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New Constraction Seven Degree Spline Function to Solve Sencond Order Initial Value Problem

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Abstract Our paper dedicated to find approximate solution of second order initial value problem by seven degree lacunary spline function of type (0, 1, 6). The convergence analysis of given method has studied. Numerical illustrations have given with example for calculating absolute error between spline functions and exact solution of second order initial value problem with their derivatives.

Keywords: *spline functions, second order initial value problem approximation solution, convergence analysis*

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1. Introduction

With the advent of computer, splines have gained more importance. Aryan [2] solved two point boundary value problem by using ninth degree lacunary spline function of the type (0, 1, 3, 5, 7), while Karwan and Aryan [8] have studied second order initial value problem by lacunary spline function of type (0, 2, 4, 5), and Abbas [1] in his topic discussed approximation solution by lacunary interpolation of type (0, 2, 4). Lacunary interpolating by deficient spline of type (0, 1, 3, 5) exhibited by Saeed [11]. Saeed and Jwamer [12] devoted their paper to lacunary interpolation by spline function of type (0, 1, 4). L. L. [9] raised the idea of basic theory spline function. Jwamer and Karim [7] showed sextet lacunary solution of fourth order initial value problem. J. Karwan [6] found approximation solution of second order initial value problem by spline function. Fazal [5] investigates numerical solution of fourth order initial value problem. Splines and differentia equation taked by [4] Gianluca. De Boor spoke on apractical guid to spline [3]. [10] M. K. resolved numerical solution of differential equation.

In our paper we are trying to solve second order initial value problems

$$y'' = f\left(x, y, y'\right), y\left(x_0\right) = y_1 \qquad (1)$$
$$y'\left(x_0\right) = y'_2$$

Where $f \in C^{n-1}[0,1] \times R^2, n \geq 2$ and that it satisfies the Lipschitz condition

$$\left|f^{(r)}\left(x, y_1, y'_1\right) - f^{(r)}\left(x, y_2, y'_2\right)\right|$$
$$\leq L\left(\left|y_1 - y_2\right| + \left|y'_1 - y'_2\right|\right), r = 0,1,2,\ldots,n-1$$

For all $x \in [0,1]$, and all reals $y_1, y_2, \grave{y}_1$ and $\grave{y}_2$, where r=0, 1, 2,.., n-1, and L is lipschitz constant. Constructing lacunary spline of degree seven of the type (0, 1, 6). Existence and uniqueness of spline function of degree seven have discussed, and convergence and error bound have studied. An illustration example used to show the convergence the lacunary spline function to the exact solution, and the error bound numerically calculated .

The lacunary interpolation problem, which we have searched for in this occupation comprised in finding the seven degree spline $S(x)$ of deficiency six, interpolating data has given on the function value and one and sixth derivatives in the interval [0,1].

In the following section spline function of degree seven has offered which interpolates the lacunary data(0, 1, 6). The results concerning existence and uniqueness of the spline function of degree seven are coming in section 3. And convergence and error bounds have studied in section 4. Finally in section 5 the demonstration of the convergence of the particular lacunary spline function, and numerical example have been given.

2. Descriptions of the Method

In order to introduce seven degree spline interpolation for one dimensional and given sufficiently smooth function *f* defined on [0,1]

Let $x_0, x_1, \ldots, x_n$ be n+1 grid points in the interval [0,1] such that $x_i = x_0 + ih$, $x_0 = 0$, $x_n = 1$, i=0, 1,…, n; $h = \frac{1}{n}$ is the distance of each subintervals, so $0 = x_0 < x_1 < x_2 \ldots < x_n = 1$, Is the uniform partition of [0,1]

A seven degree spline interpolation $S(x)$ for one dimensional on the interval $[x_0, x_1]$ defined as

$$S_0(x)=y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2}y_0''$$
$$+\frac{(x-x_0)^3}{6}y_0'''+(x-x_0)^4a_{0,4}+(x-x_0)^5a_{0,5} \quad (2)$$
$$+\frac{(x-x_0)^6}{720}y^{(6)}+(x-x_0)^7a_{0,7}$$

where $a_{0,j}, j=4,5,7$, are unknown to be determined. Let a seven degree spline interpolation $S(x)$ on subintervals $[x_i,x_{i+1}], i=1,2,\ldots,n-2$ which is denoted by $S_i(x)$ regarded as follows:

$$S_i(x)=y_i+(x-x_i)y'_i+(x-x_i)^2a_{i,2}$$
$$+(x-x_i)^3a_{i,3}+(x-x_i)^4a_{i,4}+(x-x_i)^5a_{i,5} \quad (3)$$
$$+\frac{(x-x_i)^6}{720}y^{(6)}+(x-x_i)^7a_{i,7}$$

where $a_{i,j}, i=1,2,\ldots,n-1, j=2,3,4,5,7$ are unknown to be determined.

3. Existence and Uniqueness of the Spline Function

In this section, we are supplying the existence and uniqueness theorem for lacunary spline function of degree seven of the type (0, 1, 6).

Theorem (3.1):

Given the real numbers $y(x_i)$, $y'(x_i)$ and $y^{(6)}(x_i)$ for i=0,1, 2,…,n, then there exist a unique spline of degree seven as given in the equation (2), (3) such that

$$\left.\begin{array}{l} S(x_i)=y(x_i) \\ S^{(r)}(x_i)=y^{(r)}(x_i),\ r=1,6 \end{array}\right\} for\ i=0,1,2,\ldots,n \quad (4)$$
$$and\ S''(x_0)=y''(x_0)\ and\ S'''(x_0)=y'''(x_0)$$

Proof:

The spline function $S(x)$ is defined as follows:

$$S(x)=\begin{cases} S_0(x), \text{ where } x\in[x_0,x_1] \\ S_1(x), \text{ where } x\in[x_i,x_{i+1}]; i=0,1,2,\ldots,n-2 \end{cases}$$

