\epsilon}^{t} 2Ht^{2H-1}\left(K_{F_2}(\tau)\right)^2\left(\Omega_{F_2}(r+\frac{s^{\eta_2-\beta}r}{\Gamma(\eta_2-\beta+1)})\right)^2 ds$$

Therefore, letting $\epsilon\to 0$, we can see that there are relative compact sets arbitrarily close to the set $\{u(t): u \in \emptyset_2(B_l)\}$ is relative compact in β.

Step (6):- Now to show that $\emptyset_2$ has a closed graph

Let $x_n \to x_*$, $x_n \in B_L$, $u_n \in \emptyset_2(x_n)$ and $u_n \to u_*$, we aim to show that $u_* \in \emptyset_2(x_*)$ indeed, $u_n \in \emptyset_2(x_n)$ means that there exists $\sigma_{1,2,n} \in N_{F_{1,2}}$ such that

$$u(t)=\int_0^t S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)$$

$$+\int_0^t S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s).$$

There exists $\sigma_{1,2,n} \in N_{F_{1,2,x}}$, thus

$$u_n(t)=\int_0^t S(t-s)\sigma_{1,n}(s)dw + \int_0^t S(t-s)\sigma_{2,n}(s)dw^H. \tag{3.13}$$

We must prove that there exists $\sigma_1{}^*, \sigma_2^* \in N_{F,x_*}$ such that

$$u_*(t)=\int_0^t S(t-s)\sigma_1^*(s)dw + \int_0^t S(t-s)\sigma_2^*(s)dw^H. \tag{3.14}$$

Suppose the liner continuous operator

$$\Gamma_{1,2}: L^2(J,H)\to C(J,H).$$

From lemma (1.7) it follows that $\Gamma_{1,2}N_{F_{1,2},x}$ is closed graph operator and we have

$$\left\|u_n(t)-u_*(t)+\int_0^t S(t-s)(\sigma_n(s)-\sigma_*(s))dw^H\right\|\to 0$$

as $n\to\infty$, thus

$$u_n(t)-\int_0^t S(t-s)\sigma_n(s)(s)dw^H \in \Gamma_{1,2}N_{F_{1,2},x}.$$

Since $u_n \to u_*$, it follows from Lemma (1.7) that

$$u_*(t)-\int_0^t S(t-s)\sigma_*(s)dw^H \in \Gamma_{1,2}N_{F_{1,2},x}.$$

That is, there exists a $\sigma_1^* \in N_{F1,2,x}$ such that

$$u_*(t)-\int_0^t S(t-s)\sigma_*(s)dw^H = \int_0^t S(t-s)\sigma_1^*(s)dw.$$

By using lemma (1.7) therefore $\emptyset_2$, has a close graph and therefore $\emptyset_2$ is u.s.c.

Step (7):- The operator inclusion $x \in \emptyset_1(x)+\emptyset_2(x)$ has a solution in $B[0,r]$. Define an open ball B (0, r) in, where r satisfies the inequality given in (4). We need to show that the system (3.1) has Least one mild solution, for $\lambda u \in \emptyset_1 x+\emptyset_2 x$ for some $\lambda>1$ with $E\|x\|^2=r$, then, we have

$$x(t)=\left(C(t)x(0)\right)+\lambda^{-1}\left(S(t)[x'(0))-g(s,x(0))]\right)$$

$$+\lambda^{-1}\left(\int_0^t S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\right)$$

$$+\lambda^{-1}\left(\int_0^t S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\right)$$

$$+\lambda^{-1}\left(\int_0^t C(t-s)g\left(s,x(s)\right)ds\right)$$

$$\mathrm{E}\|x(t)\|^2 \le 6\|C(t)x(0)\|^2+12\|S(t)\|^2\left[\|x'(0)-g(0,x(0))\|^2\right]$$

