American Journal of Numerical Analysis, 2022, Vol. 6, No. 1, 1-6
Available online at http://pubs.sciepub.com/ajna/6/1/1
Published by Science and Education Publishing
DOI:10.12691/ajna-6-1-1

Analytic Solution of the Finite-dimensional Richards Equation under Cauchy Von-Neumann Condition by the New Algorithm SBA

Ben-Sthal Sakoma Yelingue*, Wenddabo Olivier Sawadogo, Youssouf Pare, Blaise Some

Laboratoire d'Analyse Numérique d'Informatique et de BIOmathematique,
Universite de Ouaga 1, Joseph KI Zerbo, 03 BP 7021 Ouagadougou 03, Burkina Faso
*Corresponding author: bensthal@gmail.com

Received October 17, 2022; Revised November 22, 2022; Accepted December 04, 2022

Abstract In this paper, the SBA algorithm are used to solve and simulate the exact solution of 2*D* Richard's Modified Equation at plan flow under Cauchy Von-Neumann condition's type using the same linearization technical to [1] where all scaling parameters of the initial model have been taken into account.

Keywords: *Richards' equation, analytical solution, Cauchy Von-Neumann condition's, SBA algorithm, unsaturated zone*

Cite This Article: Ben-Sthal Sakoma Yelingue, Wenddabo Olivier Sawadogo, Youssouf Pare, and Blaise Some, "Analytic Solution of the Finite-dimensional Richards Equation under Cauchy Von-Neumann Condition by the New Algorithm SBA." *American Journal of Numerical Analysis*, vol. 6, no. 1 (2022): 1-6. doi: 10.12691/ajna-6-1-1.

1. Introduction

We consider the following general equation of the flow, the Richards equation, the combination of the Darcy equation and the next mass conservation equation [2,5]. This model is associated with initial conditions and boundary conditions In two-dimensional space:

$$\begin{cases} \dfrac{\partial \theta(h)(x,z,t)}{\partial t} = \vec{\nabla}.\left[K(h)(x,z,t)\nabla(h(x,z,t)-z)\right], \\ \quad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \\ h(x,z,0)=h_{CI}(x,z), \quad (x,z)\in\delta\Omega \\ h(x,z,t)=h_{CL}(x,z,t), \quad \begin{cases}(x,z)\in\delta\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{1}$$

With

$$\begin{cases} \theta(h)[L^3L^{-3}]: & * \\ h[L]: & ** \\ z[L]: & *** \\ K(h)[LT^{-1}]: & **** \end{cases} \tag{2}$$

(*), (**), (***) and (****) Denote respectively the ***Water Density***, the ***Hydraulic Pressure***, the ***Positively Downward Depth*** and the ***Hydraulic Conductivity***. This model can be written as a function of the potential or the humidity because these two variables are connected by the retention equation. Its resolution requires knowledge of two functions describing the hydrodynamic properties of the soil ($h(\theta)$: Hydraulic retention curve and $K(h)$: Hydraulic conductivity curve) [2].

Depending on the actual saturation, the two-dimensional $(2D)$ equation is written:

$$\begin{cases} C(h)\dfrac{\partial h}{\partial t} = \dfrac{\partial}{\partial x}\left[K(h)\dfrac{\partial h}{\partial x}\right] + \dfrac{\partial}{\partial z}\left[K(h)\left(\dfrac{\partial h}{\partial z}-1\right)\right], \\ \quad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \\ h(x,z,0)=h_{CI}(x,z), \quad (x,z)\in\delta\Omega \\ h(x,z,t)=h_{CL}(x,z,t), \quad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{3}$$

As for the Richards 1D model, Richards' two-dimensional models above are also highly nonlinear due to the Hydraulic Conductivity $K(h)$ and the Soil Retention function $\theta(h)$. In the following, we intend to extend our technique of linearization of the functions $K(h)$ and $C(h)$ that allowed the modification of the equation of Richards 1D [1] to the model of Richards $(2D)$ under Cauchy Von-Neumann condition's type and so that the exact solutions are determined by the algorithm SBA. It is a strongly nonlinear parabolic PDE whose existence and uniqueness of the solution are proven in [4,6]. SBA Method [3,6] has been used to determine the analytical solution after linearization of the functions $K(h)$ and $C(h)$. Many digital methods does not converge because of the strong nonlinearity if we want to solve the Richards equation. So it uses the SBA method to the advantage didn't discredited and maintains the physical properties of the model parameters and converges despite the nonlinearity.