Where the coefficients of these polynomials are to be determined by the following condition

$$\left.\begin{array}{l} S_i(x_{i+1})=S_{i+1}(x_{i+1})=y_{i+1} \\ S_i^{(r)}(x_{i+1})=S_{i+1}^{(r)}(x_{i+1})=y_{i+1}^{(r)}, r=1,6 \\ S_i''(x_{i+1})=S_{i+1}''(x_{i+1})\, and\, S_i'''(x_{i+1})=S_{i+1}'''(x_{i+1}) \end{array}\right\}, \quad (5)$$
$$i=0,1,2,\ldots,n-2$$

$$and\ S_{n-1}(x_n)=y_n, S_{n-1}^{(r)}(x_n)=y_n^{(r)}, r=1,6\ldots \quad (6)$$

To find uniquely the coefficients in $S_0(x)$ of equation (2) by using the condition (5) where $i=0$. Let $h=x_{i+1}-x_i$ we obtain the following

$$h^4a_{0,4}+h^5a_{0,5}+h^7a_{0,7}$$
$$=y_1-y_0-hy_0'-\frac{h}{2}y_0''-\frac{h^3}{6}y_0'''-\frac{h^6}{720}y_0^{(6)} \quad (7)$$

$$h^4a_{0,4}+h^5a_{0,5}+h^7a_{0,7}$$
$$=y_1-y_0-hy_0'-\frac{h}{2}y_0''-\frac{h^3}{6}y_0'''-\frac{h^6}{720}y_0^{(6)} \quad (8)$$

$$5040ha_{0,7}=y_1^{(6)}-y_0^{(6)} \quad (9)$$

From (3) we have

$$S_1(x_1)=y_1, S_1'(x_1)=y'_1,?$$
$$S_1''(x_1)=a_{1,2},\ S_1'''(x_1)=6a_{1,3},?$$
$$S_1^{(4)}(x_1)=24a_{1,4}, S_1^{(5)}(x_1)=120a_{1,5}$$

And from (5) and (6)

$$S_0(x_1)=S_1(x_1)=y_1, S_0'(x_1)=S'_1(x_1)=y',?$$
$$S_0^{(6)}(x_1)=S_1^{(6)}(x_1)=y_1^{(6)}$$

Now since $\begin{bmatrix} 1 & h & h^3 \\ 4 & 5h & 7h^3 \\ 0 & 0 & 1 \end{bmatrix} \neq 0$

Then by solving the above equations (7)-(9),

$$a_{0,4}=\frac{5}{h^4}[y_1-y_0]-\frac{1}{h^3}\left[y'_1+4y_0'\right]$$
$$-\frac{3}{2h^2}y_0''-\frac{1}{3h}y_0'''+\frac{h^2}{5040}\left[2y_1^{(6)}+5y_0^{(6)}\right] \quad (10)$$

$$a_{0,5}=-\frac{4}{h^5}[y_1-y_\circ]+\frac{1}{h^4}\left[y'_1+3y_0'\right]$$
$$+\frac{1}{h^3}y_0''+\frac{1}{6h^2}y_0'''-\frac{h}{5040}\left[3y_1^{(6)}+11y_0^{(6)}\right] \quad (11)$$

$$a_{0,7}=\frac{1}{5040h}\left[y_1^{(6)}-y_0^{(6)}\right] \quad (12)$$

From the boundary condition (6)

$$2a_{1,2}-y_0''-hy_0'''+\frac{h^4}{24}y_0^{(6)}$$
$$=12h^2a_{0,4}+20h^3a_{0,5}+42h^5a_{0,7} \quad (13)$$

$$6a_{1,3}-y_0'''-\frac{h^3}{6}y_0^{(6)}=24ha_{0,4}+60h^2a_{0,5}+210h^4a_{0,7} \quad (14)$$

By substituting these values $(11)-(12)$ in equation (13) and (14) we get

$$a_{1,2}=-\frac{10}{h^2}[y_1-y_0]+\frac{2}{h}\left[2y'_1+3y_0'\right]$$
$$+\frac{h^4}{420}\left[\frac{1}{3}y_1^{(6)}+\frac{1}{3}y_0^{(6)}\right]+\frac{3}{2}y_0''+\frac{h}{6}y_0''' \quad (15)$$

$$a_{1,3}=-\frac{20}{h^3}[y_1-y_0]+\frac{2}{h^2}\left[3y'_1+7y_0'\right]$$
$$+\frac{h^3}{5040}\left[43y_1^{(6)}+15y_0^{(6)}\right]?\frac{4}{h}y_0''+\frac{1}{2}y_0''' \quad (16)$$

Now we are trying to find the coefficients of $S_i(x)$ for $i=1,2,\ldots,n-1$ which defined in equation (3) so we have,

$$a_{i,2}+ha_{i,3}+h^2a_{i,4}+h^3a_{i,5}+h^5a_{i,7} = \frac{1}{h^2}[y_{i+1}-y_i-hy_i^{'}-\frac{h^6}{720}y_i^{(6)}] \quad (17)$$

$$2a_{i,2}+3ha_{i,3}+4h^2a_{i,4}+5h^3a_{i,5}+7h^5a_{i,7} = \frac{1}{h}[y_{i+1}^{'}-y_i^{'}-\frac{h^5}{120}y_i^{(6)}] \quad (18)$$

$$a_{i,7}=\frac{1}{5040h}\left[y_{i+1}^{(6)}-y_i^{(6)}\right] \quad (19)$$

Since $\begin{bmatrix}1 & h & h^3\\ 4 & 5h & 7h^3\\ 0 & 0 & 1\end{bmatrix}\neq 0$

So above system has the following unique solution:

$$a_{i,4}=\frac{5}{h^4}[y_{i+1}-y_i]-\frac{1}{h^3}[y_{i+1}^{'}+4y_i^{'}] +\frac{h^2}{2520}\left[y_{i+1}^{(6)}-\frac{5}{2}y_i^{(6)}\right]-\frac{3}{h^2}a_{i,2}-\frac{2}{h}a_{i,3} \quad (20)$$

$$a_{i,5}=-\frac{4}{h^5}[y_{i+1}-y_i]+\frac{1}{h^4}[y_{i+1}^{'}+3y_i^{'}] -\frac{h}{5040}\left[3y_{i+1}^{(6)}+11y_i^{(6)}\right]+\frac{2}{h^3}a_{i,2}+\frac{1}{h^2}a_{i,3} \quad (21)$$

$$a_{i,7}=\frac{1}{5040h}\left[y_{i+1}^{(6)}-y_i^{(6)}\right] \quad (22)$$

From the fact that $S''_i(x_{i+1})=y''_{i+1}$ and $S'''_i(x_{i+1})=y'''_{i+1}$ we have

$$S''_i(x_{i+1})=y''_{i+1}=2a_{i,2}+6ha_{i,3}+12h^2a_{i,4} +20h^3a_{i,5}+\frac{h^4}{24}y_i^{(6)}+42h^5a_{i,7}$$

$$a_{i,2}-a_{i+1,2}+3ha_{i,3}+6h^2a_{i,4} +10h^3a_{i,5}+21h^5a_{i,7}=-\frac{h^4}{48}y_i^{(6)} \quad (23)$$

$$S'''_i(x_{i+1})=6a_{i+1,3}$$

$$=6a_{i,3}+24ha_{i,4}+60h^2a_{i,5}+\frac{h^3}{6}y_i^{(6)}+210h^4a_{i,7}$$

$$a_{i,3}-a_{i+1,3}+4ha_{i,4}+10ha_{i,5}+35h^4a_{i,7}=-\frac{h^3}{36}y_i^{(6)} \quad (24)$$