$$+6\left\|\int_0^t S(t-s)F_1\left(s,x(s),D^\alpha x(s)\right)dw(s)\right\|^2$$

$$+6\left\|\int_0^t S(t-s)F_2\left(s,x(s),D^\beta x(s)\right)dw^H(s)\right\|^2$$

$$+12\left\|\int_0^t C(t-s)g\left(s,x(s)\right)ds\right\|^2$$

$$\le 6\|C(t)x(0)\|^2+12\|S(t)\|^2\|[x'(0)-g(0,x(0))\|^2$$

$$+6\|S(t-s)\|^2\left\|\int_0^t S(t-s)F_1\left(s,x(s),I^{\eta_1-\alpha}x(s)\right)dw(s)\right\|^2$$

$$+6\left\|\int_0^t S(t-s)F_2\left(s,x(s)\right)I^{\eta_2-\beta}x(s)dw^H(s)\right\|^2$$

$$+12\left\|\int_0^t C(t-s)g\left(s,x(s)\right)ds\right\|^2.$$

From assumption (1-6), we get

$$\le 6M^{\wedge 2}E\|x(0)\|^2+12N^{\wedge 2}\left(E\|x'(0)\|^2+L_3\right)$$

$$+6N^{\wedge 2}E\left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\|\\ +\frac{S^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha)}\\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|\end{pmatrix}^2 dw(s)\right)$$

$$+6N^{\wedge 2}E\left(\int_0^t K_{F_2}(\tau)\Omega_{F_2}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}\|x(s)\|\\ +\frac{S^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\ \mathrm{Sup}_{0\le\tau\le s}\|y(\tau)\|\end{pmatrix}^2 dw^H(s)\right)$$

$$+12M^{\wedge 2}b\|g(s,x(s))-g(s,0)+g(s,0)\|^2.$$

From Lemma (1-8) and (1-14), we obtain

$$\le 6M^{\wedge 2}Ex(0)^2+12N^{\wedge 2}\left(Ex'(0)^2+L_3\right)$$

$$+6N^{\wedge 2}E\left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}x(s)\\ +\frac{S^{\eta_1-\alpha}}{\Gamma(\eta_1-\alpha)}\\ \mathrm{Sup}_{0\le\tau\le s}y(\tau)\end{pmatrix}^2 dw\right)$$

$$+6N^{\wedge 2}E\left(\int_0^t K_{F_2}(\tau)\Omega_{F_2}\begin{pmatrix}\mathrm{Sup}_{0\le\tau\le s}x(s)\\ +\frac{S^{\eta_2-\beta}}{\Gamma(\eta_2-\beta+1)}\\ \mathrm{Sup}_{0\le\tau\le s}y(\tau)\end{pmatrix}^2 dw^H(s)\right)$$

$$+12M^{\wedge 2}\left(Ex(t)^2+L_4\right).$$

Now,

$$\theta^{\wedge}(t)=\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}\mathrm{Sup}_{0\le\tau\le s}\left(x(s)+y(s)\right)$$

where, $\eta=\max\{\eta_1,\eta_2\}$

$$\theta^{\wedge}(t)\le\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}$$

$$\left\{\begin{matrix}\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}6N^{\wedge 2}\\ \left(\int_0^t K_{F_1}(\tau)\Omega_{F_1}\theta^{\wedge}(s)ds\right)\end{matrix}\right\}$$

$$+\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}$$

$$\left\{6N^{\wedge 2}\left(2Ht^{2H-1}\int_0^t\left(K_{F_2}(\tau)\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right)ds\right)\right\}.$$

From integral in both sides, we get

$$\le\max\left\{1,T^{\eta-\alpha}/\Gamma(\eta-\alpha+1)\right\}$$

$$\left\{\begin{matrix}\int_0^t H(s)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\\ \left(\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right)\end{matrix}\right\}$$

$$\le C_3\int_0^t H(s)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right)$$

If we put,

$$\mu(t)=C_3\int_0^t H(s)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right).$$