2. The Richards Equation Modified in Dimension Two of Space (REM2D)

2.1. Theory of the SBA Algorithm

General theory of algorithm SBA and the case of Cauchy Von-Neumann in finite dimension space can be funded in [3,4,7,9,10].

2.2. Technical of Linearization of the Functions *C* (*h*) and *K* (*h*)

This technical can be funded in [1].

2.3. The Richards's Equation Modified in two Dimensions

In this part, we will extend to 2D Richards models the same linearization techniques used to modify the 1D Richards model [1].

Let us express equation (3) in h as a function of suction $\Psi = |h|$

$$\begin{cases} C(|h|)\dfrac{\partial(|h|)}{\partial t} \\ = \dfrac{\partial}{\partial x}\left[K(|h|)\dfrac{\partial(|h|)}{\partial x}\right] + \dfrac{\partial}{\partial z}\left[K(|h|)\left(\dfrac{\partial(|h|)}{\partial z} - 1\right)\right], \\ \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \\ |h(x,z,0)| = |h_{CI}(x,z)|, \qquad (x,z)\in\partial\Omega \\ |h(x,z,t)| = |h_{CL}(x,z,t)|, \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \end{cases} \tag{4}$$

The system (4) is equivalent to the following system:

$$\begin{cases} C(\Psi)\dfrac{\partial(\Psi)}{\partial t} \\ = \dfrac{\partial}{\partial x}\left[K(\Psi)\dfrac{\partial(\Psi)}{\partial x}\right] + \dfrac{\partial}{\partial z}\left[K(\Psi)\left(\dfrac{\partial(\Psi)}{\partial z} - 1\right)\right], \\ \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \\ \Psi(x,z,0) = \Psi_{CI}(x,z), \quad (x,z)\in\partial\Omega \\ \Psi(x,z,t) = \Psi_{CL}(x,z,t), \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \end{cases} \tag{5}$$

According to (5) we can successively pose:

$$f(\Psi) = C(\Psi)\frac{\partial\Psi}{\partial t} - \frac{\partial}{\partial x}\left[K(\Psi)\frac{\partial\Psi}{\partial x}\right] - \frac{\partial}{\partial Z}\left[K(\Psi)\left(\frac{\partial\Psi}{\partial Z} - 1\right)\right] \tag{6}$$

$$f(\Psi) = C(\Psi)\frac{\partial\Psi}{\partial t} - \frac{\partial K(\Psi)}{\partial x}\frac{\partial\Psi}{\partial x} - K(\Psi)\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial K(\Psi)}{\partial z}\frac{\partial\Psi}{\partial z} - K(\Psi)\frac{\partial^2\Psi}{\partial z^2} + \frac{\partial K(\Psi)}{\partial z} \tag{7}$$

$$f(\Psi) = C(\Psi)\frac{\partial\Psi}{\partial t} - K(\Psi)\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial z^2}\right] - \left[\frac{\partial K(\Psi)}{\partial x}\frac{\partial\Psi}{\partial x} + \frac{\partial K(\Psi)}{\partial z}\frac{\partial\Psi}{\partial z}\right] + \frac{\partial K(\Psi)}{\partial z} \tag{8}$$

$$f(\Psi) = C(\Psi)\frac{\partial\Psi}{\partial t} - K(\Psi)\Delta\Psi - \left[\frac{\partial K(\Psi)}{\partial x}\frac{\partial\Psi}{\partial x} + \frac{\partial K(\Psi)}{\partial z}\frac{\partial\Psi}{\partial z}\right] + \frac{\partial K(\Psi)}{\partial z} \tag{9}$$

Avec $\Delta\Psi = \dfrac{\partial^2\Psi}{\partial x^2} + \dfrac{\partial^2\Psi}{\partial z^2}$.

So the equation (5) can be rewritten as follows:

$$\begin{cases} C(\Psi)\dfrac{\partial\Psi}{\partial t} \\ = K(\Psi)\Delta\Psi + \left[\dfrac{\partial K(\Psi)}{\partial x}\dfrac{\partial\Psi}{\partial x} + \dfrac{\partial K(\Psi)}{\partial z}\dfrac{\partial\Psi}{\partial z}\right] - \dfrac{\partial K(\Psi)}{\partial z}, \\ \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \\ \Psi(x,z,0) = \Psi_{CI}(x,z), \quad (x,z)\in\partial\Omega \\ \Psi(x,z,t) = \Psi_{CL}(x,z,t), \quad \begin{cases} (x,z)\in\Omega \\ t\in[0,T] \end{cases} \end{cases} \tag{10}$$

Note: Boundary conditions will take into account flow types and their applications in practice. Let us express the models (10) according to the variable $S = h_g^{-1}\Psi$ defined in the previous parts. What is written also $\Psi = h_g S$.