Substituting the values of $a_{i,4}, a_{i,5}$ and $a_{i,7}$ in (23) and (24) we get

$$2a_{i,2}-2a_{i+1,2}=-6ha_{i,3}$$

$$-12h^2\left\{\frac{5}{h^4}[y_{i+1}-y_i]-\frac{1}{h^3}[y'_{i+1}+4y'_i] +\frac{h^2}{2520}\left[y_{i+1}^{(6)}+\frac{5}{2}y_i^{(6)}\right]-\frac{3}{h^2}a_{i,2}+\frac{2}{h}a_{i,3}\right\}$$

$$-20h^3\left\{\frac{-4}{h^5}[y_{i+1}-y_i]+\frac{1}{h^4}[y'_{i+1}+3\dot{y}_i] -\frac{h}{5040}\left[3y_{i+1}^{(6)}+11y_i^{(6)}\right]+\frac{2}{h^3}a_{i,2}+\frac{1}{h^2}a_{i,3}\right\}$$

$$-42h^5\left\{\frac{1}{5040h}\left[y_{i+1}^{(6)}-y_i^{(6)}\right]\right\}-\frac{h^4}{24}y_i^{(6)}$$

$$2a_{1,2}-2a_{i+1,2}=-6ha_{i,3}-\frac{60}{h^2}[y_{i+1}-y_i]$$

$$+\frac{12}{h}y_{i+1}^{'}+\frac{48}{h}y_i^{'}-\frac{h^4}{210}y_{i+1}^{(6)}-\frac{1}{84}y_i^{(6)}$$

$$+36a_{i,2}+24ha_{i,3}+\frac{80}{h^2}[y_{i+1}-y_i]-\frac{20}{h}y_{i+1}^{'}$$

$$-\frac{60}{h}y'_i+\frac{3h^4}{252}y_{i+1}^{(6)}+\frac{11h^4}{252}y_i^{(6)}-40a_{i,2}$$

$$-20ha_{i,3}-\frac{h^4}{120}y_{i+1}^{(6)}+\frac{h^4}{120}y_i^{(6)}-\frac{h^4}{24}y_i^{(6)}$$

$$-2a_{i+1,2}+6a_{i,2}+2a_{i,3}$$

$$=\frac{20}{h^3}[y_{i+1}-y_i]-\frac{4}{h}[2y_{i+1}^{'}+3y_i^{'}]-\frac{h^4}{2520}\left[3y_{i+1}^{(6)}+4y_i^{(6)}\right]$$

$$a_{i+1,2}-3a_{i,2}-a_{i,3}=-\frac{10}{h^3}[y_{i+1}-y_i] +\frac{2}{h}\left[2y'_{i+1}+3y'_i\right]+\frac{h^4}{5040}\left[3y_{i+1}^{(6)}+4y_i^{(6)}\right] \quad (25)$$

$$6a_{i,3}-6a_{i+1,3}$$

$$=24h\left\{[y_{i+1}-y_i]-\frac{1}{h^3}[y_{i+1}^{'}+4y_i^{'}] +\frac{h^2}{2520}\left[y_{i+1}^{(6)}+\frac{5}{2}y_i^{(6)}\right]-\frac{3}{h^2}a_{i,2}-\frac{2}{h}a_{i,3}\right\}$$

$$-60h^2\left\{-\frac{4}{h^5}[y_{i+1}-y_i]+\frac{1}{h^4}[y_{i+1}^{'}+3y'_i] -\frac{h}{5040}\left[3y_{i+1}^{(6)}+11y_i^{(6)}\right]+\frac{2}{h^3}a_{i,2}+\frac{1}{h^2}a_{i,3}\right\}$$

$$-210h^2\left\{\frac{1}{5040h}\left[y_{i+1}^{(6)}-y_i^{(6)}\right]\right\}-\frac{h^3}{36}y_i^{(6)}$$

$$=-\frac{120}{h^3}[y_{i+1}-y_i]+\frac{24}{h^2}y_{i+1}^{'}+\frac{96}{h^2}y'_i$$

$$-\frac{h^3}{105}y_{i+1}^{(6)}-\frac{h^3}{42}y_i^{(6)}+\frac{72}{h}a_{i,2}+48a_{i,3}$$

$$+\frac{240}{h^3}[y_{i+1}-y_i]-\frac{60}{h^2}y_i^{'}-\frac{180}{h^2}y_i^{'}$$

$$+\frac{3h^3}{84}y_{i+1}^{(6)}+\frac{11h^3}{84}y_i^{(6)}-\frac{120}{h}a_{i,2}$$

$$-60a_{i,3}-\frac{h^3}{24}y_{i+1}^{(6)}+\frac{h^3}{24}y_i^{(6)}-\frac{h^3}{36}y_i^{(6)}$$

$$-126a_{i,3}-6a_{i+1,3}-\frac{48}{h}a_{i,2}$$

$$=\frac{120}{h^3}[y_{i+1}-y_i]-\frac{12}{h^2}[3y_{i+1}^{'}+7y_i^{'}]$$

$$-\frac{h^3}{2520}\left[39y_{i+1}^{(6)}-94y_i^{(6)}\right]$$

$$21a_{i,3}+a_{i+1,3}+\frac{8}{h}a_{i,2} = -\frac{20}{h^3}[y_{i+1}-y_i]+\frac{2}{h^2}[3y'_{i+1}+7y'_i] -\frac{h^3}{5040}\left[78y_{i+1}^{(6)}-160y_i^{(6)}\right] \quad (26)$$

So the coefficient matrix of the system of equations (15), (16), (25) and (26) for the unknown $a_{i,2}, i=1,2,\ldots,n-1$ is a non-singular matrix and thus the coefficients $a_{i,2}, i=1,2,\ldots,n-1$ are specified uniquely, and consequently the coefficients $a_{i,4}, a_{i,5}$ and $a_{i,7}$. Hence the proof of the theorem has achieved.