Let us denote by $\theta^{\wedge}(t)$ the right hand side of above then $\mu(0)=C_2$ and $\theta^{\wedge}(t)\le\mu(t)$

$$\frac{d\mu(t)}{dt}=\frac{d}{dt}\left[C_3\int_0^t\left\{\begin{matrix}H(s)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\\ \left(\left[\Omega_{F_1}\left(\theta^{\wedge}(s)\right)+\Omega_{F_2}\left(\theta^{\wedge}(s)\right)\right]ds\right)\end{matrix}\right\}\right]$$

$$=C_3H(t)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\left[\Omega_{F_1}\left(\theta^{\wedge}(t)\right)+\Omega_{F_2}\left(\theta^{\wedge}(t)\right)\right]\right).$$

Thus

$$\mu'(t)=C_3\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(H(t)\left[\Omega_{F_1}\left(\theta^{\wedge}(t)\right)+\Omega_{F_2}\left(\theta^{\wedge}(t)\right)\right]\right)$$

$$\le C_3H(t)\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\left[\Omega_{F_1}(\mu(t))+\Omega_{F_2}(\mu(t))\right]\right),$$

therefore

$$\frac{\mu'(t)}{\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\Omega_{F_1}(\mu(t))+\Omega_{F_2}(\mu(t))\right)}\le C_3H(t).$$

By taking the integral in both sides we get

$$\int_0^t\frac{\mu'(s)}{\frac{1}{\left(1-12M^{\wedge 2}L_4\right)}\left(\Omega_{F_1}(\mu(s))+\Omega_{F_2}(\mu(s))\right)}ds$$

$$\le C_3\int_0^t H(s)ds$$

$$\mu(t)=\tau,\ \mu(0)=C_2,\ 0<s<t.$$

So,

$$\int_{c_2}^{\mu(t)} \frac{1}{\left(1-12M^{\wedge 2} L_4\right)} \frac{ds}{\left(\Omega_{F_1}(s)+\Omega_{F_2}(s)\right)} \leq C_{23}\int_0^t H(s)ds.$$

From theorem (3-1), we get

$$\int_{c_2}^{\mu(t)} \frac{ds}{\frac{1}{\left(1-12M^{\wedge 2} L_4\right)}\left(\Omega_{F_1}(s)+\Omega_{F_2}(s)\right)} \leq C_3\int_0^t H(s)ds$$

$$\leq C_3 \frac{1}{C_3}\int_{c_2}^{\infty} \frac{ds}{\frac{1}{\left(1-12M^{\wedge 2} L_4\right)}\left(\Omega_{F_1}(s)+\Omega_{F_2}(s)\right)}$$

$$\int_{c_2}^{\mu(t)} ds \leq C_3\int_0^t \frac{1}{\left(1-12M^{\wedge 2} L_4\right)}\begin{pmatrix}\Omega_{F_1}(s)\\ +\Omega_{F_2}(s)\end{pmatrix} H(s)ds < \int_{c_2}^{\mu(t)} ds$$

$$\mu(t)-C_2 \leq C_3\int_0^t \frac{1}{\left(1-12M^{\wedge 2} L_4\right)}\begin{pmatrix}\Omega_{F_1}(s)\\ +\Omega_{F_2}(s)\end{pmatrix} H(s)ds < \infty.$$

Hence $\theta^{\wedge}(t)$ is bounded and then $(\|x(t)\|)$ is bounded then the $\emptyset_2$ maps bounded sets in to bounded sets into bounded sets in B.

5. Example (3.1)

Consider the problem

$$\frac{\partial}{\partial t}\left[u_t(t,x)+G(t,x,u(t,x))\right]$$
$$\in u_{xx}(t,x)+\widehat{F}_1\left(t,x,u(t,x),D^{\alpha}u(t,x)\right)dw(t)$$
$$+\widehat{F}_2\left(t,x,u(t,x),D^{\beta}u(t,x)\right)dw^H(t),$$
$$t\in I=[0,T], x\in[a,b].$$
$$u(t,a)=u(t,b)=0,$$
$$t\in I, \frac{1}{2}<H<1$$
$$u(0,x)=u^0, x\in[a,b]$$
$$u'(0,x)=u^1, x\in[a,b].$$