In this case, (9) becomes after reduction:

$$f(S) = h_g^2 C(S)\frac{\partial S}{\partial t} - h_g^2 K(S)\Delta S - h_g^2\left[\frac{\partial K(S)}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial K(S)}{\partial z}\frac{\partial S}{\partial z}\right] + h_g\frac{\partial K(S)}{\partial z} \tag{11}$$

Therefore, the equation (10) can be rewritten in the following way after simplifications:

$$\begin{cases} C(S)\dfrac{\partial S}{\partial t} = K(S)\Delta S + \left[\begin{array}{l}\dfrac{\partial K(S)}{\partial x}\dfrac{\partial S}{\partial x} \\ +\dfrac{\partial K(S)}{\partial z}\dfrac{\partial S}{\partial z}\end{array}\right] - \dfrac{1}{h_g}\dfrac{\partial K(S)}{\partial z}, \\ \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \\ S(x,z,0) = S_{CI}(x,z), \qquad (x,z)\in\delta\Omega \\ S(x,z,t) = S_{CL}(x,z,t), \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{12}$$

By decomposing into (12) $C(S)\frac{\partial S}{\partial t}$, $\frac{\partial K(S)}{\partial x}\frac{\partial S}{\partial x}$, $K(S)\frac{\partial^2 S}{\partial x^2}$, $\frac{\partial K(S)}{\partial z}\frac{\partial S}{\partial z}$, $K(S)\frac{\partial^2 S}{\partial z^2}$, $\frac{\partial K(S)}{\partial z}$ we obtains successively:

$$C(S)\frac{\partial S}{\partial t} = C\frac{\partial S}{\partial t} + Cg(S)\frac{\partial S}{\partial t} + Ch(S)\frac{\partial S}{\partial t} \tag{a}$$

$$\frac{\partial K(S)}{\partial x}\frac{\partial S}{\partial x} = K_s\omega\frac{\partial(S^n)}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial\varphi(S)}{\partial x}\frac{\partial S}{\partial x} \tag{b}$$

$$K(S)\frac{\partial^2 S}{\partial x^2} = K_s\frac{\partial^2 S}{\partial x^2} + \left[K_s\omega(S^n)+\varphi(S)\right]\frac{\partial^2 S}{\partial x^2} \tag{c}$$

$$\frac{\partial K(S)}{\partial x} = K_s\omega\frac{\partial(S^n)}{\partial y}\frac{\partial S}{\partial t} + \frac{\partial\varphi(S)}{\partial x} \tag{d}$$

$$\frac{\partial K(S)}{\partial z}\frac{\partial S}{\partial z} = K_s\omega\frac{\partial(S^n)}{\partial z}\frac{\partial S}{\partial z} + \frac{\partial\varphi(S)}{\partial z}\frac{\partial S}{\partial z} \tag{e}$$

$$K(S)\frac{\partial^2 S}{\partial z^2} = K_s\frac{\partial^2 S}{\partial z^2} + \left[K_s\omega(S^n)+\varphi(S)\right]\frac{\partial^2 S}{\partial z^2} \tag{f}$$

$$\frac{\partial K(S)}{\partial z} = K_s\omega\frac{\partial(S^n)}{\partial z} + \frac{\partial\varphi(S)}{\partial z} \tag{g}$$

So by combining the relations $(a),(b),\ldots,(g)$, the system (12) becomes:

$$\begin{cases} C\dfrac{\partial S}{\partial t} + Cg(S)\dfrac{\partial S}{\partial t} + Ch(S)\dfrac{\partial S}{\partial t} \\ = K_s\Delta S + \left[K_s\omega(S^n)+\varphi(S)\right]\Delta S - \\ -K_S\omega\left[\dfrac{\partial(S^n)}{\partial x}\dfrac{\partial S}{\partial x} + \dfrac{\partial\varphi(S)}{\partial x}\dfrac{\partial S}{\partial x} + \dfrac{\partial(S^n)}{\partial z}\dfrac{\partial S}{\partial z} + \dfrac{\partial\varphi(S)}{\partial z}\dfrac{\partial S}{\partial z}\right] \\ --\dfrac{1}{h_g}\left[K_s\left(\dfrac{\partial(S^n)}{\partial x} + \dfrac{\partial(S^n)}{\partial z}\right) + \left(\dfrac{\partial\varphi(S)}{\partial x} + \dfrac{\partial\varphi(S)}{\partial z}\right)\right], \\ \qquad \begin{cases}(x,.z)\in\Omega \\ t\in[0,T]\end{cases} \\ S(x,z,0) = S_{CI}(x,z) \qquad (x,z)\in\delta\Omega \\ S(x,z,t) = S_{CI}(x,z,t) \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{13}$$