4. Convergence and Error Bound

This section, includes studying the error bound of the spline function $S(x)$of degree seven which is defined in section (1) and it is a solution of problem (1).

Theorem4.1:

let $y\in C^7[0,1]$ and $S(x)$ be a unique spline function of degree seven which is the solution of the problem (1). Then for $x\in[x_\circ,x_1]$, the following error bounds are holds

$$s_0^{(r)}(x)-y^{(r)}(x)\le\begin{cases} w_7(f;h) & r=0\\ hw_7(f;h) & r=1\\ \frac{h^2}{6}w_7(f;h) & r=2\\ \frac{5}{21}h^3w_7(f;h) & r=3\\ \frac{11}{168}h^4w_7(f;h) & r=4\\ \frac{103}{2520}h^rw_7(f;h) & r=5,6,7\end{cases}$$

Where $w_7(f;h)$ denotes the modulus of continuity of $y^{(7)}$, where $\|f(x)\|=max\{|f(x)|;0\le x\le 1\}$

Proof:

Let $x\in[x_0,x_1]$ the Taylor's expansion formed about $x=x_0$ for $y\in C^7[0,1]$

$$y(x)=y(x_0)+(x-x_0)y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{6}y'''(x_0)+\frac{(x-x_0)^4}{24}y^{(4)}(x_O)+\frac{(x-x_0)^5}{120}y^{(5)}(x_0)+\frac{(x-x_0)^6}{720}y^{(6)}(x_0)+\frac{(x-x_o)^7}{5040}y^{(7)}(x_0)$$

where $x_0<\gamma<x_1$

First to find $\|s_0^{(7)}(x)-y^{(7)}\|$ from (2)

$$s_0^{(7)}(x)=5040a_{o,7} \quad (27)$$

By using (27) and (12)

$$\left|s_0^{(7)}(x)-y^{(7)}(x)\right|=\left|5040a_{0,7}-y^{(7)}(x)\right| =\left|5040\left\{\frac{1}{5040h}\left[y_1^{(6)}-y_0^{(6)}\right]-y^{(7)}(x)\right\}\right| =\left|\frac{1}{h}\left[y_0^{(6)}+hy_0^{(7)}-y_0^{(6)}-y_{(\gamma)}^{(7)}\right]\right| =\left|y_0^{(7)}-y_{(\gamma)}^{(7)}\right|=w_7(h;f) \quad (28)$$

Since $S^{(r)}(x_i)=y^{(r)}(x_i)$for $r=1,6, i=0,1,2,\ldots,n$ so we have

$$s_0^{(6)}(x_0)=y^{(6)}(x_0)\ or\ s_0^{(6)}(x)-y^{(6)}(x_0)=0 \quad (29)$$

$$\left|s_0^{(6)}(x)-y^{(6)}(x)\right|=\left|\int_{x_0}^{x}(s_0^{(7)}(s)-y^{(7)}(s))ds\right| \le\int_{x_0}^{x}\left|(s_0^{(7)}(s)-y^{(7)}(s))\right| \le ds\int_{x_0}^{x}w_7(h;f)=hw_7(h;f) \quad (30)$$

To find $|s_0^{(5)}(x)-y^{(5)}(x)|$ we need the following

Using Taylor's series expansion on $y^{(5)}(x)$ a bout $x=x_1$

$$y^{(5)}(x)=y^{(5)}(x_0)+(x-x_0)y^{(6)}(x_0)+\frac{(x-x_0)^2}{2}y^{(7)}(x_0) \quad (31)$$

From (2)

$$S_0^{(5)}(x)=120a_{05}+(x-x_0)y^{(6)}(x_0)+2520h^2a_{0,7} \quad (32)$$

Then from (10)-(12) and(31), (32)we get

$$\left|s_0^{(5)}(x)-y^{(5)}(x)\right|=\left|\begin{matrix}120a_{0,5}+(x-x_0)y^{(6)}(x_0)\\+2520a_{0,7}h^2-y^{(5)}(x)\end{matrix}\right| =\left|\frac{7}{42}y_0^{(7)}-\frac{7}{42}y(x_0)\right|\le\frac{h^2}{6}\left|y_0^{(7)}(\alpha_1)-y_0^{(7)}(\alpha_2)\right| \le\frac{h^2}{6}w_7(f;h) \quad (33)$$

where $x_0<\alpha_1<\alpha_2<x_1$.

$$\left|s_0^{(4)}(x)-y^{(4)}(x)\right| =24a_{0,4}+120a_{0,5}+\frac{(x-x_0)^2}{2}y^{(6)}(x_0) +840h^3a_{0,7}-y^{(4)}(x_0)-hy^{(5)}(x_0) -\frac{h^2}{2}y^{(6)}(x_0)-\frac{h^3}{6}y^{(7)}(x_0)\le\frac{5}{21}h^3w_7(f;h) \quad (34)$$

To find $|s_0'''(x_0)-y'''(x_0)|$, from (10)-(12), (2) and Taylor's series expansion on $y'''(x)$ about $x=x_1$

$$\left|s_0'''(x) - y'''(x_0)\right|$$

$$= \left| \begin{array}{l} y_0''' + 24ha_{0,4} + 60h^2 a_{0,5} \\ + \dfrac{(x-x_0)^3}{6} y^{(6)}(x_0) + 210h^4 a_{0,7} \\ -y'''(x_0) - hy^{(4)}(x_0) - \dfrac{h^2}{2} y^{(5)}(x_0) \\ - \dfrac{h^3}{6} y^{(6)}(x) - \dfrac{h^4}{24} y^{(7)}(x_0) \end{array} \right| \quad (35)$$

$$\leq \frac{11h^4}{168} w_7(f;h)$$

To find $|s_0''(x) - y''(x_0)|$ from equations (10) – (12), (2) and Taylor's series $y''(x)$ about $x = x_1$

$$\left|s_0''(x) - y''(x_0)\right|$$

$$= \left| \begin{array}{l} y''_0 + hy_0''' + 1?\ h^2 a_{0,4} + \quad h^3 a_{0,5} \\ + \dfrac{(x-x_0)^4}{24} y^{(6)}(x_0) + 42h^5 a_{0,7} \\ -y''_0(x_0) - hy'''(x_0) - \dfrac{h^2}{2} y^{(4)}(x_0) \\ - \dfrac{h^3}{6} y^{(5)}(x_0) - \dfrac{h^4}{24} y^{(6)}(x_0) \dfrac{h^5}{120} y^{(7)}(x_0) \end{array} \right| \quad (36)$$

$$\leq \frac{103h^5}{2520} \left| y^{(7)}(\gamma_1) - y^{(7)}(\gamma_2) \right| = \frac{103h^5}{2520} w_7(f;h)$$

where $x_\circ < \gamma_1 < \gamma_2 < x_1$.