In the space $X=L^2([0,\pi])$. This problem is the abstract setting of (3.1). To we define the operator $Ay=y''$ with domain

$$D(A):=\{y\in H^2([0,\pi]): y(0)=y(\pi)=0\}.$$

The operator A has a discrete spectrum with $-n^2, n=1,2,\ldots.$ as eigenvalues and $z_n(s)=\sqrt{2/\pi}\, sin(ns)$, $n=1,2,\ldots$, as their corresponding normalized eigenvectors. So we may write

$$Ay=-\sum_{n=1}^{\infty} n^2(y,z_n)z_n, y\in D(A).$$

Since $-A$ is positive and self-adjoint in $L^2([0,\pi])$, the operator A is the infinitesimal generator of a strongly continuous cosine family $C(t), t\in R$ which has the form

$$C(t)y=\sum_{n=1}^{\infty}\cos(nt)(y,z_n)z_n, y\in X.$$

The associated sine family is

$$C(t)y=\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}(y,z_n)z_n, y\in X.$$

For $u,v\in C([0,T]; X)$ and x ∈ [a, b], defining the operators

$$g(t,u)(x):=G(t,x,u(t,x)),$$
$$F_{1,2}(t,u,v)(x):=F_{1,2}(t,x,u(t,x),v(t,x)),$$

allows us to write (3.12) abstractly as

$$\frac{d}{dt}\left[u'(x)+g(t,u(t))\right]$$
$$=Au(t)+F_1\left(t,u(t),D^{\alpha}u(t)\right)dw$$
$$+F_2\left(t,x,u(t),D^{\beta}u(t)\right)dw^H(t),$$
$$u(0)=u^0, u'(0)=u^1.$$

Under appropriate conditions on $\widehat{F}_{1,2}, G$, which make the conditions (3.1) hold for the corresponding functions $F_{1,2}, g$. Theorem (3.1) ensures the existence of a mixed stochastic mild solution to problem (3.1). Some special cases of this problem may be found in models of some phenomena with hereditary properties [5,8,15].

6. Stability for the Mild Solution of Fractional Inclusion Formulation Problem (3.1)

The following theorem investigate the stability of the inclusion equation (3.1) by using Gran will Bellman inequality via cosine dynamical system. We need to investigate the definition (3.1) on the inclusion problem (3.1).

Definition(3.5):

The solution $x(t,0,\emptyset,\psi)$ of the system (3.1) in said to be stable, if for any $\in>0$, there exists a number $\delta=\delta(\in)>0$, such that for any other solution $y(t,0,\psi)$ of the system (3.1) satisfying $\|\emptyset_1-\psi_1\|=\delta_1, \|\emptyset_2-\psi_2\|=\delta_2$, then $\|x(t,0,\emptyset)-y(t,0,\emptyset)\|<\in, x(t,0,\emptyset)$ is said to be asymptotically stable if it stable and if there is a constant $\delta_1,\delta_2>0$ such that $\|\emptyset_1-\psi_1\|<\delta_1, \|\emptyset_2-\psi_2\|<\delta_2$, then

$$\lim_{t\to\infty}\left\|x(t,0,\emptyset)-y(t,0,\emptyset)\right\|<r.$$

Theorem (3.3):

Assume the hypotheses (1-6) are holds for the system (3.1) with $\|F_{1,2}(s,x(s),D^{\alpha}x(s))-F_{1,2}(s,y(s),D^{\alpha}y(s))\| < M_{1,2}\|x(t)-y(t)\|$ and has an stabile mixed stochastic mild solution.