By asking:

$$A = \left[K_s\omega(S^n)+\varphi(S)\right]$$

$$B = \left[\frac{\partial(S^n)}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial\varphi(S)}{\partial x}\frac{\partial S}{\partial x} + \frac{\partial(S^n)}{\partial z}\frac{\partial S}{\partial z} + \frac{\partial\varphi(S)}{\partial z}\frac{\partial S}{\partial z}\right]$$

$$E = \left(\frac{\partial(S^n)}{\partial x} + \frac{\partial(S^n)}{\partial z}\right)$$

$$D = \left(\frac{\partial\varphi(S)}{\partial x} + \frac{\partial\varphi(S)}{\partial z}\right)$$

So the system (13) is equivalent to the following system:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{K_s}{C}\Delta S + \dfrac{1}{C}\left[-Cg(S)\dfrac{\partial S}{\partial t} - Ch(S)\dfrac{\partial S}{\partial t}\right] \\ +\dfrac{A}{C}\Delta S - \dfrac{K_S\omega}{C}B - \dfrac{C}{h_g}\left[K_s\omega E + D\right], \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \\ S(x,z,0) = S_{CI}(x,z) \qquad (x,z)\in\delta\Omega \\ S(x,z,t) = S_{CI}(x,z,t) \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{14}$$

where

$$C = \frac{\theta_s(2-m)}{h_g}$$

Note: We have once again noticed that all the scale parameters contained in the functions $K(h)$ and $C(h)$ are fully taken into account in the modified $2D$ and $3D$ Richards models and its parameters are: K_s, θ_s, h_g as well as shape parameters such as: m and n where $n = f(m)$ and vice versa.

2.4. Theoretical Exact Solution of REM2D under Cauchy Von-Neumann Condition

By SBA algorithm and simulation

In this part, we will theoretically solve the ERM2D by the SBA algorithm with the Neumann boundary conditions [9].

We consider the following model defined previously:

$$\begin{cases} \dfrac{\partial S}{\partial t} = \dfrac{K_s}{C}\Delta S + \dfrac{1}{C}\left[-Cg(S)\dfrac{\partial S}{\partial t} - Ch(S)\dfrac{\partial S}{\partial t}\right] \\ +\dfrac{A}{C}\Delta S - \dfrac{K_S\omega}{C}B - \dfrac{C}{h_g}\left[K_s\omega E + D\right], \\ \begin{cases}(x,z)\in\Omega = [a,b]\times[c,d] \\ \qquad t\in[0,T]\end{cases} \\ S(x,z,0) = S_{CI}(x,z) = h(x,z) \qquad (x,z)\in\delta\Omega \\ S(x,z,t) = S_{CI}(x,z,t) \\ = \begin{cases} S_x(a,z,t) = f_1(a,z,t) \\ S_x(b,z,t) = f_2(b,z,t) \\ S_z(c,z,t) = g_1(c,z,t) \\ S_z(d,z,t) = g_2(d,z,t) \end{cases}; \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases} \end{cases} \tag{15}$$

With ΔS , A , B , E , D Defined as above.

Note: In our case the source function is assumed to be null, i.e. $f \equiv 0$.

The model (15) can also be written:

$$\begin{cases} S_t(x,z,t) = \frac{K_s}{C}\Delta S + \frac{K_s}{C}S \\ +\frac{1}{C}\left[-Cg(S)\frac{\partial S}{\partial t} - Ch(S)\frac{\partial S}{\partial t}\right] \\ -\frac{K_S}{C}S + \frac{A}{C}\Delta S - \frac{K_S\omega}{C}B - \frac{C}{h_g}\left[K_s\omega E + D\right], \\ \qquad \begin{cases}(x,z)\in\Omega = [a,b]\times[c,d] \\ t\in[0,T]\end{cases} \\ S(x,z,0) = S_{CI}(x,z) = h(x,z) \qquad (x,z)\in\partial\Omega \\ S(x,z,t) = S_{CI}(x,z,t) \\ = \begin{cases} S_x(a,z,t) = f_1(a,z,t) \\ S_x(b,z,t) = f_2(b,z,t) \\ S_z(c,z,t) = g_1(c,z,t) \\ S_z(d,z,t) = g_2(d,z,t)\end{cases}; \qquad \begin{cases}(x,z)\in\Omega \\ t\in[0,T]\end{cases}\end{cases} \tag{16}$$