Since $s^{(r)}(x_i) = y^{(r)}(x_i)$ for r=1,6, i=0,1,…,n, so we have $s'_0(x_0) = y'(x_0)$ or $s'_0(x_0) - y'(x_0) = 0$,and from (36), then

$$\left|s_0'(x) - y'(x)\right| = \left| \int_{x_0}^{x} (S_0''(s) - y''(s)) ds \right|$$

$$\leq \int_{x_0}^{x} \left| \left(S''_0(s) - y''(s) \right) \right| ds \quad (37)$$

$$\leq \int_{x_0}^{x} \frac{103h^5}{2520} w_7(f;h) = \frac{103h^6}{2520} w_7(f;h)$$

Also from (4) and (37) we have $s_0(x_0) - y(x_0) = 0$, then

$$\left|s_0(x) - y_0(x)\right| = \left| \int_{x_0}^{x} (s'_0(t) - y_0'(t)) dt \right|$$

$$\leq \int_{x_0}^{x} \left| (s'\,'_0(t) - y_0(t) \right| dt \quad (38)$$

$$\leq \frac{103}{2520} \int h^6 w_7(f;h) = \frac{103h^7}{2520} w_7(f;h)$$

Lemma 4.1:

Let $y \in C^7[0,1]$, then $\left|e_{i,2}\right| \leq K_i h^5 w_7(f,h)$ for $i = 1,2,\ldots,n-1$ where

$$e_{i,2} = 2a_{i,2} - y_i \quad (39)$$

K_i depend on the numbers of interval

Proof:

For $y \in C^7[0,1]$, then from Taylor's expansion formula, we have

$$y(x) = y(x_i) + (x - x_i) y'(x_i) + \frac{(x-x_i)^2}{2} y''(x_i) + \ldots + \frac{(x-x_i)^7}{5040} y^{(7)}(\theta_i),$$

where $x_i < \theta < x_{i+1}$ and similar expressions for the derivatives for $y(x)$ can be used.

Now if $i = 1$, then from equations (15) and (16) we obtain

$$e_{1,2} = 2a_{1,2} - y_1 = -\frac{h^5}{252} y^{(7)}(\theta_{1,1}) + \frac{h^5}{90} y^{(7)}(\theta_{2,1}) + \frac{h^5}{840} y^{(7)}(\theta_{3,1}) - \frac{h^5}{120} y^{(7)}(\theta_{4,1})$$

where

$$x_i < \theta_{1,1} < \theta_{2,1} < \theta_{3,1} < \theta_{4,1} < x_1$$

$$or \left|e_{1,2}\right| \leq \frac{31}{2520} h^5 w_7(f;h)$$

$$so\ K_1 = \frac{31}{2520}.$$

If $i = 2$, then from equations (15)-(17), and using (39) we obtain

$$\left|e_{2,2}\right| \leq \frac{239}{5040} h^5 w_7(f;h)\ so\ K_2 = \frac{239}{5040}.$$

By the same way aforementioned above and using the step before K_i we can show that the inequality

$$\left|e_{i,2}\right| \leq K_i h^5 w_7(f,h)\ for\ i = 1,2,\ldots,n-1.$$

Hence the proof have completed.

Lemma 4.2:

Let $y \in C^7[0,1]$, then $\left|e_{i,3}\right| \leq K'_i h^4 w_7(f,h)$ for $i = 1,2,\ldots,n-1$

where

$$e_{i,3} = 6a_{i,3} - y'''_i \quad (40)$$

and K'_i depend on the numbers of intervals.

Proof:

For $y \in C^7[0,1]$, then from Taylor's expansion formula, we have

$$y(x) = y(x_i) + (x - x_i) y'(x_i) + \frac{(x-x_i)^2}{2} y''(x_i) + \ldots + \frac{(x-x_i)^7}{5040} y^{(7)}(\varphi_i),$$

where $x_i < \varphi < x_{i+1}$ and similar expressions for the derivatives for $y(x)$ can be used.

Now if $i = 1$, then from equations (16), and (40) we obtain

$$e_{1,3} = 6a_{i,3} - y'''_i = -\frac{120h^4}{5040} y^{(7)}(\varphi_{1,1}) + \frac{36h^4}{720} y^{(7)}(\varphi_{2,1}) + \frac{13h^4}{840} y^{(7)}(\varphi_{3,1}) - \frac{h^4}{24} y^{(7)}(\varphi_{4,1})$$

where $x_i < \varphi_{1,1} < \varphi_{2,1} < \varphi_{3,1} < \varphi_{4,1} < x_1$ then $|e_{1,3}| \le \frac{11}{1680} h^4 w_7(f;h)$ so $K'_1 = \frac{11}{1680}$.

Also if $i = 2$, then from equation (15), (16), and (18) and using (40) we obtain

$$|e_{3,3}| \le \frac{213}{560} h^4 w_7(f,h) \text{ so } K'_2 = \frac{213}{560}.$$

By same way in above and using the step before K'_i we can show that the inequality

$$|e_{i,3}| \le K'_i h^4 w_7(f;h)$$

for $i = 1,2,\dots,n-1$ this completes the proof of the lemma (4.2).

Theorem 4.2:

Let $y \in C^7[0,1]$ and $S(x)$ be a unique spline function of degree seven which is a solution of the problem (4). Then for $x \in [x_i, x_{i+1}]$, $i = 1,2,\dots,n-1$ the following error bounds are holds:

$$s_i^{(r)}(x) - y^{(r)}(x) \le \begin{cases} w_7(f;h) & r=0 \\ hw_7(f;h) & r=1 \\ 120h^2K_iw_7(f;h) + 20h^2K'_i w_7(f;h) + \frac{h^2}{2} w_7(f;h), & r=2 \\ 84h^3K_iw_7(f;h) + 12h^3K'_i w_7(f;h) + \frac{3h^3}{10} w_7(f;h), & r=3 \\ 24h^4K_iw_7(f;h) + 2h^4K'_i w_7(f;h) + \frac{11h^4}{120} w_7(f;h), & r=4 \\ 2h^5K_iw_7(f;h) + \frac{4}{6}h^5K'_i w_7(f;h) + \frac{7h^5}{360} w_7(f;h), & r=5 \\ 2h^6K_iw_7(f;h) + \frac{4}{6}h^6K'_i w_7(f;h) + \frac{7h^6}{360} w_7(f;h), & r=6 \\ 2h^7K_iw_7(f;h) + \frac{4}{6}h^7K'_i w_7(f;h) + \frac{7h^7}{360} w_7(f;h), & r=7 \end{cases}$$