Proof:

Let $x(t) = x(t, 0, \emptyset_1, \emptyset_2)$ and $y(t) = y(t, 0, \psi_1, \psi_2)$ be a two solutions of equation (3.1) such that

$$x(t)=C(t)\varnothing_1-S(t)\left[\varnothing_2-g(0,\varnothing_1)\right]$$
$$+\int_0^t S(t-s)F_1(s,x(s),D^{\alpha}x(s))dw(s)$$
$$+\int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)$$
$$+\int_0^t C(t-s)g\left(s,x(s)\right)ds.$$

and,

$$y(t)=C(t)\psi_1-S(t)\left[\psi_2-g(0,\psi_1)\right]$$
$$+\int_0^t S(t-s)F_1(s,x(s),D^{\alpha}x(s))dw(s)$$
$$+\int_0^t S(t-s)F_2\left(s,x(s),D^{\beta}x(s)\right)dw^H(s)$$
$$+\int_0^t C(t-s)g\left(s,x(s)\right)ds.$$

Thus,

$$\|x(t)-y(t)\|\le\|C(t)\|\begin{pmatrix}\|\varnothing_1-\psi_1\|+\|S(t)\|\|\varnothing_2-\psi_2\|\\+\|g(0,\psi_1)-g(0,\psi_2)\|\end{pmatrix}$$
$$+\int_0^t\|S(t-s)\|\left\|\begin{matrix}F_1\left(s,x(s),D^{\alpha}x(s)\right)\\-F_1\left(s,y(s),D^{\alpha}y(s)\right)\end{matrix}\right\|dw(s)$$
$$+\int_0^t\|S(t-s)\|\left\|\begin{matrix}F_2\left(s,x(s),D^{\beta}x(s)\right)\\-F_2(s,y(s),D^{\beta}y(s))\end{matrix}\right\|dw^H(s)$$
$$+\int_0^t\|C(t-s)\|\left\|g\left(s,x(s)\right)-g\left(s,y(s)\right)\right\|ds$$
$$\le M^{\wedge 2}\left(\|\varnothing_1-\psi_1\|+N^{\wedge 2}(1+M_2)\|\varnothing_2-\psi_2\|\right)$$
$$+\int_0^t N^{\wedge 2}\left\|F_1\left(s,x(s),D^{\alpha}x(s)\right)-F_1\left(s,y(s),D^{\alpha}y(s)\right)\right\|dw(s)$$
$$+\int_0^t N^{\wedge 2}\left\|F_2\left(s,x(s),D^{\beta}x(s)\right)-F_2\left(s,y(s),D^{\beta}y(s)\right)\right\|dw^H(s)$$
$$+N^{\wedge 2}M_2\|Tx(t)-y(t)\|$$
$$\le M^{\wedge 2}\left(\|\varnothing_1-\psi_1\|+N^{\wedge 2}(1+M_2)\|\varnothing_2-\psi_2\|\right)$$
$$+N^{\wedge 2}TM_3Tr(Q)\|x(t)-y(t)\|^2$$
$$+N^{\wedge}M_3Ht^{2H-1}T\|x(t)-y(t)\|^2$$
$$+N^{\wedge 2}M_2T\|x(t)-y(t)\|^2.$$

Then,

$$\left(1-\left(N^{\wedge 2}TM_3Tr(Q)+\begin{pmatrix}N^{\wedge 2}M_3Ht^{2H-1}T\\+N^{\wedge 2}M_2T\end{pmatrix}\right)\right)\|x(t)-y(t)\|^2$$
$$\le M^{\wedge 2}\left(\|\varnothing_1-\psi_1\|+N^{\wedge 2}(1+M_2)\right)\|\varnothing_2-\psi_2\|\|x(t)-y(t)\|^2$$
$$\le\frac{M^{\wedge 2}\left(\varnothing_1-\psi_1+N^{\wedge 2}(1+M_2)\varnothing_2-\psi_2\right)}{\left(1-\left(N^{\wedge 2}TM_3Tr(Q)+N^{\wedge 2}M_3Ht^{2H-1}T+N^{\wedge 2}M_2T\right)\right)}.$$

Where $1 > \left(N^{\wedge 2}TM_3Tr(Q) + N^{\wedge 2}M_3Ht^{2H-1}T + N^{\wedge 2}M_2T\right)$.

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