By asking the following operators and their inverses:

$$L_t(.) = \frac{\partial(.)}{\partial t} \Rightarrow L_t^{-1}(.) = \int_0^t (.)du \tag{17}$$

$$L_x(.) = \frac{\partial(.)}{\partial x} \Rightarrow L_x^{-1}(.) = \int_0^x (.)dr \tag{18}$$

$$L_z(.) = \frac{\partial(.)}{\partial z} \Rightarrow L_z^{-1}(.) = \int_0^z (.)ds \tag{19}$$

We consider

$$L_t(S) = L(S) + N(S) \tag{1*}$$

Or $L(S) = \frac{K_S}{C}S$

and

$$\begin{aligned} N(S) &= \frac{K_s}{C}\Delta S + \frac{1}{C}\left[-Cg(S)\frac{\partial S}{\partial t} - Ch(S)\frac{\partial S}{\partial t}\right] \\ &-\frac{K_S}{C}S + \frac{A}{C}\Delta S - \frac{K_S\omega}{C}B - \frac{C}{h_g}\left[K_s\omega E + D\right]\end{aligned} \tag{20}$$

• Considering first the operator L_t and his inverse L_t^{-1} equality (1*) becomes:

$$L_t^{-1}L_t(S) = L_t^{-1}L(S) + L_t^{-1}N(S) \tag{2*}$$

$$\int_0^t \frac{\partial S(x,z,\rho)}{\partial t}d\rho = \frac{K_S}{C}L_t^{-1}S + L_t^{-1}N(S)$$

$$S(x,z,t) = S(x,z,0) + \frac{K_S}{C}L_t^{-1}S + L_t^{-1}N(S) \tag{3*}$$

• Considering then the operator L_x and his inverse L_x^{-1} equality (1*) becomes:

$$L_x^{-1}L_t(S) = \frac{K_S}{C}L_x^{-1}S + L_x^{-1}N(S) \tag{4*}$$

By asking $N(S) = S_{xx} + N'(S)$ with $N'(S) = N(S) - S_{xx}$

Then (4*) becomes:

$$L_x^{-1}L_t(S) = \frac{K_S}{C}L_x^{-1}S + L_x^{-1}S_{xx} + L_x^{-1}N'(S)$$

$$L_x^{-1}S_{xx} + L_x^{-1}\left(\frac{K_S}{C}S + N'(S) - S_t\right) = 0$$

$$\int_a^b \frac{\partial^2 S(x,z,t)}{\partial x^2}dx + L_x^{-1}\left(\frac{K_S}{C}S + N'(S) - S_t\right) = 0$$

$$\left.\frac{\partial S(x,z,t)}{\partial x}\right]_a^b + L_x^{-1}\left(\frac{K_S}{C}S + N'(S) - S_t\right) = 0$$

$$S_x(b,z,t) - S_x(a,z,t) + L_x^{-1}\left(\frac{K_S}{C}S + N'(S) - S_t\right) = 0 \tag{5*}$$

• Considering then the operator L_z and his inverse L_z^{-1} equality (1*) becomes:

$$L_z^{-1}L_t(S) = \frac{K_S}{C}L_z^{-1}S + L_z^{-1}N(S) \tag{6*}$$

By asking $N(S) = S_{zz} + N''(S)$ avec $N''(S) = N(S) - S_{zz}$.

Then (6*) becomes:

$$L_z^{-1}L_t(S) = \frac{K_S}{C}L_z^{-1}S + L_z^{-1}S_{zz} + L_z^{-1}N''(S)$$

$$L_z^{-1}S_{zz} + L_z^{-1}\left(\frac{K_S}{C}S + N''(S) - S_t\right) = 0$$

$$\int_a^b \frac{\partial^2 S(x,z,t)}{\partial z^2}dz + L_z^{-1}\left(\frac{K_S}{C}S + N''(S) - S_t\right) = 0$$

$$\left.\frac{\partial S(x,z,t)}{\partial z}\right]_a^b + L_z^{-1}\left(\frac{K_S}{C}S + N''(S) - S_t\right) = 0$$

$$S_z(x,d,t) - S_z(x,c,t) + L_z^{-1}\left(\frac{K_S}{C}S + N''(S) - S_t\right) = 0 \tag{7*}$$