Proof:

Let $x \in [x_i, x_{i+1}]$, $i = 1,2,\dots,n-1$

From equation (2) and Taylor's expansion formula we get

$$s_i^{(7)}(x) = 5040a_{i,7} \tag{41}$$

$|s_i^{(7)}(x) - y^{(7)}(x)| = |5040a_{i,7} - y^{(7)}(x|$, and from (22)

$$\begin{aligned} &= \left|5040\left[\frac{1}{5040h}\left[y_{i+1}^{(6)} - y_i^{(6)}\right] - y^{(7)}(x)\right]\right| \\ &= \left|\frac{1}{h}\left[y_i^{(6)} + h y_i^{(7)} - y_i^{(6)}\right] - y^{(7)}(x)\right| \\ &= \left|y_i^7 - y_i^7(\varphi)\right| \le w_7(f;h) \end{aligned} \tag{42}$$

By (5), $s_i^{(r)}(x_i) = y_i^{(r)}(x_i)$, $r = 1,6, i = 0,1,\dots,n$ so we have

$$s_i^{(6)}(x_i) = y_i^{(6)}(x) \text{ or } s_i^{(6)}(x_i) - y_i^{(6)}(x) = 0$$

$$\begin{aligned} \left|s_i^{(6)}(x_i) - y_i^{(6)}(x)\right| &= \left|\int_{x_i}^{x} (s_i^{(7)}(s) - y_i^{(7)}(s))ds\right| \\ &\le \int_{x_i}^{x} \left|s_i^{(7)}(s) - y_i^{(7)}(s)\right| ds = \int_{x_i}^{x} w_7(f;h) = hw_7(f;h) \end{aligned} \tag{43}$$

To find $|s_i^{(5)}(x_i) - y_i^{(5)}(x)|$ where $y_i^{(5)}(x)$ about $x = x_i$ is

$$y_i^{(5)}(x) = y_i^{(5)}(x_i) + (x - x_i) y_i^{(6)}(x_i) + \frac{(x-x_i)^2}{2} y_i^{(7)}(x_i) \tag{44}$$

And from (3)

$$s_i^{(5)}(x_i) = 120a_{i,5} + (x - x_i) y_i^{(6)}(x_i) + 2520h^2 a_{i,7},$$

then from (21), (22) and (40) we get

$$\begin{aligned} &\left|s_i^{(5)}(x_i) - y_i^{(5)}(x)\right| \\ &= \left|120a_{i,5} + (x-x_i)y_i^{(6)}(x_i) + 2520h^2a_{i,7} - y_i^{(5)}(x_i) - (x-x_i)y_i^{(6)}(x_i) - \frac{(x-x_i)^2}{2} y_i^{(7)}(x_i)\right| \\ &\le \left|120a_{i,5} + (x-x_i)y_i^{(6)}(x_i) - y_i^{(5)}(x_i) - (x-x_i)y_i^{(6)}(x_i)\right| + \left|2520h^2a_{i,7} - \frac{h^2}{2} y_i^{(7)}(x_i)\right| \\ &\le \left|\frac{120}{h^3}(2a_{i,2} - y_i'')\right| + \left|\frac{20}{h^2}(6a_{i,3} - y_i'''\right| + \left|2520h^2a_{i,7} - \frac{h^2}{2} y_i^{(7)}(x_i)\right| \\ &= \frac{120}{h^3}|e_{i,2}| + \frac{20}{h^2}|e_{i,3}| + \left|2520h^2a_{i,7} - \frac{h^2}{2} y_i^{(7)}(x_i)\right| \end{aligned} \tag{45}$$

From lemma (4.1) and (4.2) we obtain

$$\begin{aligned} \left|s_i^{(5)}(x_i) - y_i^{(5)}(x)\right| &\le \frac{120}{h^3} K_i h^5 w_7(f;h) + \frac{20}{h^2} K'_i h^4 w_7(f;h) + \frac{h^2}{2} w_7(f;h) \\ &= 120K_ih^2w_7(f;h) + 20K'_ih^2w_7(f;h) + \frac{h^2}{2} w_7(f;h) \end{aligned} \tag{46}$$

To find $|s_i^{(4)}(x_i) - y_i^{(4)}(x)|$ from Taylor's $y_i^{(4)}(x)$ about $x = x_i$ we get

$$y_i^{(4)}(x) = y_i^{(4)}(x_i) + (x - x_i) y_i^{(5)}(x_i) + \frac{(x-x_i)^2}{2} y_i^{(6)}(x_i) + \frac{(x-x_i)^3}{6} y_i^{(7)}(x_i),$$

and from (3) and (4) , (20) –(22)

$$s_i^{(4)}(x_i) = 24a_{i,4} + 120ha_{i,5} + \frac{h^2}{2} y_i^{(6)}(x_i) + 840h^3 a_{i,7},$$

hence

$$\left| s_i^{(4)}(x_i) - y_i^{(4)}(x) \right| = \left| \begin{array}{l} 24a_{i,4} + 120ha_{i,5} + \frac{h^2}{2} y_i^{(6)}(x_i) \\ +840h^3 a_{i,7} - y_i^{(4)}(x_i) - (x-x_i) y_i^{(5)}(x_i) \\ -\frac{(x-x_i)^2}{2} y_i^{(6)}(x_i) - \frac{(x-x_i)^3}{6} y_i^{(7)}(x_i) \end{array} \right| \le \left| \frac{84}{h^2}(e_{i,2}) + \frac{12}{h}(e_{i,3}) + \frac{3h^3}{10} w_7(f;h) \right|$$

Where

$$\left| 840h^3 a_{i,7} - \frac{(x-x_i)^3}{6} y_i^{(7)}(x_i) \right| \le \frac{h^3}{6} w_7(f;h) \tag{47}$$

$$\le 84h^3 K_i w_7(f;h) + 12h^3 K'_i w_7(f;h) + \frac{3h^3}{10} w_7(f;h)$$

Where $|e_{i,2}| = h^5 k_i w_7(f;h)$, $|e_{i,3}| = h^4 \grave{k}_i w_7(f;h)$.

To find $|s'''_i(x_i) - y'''(x)|$ from Taylor's $y'''(x)$ about $x = x_i$ we get

$$y'''(x) = y'''(x_i) + (x-x_i) y^{(4)}(x_i) + \frac{(x-x_i)^2}{2} y^{(5)}(x_i) + \frac{(x-x_i)^3}{6} y^{(6)}(x_i) + \frac{(x-x_i)^4}{24} y^{(7)}(x_i),$$

and from (3) and (4)

$$s'''_i(x) = y'''(x_i) + 20ha_{i,4} + 60h^2 a_{i,5} + \frac{h^3}{2} y^{(6)}(x_i) + 210h^4 a_{i,7},$$

then from (21)-(23)

$$\left| s'''_i(x) - y'''_i(x) \right| = \left| \begin{array}{l} y'''(x_i) + 20ha_{i,4} + 60h^2 a_{i,5} + \frac{h^3}{2} y^{(6)}(x_i) + 210h^4 a_{i,7} \\ -y'''(x_i) - (x-x_i) y^{(4)}(x_i) - \frac{(x-x_i)^2}{2} y^{(5)}(x_i) \\ -\frac{(x-x_i)^3}{6} y^{(6)}(x_i) - \frac{(x-x_i)^4}{24} y^{(7)}(x_i) \end{array} \right| \le \left| \frac{24}{h}(e_{i,2}) + 2(6a_{i,3} - y_i''') + \frac{11h^4}{120} w_7(f;h) \right|$$

Where

$$\left| 210h^4 a_{i,7} - \frac{h^4}{24} y_i^{(7)} \right| \le \frac{h^4}{24} w_7(f;h)$$

$$\le 24h^4 K_i w_7(f;h) + 2h^4 K'_i w_7(f;h) + \frac{11h^4}{120} w_7(f;h) \tag{48}$$

where $|e_{i,2}| = h^5 K_i w_7(f;h)$, $|e_{i,3}| = h^4 K'_i w_7(f;h)$.

To find $|s''_i(x_i) - y''_i(x)|$ from Taylor's $y''(x)$ about $x = x_i$ we get

$$y(x) = y''_i(x) + (x-x_i) y'''(x_i) + \frac{(x-x_i)^2}{2} y^{(4)}(x_i) + \frac{(x-x_i)^3}{6} y^{(5)}(x_i) + \frac{(x-x_i)^4}{24} y^{(6)}(x_i) + \frac{(x-x_i)^5}{120} y^{(7)}(x_i),$$

and from (3) and (20)-(22)

$$s''_i(x) = y''(x_i) + hy'''(x_i) + 12h^2 a_{i,4} + 20h^3 a_{i,5} + \frac{h^4}{24} y^{(6)}(x_i) + 42h^5 a_{i,7}$$

Hence

$$\left| s''(x) - y''(x) \right| = \left| \begin{array}{l} y''(x) + (x-x_i) y'''(x_i) \\ +12h^2 a_{i,4} + 20h^3 a_{i,5} + \frac{h^4}{24} y^{(6)}(x_i) \\ +42h^5 a_{i,7} - y''(x_i) - (x-x_i) y'''(x_i) \\ -\frac{(x-x_i)^2}{2} y^{(4)}(x_i) - \frac{(x-x_i)^3}{6} y^{(5)}(x_i) \\ -\frac{(x-x_i)^4}{24} y^{(6)}(x_i) - \frac{(x-x_i)^5}{120} y^{(7)}(x_i) \end{array} \right| \le \left| \begin{array}{l} 2(2a_{i,2} - y_i'') + (\frac{-4h}{6})(6a_{i,3} - y'''_i) \\ +\frac{7h^5}{36} w_7(f;h) \end{array} \right|$$

Where $\left| 42h^5 a_{i,7} - \frac{h^5}{120} y_i^{(7)} \right| \le \frac{h^3}{120} w_7(f;h)$.

Hence

$$\left| s''(x) - y''(x) \right| \le 2h^5 k_i w_7(f;h) + \frac{4}{6} h^5 \grave{k}_i w_7(f;h) + \frac{7h^4}{360} w_7(f;h) \tag{49}$$

where $|e_{i,2}| = h^5 k_i w_7(f;h)$, $|e_{i,3}| = h^4 \grave{k}_i w_7(f;h)$.

To find $|s'_i(x) - y'_i(x)|$ from (4) and (49) we have $s'_i(x) - y'(x_i) = 0$ from which we obtain

$$\left|s_i^{'}(x)-y'(x_i)\right|=\left|\int_{x_i}^{x}(s_i^{''}(t)-y''(t))dt\right|$$

$$\le\int_{x_i}^{x}\left|(s_i^{''}(t)-y_i^{''}(t))\right|dt\le 2\int_{x_i}^{x}h^5K_iw_7(f;h) \quad (50)$$

$$+\frac{4}{6}\int_{x_i}^{x}h^5K'w_7(f,h)+\frac{7}{360}\int_{x_i}^{x}h^5w_7(f,h)$$

$$=2h^6K_iw_7(f;h)+\frac{4}{6}h^5K_i^{'}w_7(f;h)+\frac{7}{360}h^5w_7(f;h).$$

To find $|s_i(x)-y(x)|$ from (4) and (50) we have $s_i(x)-y(x_i)=0$ from which we obtain

$$\left|s_i(x)-y_i(x)\right|=\left|\int_{x_i}^{x}(s_i^{'}(t)-y'(t))dt\right|\le\int_{x_i}^{x}\left|s_i^{'}(t)-y^{'(t)}\right|dt$$

$$=2\int_{x_i}^{x}h^6K_iw_7(f;h)dt+\frac{4}{6}\int_{x_i}^{x}h^5K_i^{'}w(f;h)dt+\frac{7}{360}\int_{x_i}^{x}h^6w_7(f;h)$$

$$2h^7K_iw_7(f;h)+\frac{4}{6}h^7K_i^{'}w_7(f;h)+\frac{7}{360}h^7w_7(f;h).$$

Thus the proof has completed for $x\in[x_i,x_{i+1}], i=1,2,\dots,n-1$.

5. Numerical Conclusion

In this section we are performing numerical result to show the convergence of the spline function of degree seven which constructed in section 4 to the second order initial value problem.

Example:

Consider the second order initial value problem

$$y''=4y+12x, y(0)=4, y'(01)=1$$

where x$\in[0,1]$ *with the exact solution*

$$y(x)=3e^{2x}+e^{-2x}-3x.$$

Solution:

Let h=0.1 , n=10.