• Combining (3*), (5*) and (7*) we get:

$$\begin{aligned} S(x,z,t) &= S(x,z,0) + \frac{K_S}{C}L_t^{-1}S + L_t^{-1}N(S) + S_x(b,z,t) \\ &-S_x(a,z,t) + S_z(x,d,t) - S_z(x,c,t) + \\ &+L_x^{-1}\left(\frac{K_s}{C}S + N'(S) - S_t\right) + L_z^{-1}\left(\frac{K_s}{C}S + N''(S) - S_t\right)\end{aligned}$$

Hence, after reduction, we obtain:

$$S(x,z,t) = S(x,z,0) + \frac{K_S}{C} L_t^{-1} S(x,z,t) + N_1(S(x,z,t) \quad (8^*)$$

Or

$$\begin{aligned} & N_1(S(x,z,t) = L_t^{-1} N(S) + S_x(b,z,t) \\ & -S_x(a,z,t) + S_z(x,d,t) - S_z(x,c,t) \\ & + L_x^{-1}\left(\frac{K_s}{C} S + N'(S) - S_t\right) + L_z^{-1}\left(\frac{K_s}{C} S + N''(S) - S_t\right) \end{aligned} \quad (22)$$

Thus we obtain the following canonical Adomian form:

$$S(x,z,t) = S(x,z,0) + \frac{K_S}{C} L_t^{-1} S(x,z,t) + N_1(S(x,z,t)) \quad (9^*)$$

Avec $\lambda = \frac{K_S}{C}$.

Applying the method of successive approximations to:

$$\begin{aligned} & S^k(x,z,t) \\ & = g^k(x,z,0) + \lambda L_t^{-1} S^k(x,z,t) + N_1(S^k(x,z,t)), k \geq 1 \end{aligned} \quad (10^*)$$

où $g^k(x,z,t) = S(x,z,t)$

By applying the Adomian algorithm to (10*), we obtain:

$$\begin{cases} S_0^k(x,z,t) = g^k(x,z,t) + N_1(S^{k-1}(x,z,t)); k \geq 1 \\ S_n^k(x,z,t) = \lambda L_t^{-1} S_{n-1}^k(x,z,t); n \geq 1 \end{cases} \quad (11^*)$$

Apply to (11*), the Piccard principle:

Consider $S^0(x,z,t)$ a root of equation $N_1(S(x,z,t)) = 0$.

- We obtain for $k = 1$ the following algorithm:

$$\begin{cases} S_0^1(x,z,t) = g^0(x,z,t) = S(x,z,0) \\ S_n^1(x,z,t) = \lambda L_t^{-1} S_{n-1}^1(x,z,t); n \geq 1 \end{cases} \quad (12^*)$$

Let's compute the approximate solution $S^1(x,z,t)$ of the problem in Step 1.

$$S_n^1(x,z,t) = \lambda \int_0^t S_{n-1}^1(x,z,t) dr$$

For $n = 1$,

$$S_1^1(x,z,t) = \lambda \int_0^t S_0^1 dr = \lambda S_0^1 \int_0^t dr = \lambda S_0^1 t = S_0^1 (\lambda t)^1$$

For $n = 2$,

$$S_2^1(x,z,t) = \lambda \int_0^t S_1^1 dr = \lambda S_0^1 \int_0^t (\lambda t) dr = \lambda^2 S_0^1 \frac{t^2}{2} = S_0^1 \frac{(\lambda t)^2}{2!}$$

For $n = 3$,

$$S_3^1(x,z,t) = \lambda \int_0^t S_2^1 dr = \lambda S_0^1 \int_0^t \frac{(\lambda t)^2}{2} dr = \lambda^3 S_0^1 \frac{t^3}{6} = S_0^1 \frac{(\lambda t)^3}{3!}$$

Recurringly

For $n = k$,

$$S_k^1(x,z,t) = \lambda \int_0^t S_{k-1}^1 dr = S_0^1 \frac{(\lambda t)^k}{k!}$$

By asking $\Phi_m^1(x,z,t) = \sum_{k=0}^{m} S_k^1(x,z,t)$.

Thereafter, the approximate solution in step 1 is:

$$S^1(x,z,t) = \lim_{m \to +\infty} \Phi_m^1(x,z,t) = S_0^1 \sum_{k=0}^{m} \frac{(\lambda t)^k}{k!} = S_0^1 e^{\lambda t}$$

Then, $S^1(x,z,t) = S_0^1 e^{\lambda t}$.

- Calculate $N_1(S^1(x,z,t))$

Suppose that S^1 is root of N_1 such that $N_1(S^1) = 0$.

- For $k = 2$, Calculate $S^2(x,z,t)$

We have the following algorithm:

$$\begin{cases} S_0^2(x,z,t) = g^0(x,z,t) = S(x,z,0) \\ S_n^2(x,z,t) = \lambda L_t^{-1} S_{n-1}^2(x,z,t); n \geq 1 \end{cases}$$

By calculating, we obtain successively for different values of n.

$$\begin{cases} S_0^2(x,z,t) = S_0^1(\lambda t)^0 \\ S_1^2(x,z,t) = S_0^1(\lambda t)^1 \\ S_2^2(x,z,t) = S_0^1 \frac{(\lambda t)^2}{2!} \\ \quad \ldots\ldots\ldots\ldots\ldots \\ S_n^2(x,z,t) = S_0^1 \frac{(\lambda t)^k}{k!} \end{cases}$$

By asking

$$\Phi_m^2(x,z,t) = \sum_{k=0}^{m} S_k^2(x,z,t) = S_0^1 \sum_{k=0}^{m} \frac{(\lambda t)^k}{k!} = S_0^1 e^{\lambda t}.$$

Therefore, the approximate solution in Step 2 is:

$$\begin{aligned} S^2(x,z,t) &= \lim_{m \to +\infty} \Phi_m^2(x,z,t) \\ &= S_0^1 \sum_{k=0}^{m} \frac{(\lambda t)^k}{k!} = S_0^1 e^{\lambda t} = S(x,z,t) e^{\lambda t} \end{aligned}$$

Then $S^2(x,z,t) = S_0^1 e^{\lambda t} = S(x,z,t) e^{\lambda t}$.

Recurringly, we obtains:

$$S^1(x,z,t) = S^2(x,z,t) = \cdots = S^k(x,z,t) = S_0^1 e^{\lambda t} = S(x,z,t) e^{\lambda t}$$

Subsequently the exact solution of the model ERM2D by SBA in the variable S is:

$$S(x,z,t) = S_0^1 e^{\lambda t} = S(x,z,0) e^{\lambda t}$$
$$or\ \lambda = \frac{K_s}{C}\ et\ C = \frac{\theta_s(2-m)}{h_g}$$

As $S = \frac{\Psi}{h_g} \Rightarrow \Psi(x,z,t) = h_g S(x,z,t)$ et

$$\Psi(x,z,0) = h_g S(x,z,0)$$

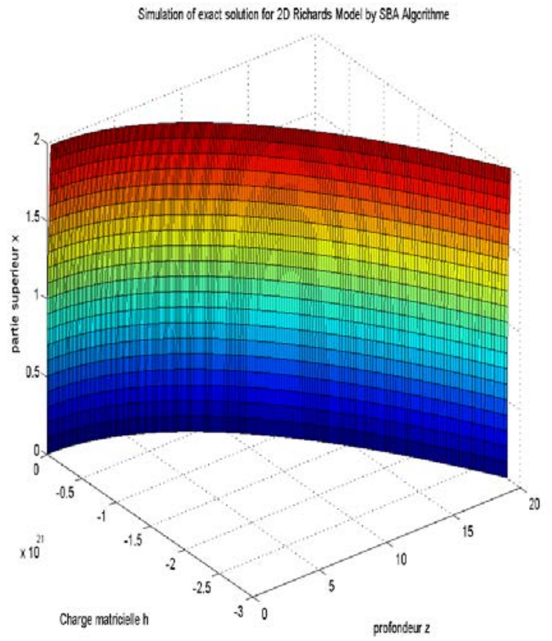
Then $\Psi(x,z,t) = \Psi(x,z,0) e^{\lambda t} = h_g S(x,z,0) e^{\lambda t}$.

Also $\Psi(x,z,t) = |h(x,z,t)| = -h(x,z,t)$; si $h \prec 0$

Finally

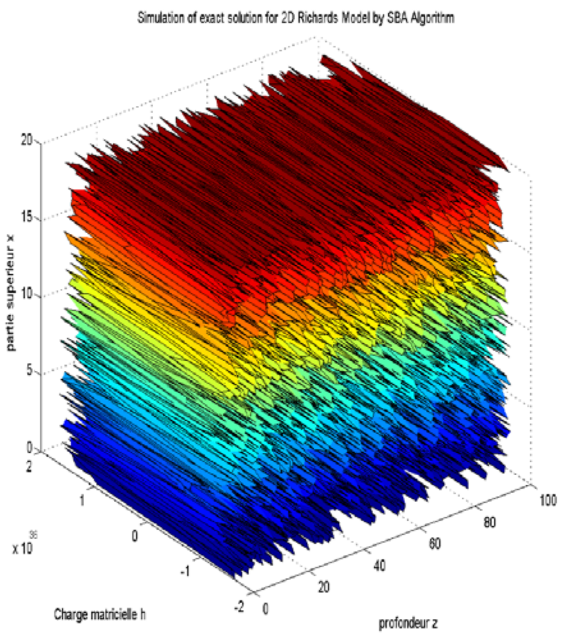
$$h(x,z,t) = -\Psi(x,z,t) = -h_g S(x,z,0)e^{\lambda t} \qquad (23)$$

Simulation:



$$(x,z) \in [0;20] \times [0;100]; t = 170$$

$$S_0(x,z,0) = \cos(-x^3 - z^2)$$



$$(x,z) \in [0;2] \times [0;20]; t = 90$$

$$S_0(x,z,0) = \left|-x^3 - z^2\right|$$

Interpretation:

Through these few simulations, we have noticed that our exact two-dimensional space solution retains all the parameters of the Richards model. These parameters can be simulated in any conditions and according to any type of soil. This solution is also transferable in practical cases after its validation through solutions obtained by numerical methods.

3. Conclusion

Our technique could be extended to the two-way Richards model while retaining all the parameters of the initial model. The exact solution was determined under the conditions of Cauchy Von Neumann and remains valid under other conditions. This approach is original because it avoids the problems often generated by numerical methods. Our solution can be adapted to the problems of localized irrigation in plan flow.

References

[1] B.S. SAKOMA, W.O. SAWADOGO, and B. SOME. Solving Richards' non linear PDE modeling dynamic from water in Unsaturated Zone by a new numerical method called SBA. ijamr. Vol 6. 1-20-29, 2017.

[2] P. Ngnepieba, F.X. Le Dimet, A; Boukong, and G. Nguetseng. Identification des parameters: une application à l'équation de Richards. ARIMA Vol1-127-157, 2002.

[3] B. Abbo, B. Some, Nouvel algorithme numérique de résolution des équations différentielles ordinaires(EDO) et des équations aux dérivées partielles(EDP) non linéaires.119p. Mathématiques appliquées. Ouagadougou : Université de Ouagadougou, 2007.

[4] S. Bisso, B.Some, Résolution et simulation numérique d'un problème de contrôle optimal gouverné par des équations aux dérivées partielles de type diffusion-réaction issues de la modélisation mathématique en traitement du cancer du cerveau. 164p. Mathématiques appliquées. Ouagadougou : Université de Ouagadougou, 2003.

[5] D. Crevoisier, J.C. Mailhol, Modélisation analytique des transferts bi-et tridirectionnels eau-soluté : Application à l'irrigation à la raie et à la micro-irrigation. 263p. Sciences de l'eau. Paris: Ecole Nationale du Génie Rural, des Eaux et Forêts, 2005.

[6] F Lafolie, C. Thirriot, Etude numérique de la résolution des équations de transfert: Application à l'irrigation localisée.260p. Sciences du sol. Avignon: Université d'Avignon, 1986.

[7] Blaise Some. "Optimisation générale et Méthodes numériques" notes de cours DEA / Mathématique. Université de Ouagadougou. 2010.

[8] B.S.SAKOMA/2009/Comparaison de SBA avec la méthode spectrale de Chebyshev et des différences finies pour la résolution des EDO et EDP fortement non linéaires/Mémoire de DEA/ Univ. Ouagadougou.

[9] Y. Pare, Résolution de quelques équations fonctionnelles par la numérique SBA (SOME BLAISE-ABBO), Thèse de Doctorat unique. Université de Ouagadougou, mai 2010, UFR/SEA, Département Mathématiques et Informatique (Burkina Faso).

[10] F. Bassono- Etude de quelques équations fonctionnelles par les méthodes: Some Blaise Abbo, décompositionnelle d'Adomian et perturbations. Université de Ouagadougou, Janvier 2013, UFR/SEA, Département Mathématiques et Informatique (Burkina Faso).