The following are absolute errors e_i for s(x) and its derivative.

The results of s_0(x) and y(x) and the errors e

x	s_0(x)	y(x)	E
0	4	4	0
0.01	4.023221761455326	4.010802693387022	$1.241906806830305\times 10^{-2}$
0.02	4.052977540396584	4.023221761729488	$2.975577866709558\times 10^{-2}$
0.03	4.089410921147630	4.037274173220328	$5.213674792730286\times 10^{-2}$
0.04	4.132676097802936	4.052977549411511	$7.969854839142376\times 10^{-2}$
0.05	4.182938075201394	4.070350172262902	$1.125879029384910\times 10^{-1}$
0.06	4.240372881521335	4.089410991455285	$1.509618900660507\times 10^{-1}$
0.07	4.305167792726000	4.110179631970488	$1.949881607555124\times 10^{-1}$
0.08	4.377521569102155	4.132676401941642	$2.448451671605134\times 10^{-1}$
0.09	4.457644704148236	4.156922300776703	$3.007224033715329\times 10^{-1}$
0.1	4.545759686082378	4.182939027558492	$3.628206585238867\times 10^{-1}$

The results of s_0'(x) and y'(x) and the errors e

X	s_0'(x)	y'(x)	E
0	1	1	0
0.01	1.323285738895434	1.160810693547024	$1.624750453484092\times 10^{-1}$
0.02	1.653489245157033	1.323285766849683	$3.302034783073504\times 10^{-1}$
0.03	1.991137759303147	1.487490212103660	$5.036475471994871\times 10^{-1}$
0.04	2.336769478127523	1.653489713276480	$6.832797648510434\times 10^{-1}$
0.05	2.690934250283532	`1.821350672381967	$8.695835779015650\times 10^{-1}$
0.06	3.054194285121044	1.991140236041939	$1.063054049079105\times 10^{-1}$
0.07	3.427124875552130	2.162926322345752	1.264198553206378
0.08	3.810315135737784	2.336777648018439	1.473537487719345
0.09	4.204368754404904	2.512763755908317	1.691604998496587
0.1	4.609904764620713	2.690955042805055	1.918949721815658

The results of s_0''(x) and y"(x) and the errors e

x	$s_0''(x)$	$y''(x)$	E
0	16	16	0
0.01	$1.633288693894130\times 10^{1}$	$1.616321077354809\times 10^{1}$	$1.696761653932101\times 10^{-1}$
0.02	$1.669190844809274\times 10^{1}$	$1.633288704691795\times 10^{1}$	$3.590214011747844\times 10^{-1}$
0.03	$1.707763499274600\times 10^{1}$	$1.650909669288131\times 10^{1}$	$5.685382998646865\times 10^{-1}$
0.04	$1.749067686605782\times 10^{1}$	$1.669191019764605\times 10^{1}$	$7.987666684117757\times 10^{-1}$
0.05	$1.793168496731045\times 10^{1}$	$1.688140068905161\times 10^{1}$	1.050284278258835
0.06	$1.840135162662548\times 10^{1}$	$1.707764396582114\times 10^{1}$	1.323707660804343
0.07	$1.890041147703789\times 10^{1}$	$1.728071852788195\times 10^{1}$	1.619692949155937
0.08	$1.942964237489029\times 10^{1}$	$1.749070560776657\times 10^{1}$	1.938936767123721
0.09	$1.998986636956269\times 10^{1}$	$1.770768920310681\times 10^{1}$	2.282177166455882
0.1	$2.058195072360869\times 10^{1}$	$1.793175611023397\times 10^{1}$	2.650194613374725

The results of $s_0'''(x)$ and y'''(x) and thr errors e

x	$s_0'''(x)$	$y'''(x)$	e
0	16	16	0
0.01	$1.729314295558173\times 10^1$	$1.664324277418810\times 10^1$	$6.499001813936368\times 10^{-1}$
0.02	$1.861395698062813\times 10^1$	$1.729314306739873\times 10^1$	1.320813913229401
0.03	$1.996455103721259\times 10^1$	$1.794996084841464\times 10^1$	2.014590188797949
0.04	$2.13470779125100\times 10^1$	$1.861395885310592\times 10^1$	2.733119059404173
0.05	$2.276373700113413\times 10^1$	$1.928540268952787\times 10^1$	3.478334311606260
0.06	$2.421677714048418\times 10^1$	$1.996456094416775\times 10^1$	4.252216196316421
0.07	$2.570849950220852\times 10^1$	$2.065170528938301\times 10^1$	5.056794212825511
0.08	$2.724126054295114\times 10^1$	$2.134711059207375\times 10^1$	5.894149950877379
0.09	$2.881747501761962\times 10^1$	$2.205105502363327\times 10^1$	6.766419993986348
0.1	$3.043961905848285\times 10^1$	$2.276382017122022\times 10^1$	7.675798887262630

The results of $s_0^{(4)}(x)$ and $y^{(4)}(x)$ and the errors e

x	$s_0^{(4)}(x)$	$y^{(4)}(x)$	e
0	64	64	0
0.01	$6.533154775576520\times 10^1$	$6.465284309419236\times 10^1$	$6.787046615728405\times 10^{-1}$
0.02	$6.676763379237094\times 10^1$	$6.533154818767180\times 10^1$	1.436085604699138
0.03	$6.831053997098398\times 10^1$	$6.603638677152524\times 10^1$	2.274153199458746
0.04	$6.996270746523129\times 10^1$	$6.676764079058418\times 10^1$	3.195066673647103
0.05	$7.172673986924178\times 10^1$	$6.752560275620644\times 10^1$	4.201147113035341
0.06	$7.360540650650192\times 10^1$	$6.831057586320644\times 10^1$	5.294830643217371
0.07	$7.560164590815155\times 10^1$	$6.912287411152781\times 10^1$	6.478771796623794
0.08	$7.771856949956116\times 10^1$	$6.996282243106627\times 10^1$	7.755747068494883
0.09	$7.995946547825076\times 10^1$	$7.083075681242724\times 10^1$	9.128708665823528
0.1	$8.232780289443475\times 10^1$	$7.172702444093586\times 10^1$	1.060077845349890

The results of $s_0^{(5)}(x)$ and $y^{(5)}(x)$ and the errors e

x	$s_0^{(5)}(x)$	$y^{(5)}(x)$	e
0	64	64	0
0.01	$6.917257182232693\times 10^1$	$6.657297109675238\times 10^1$	2.599600725574547
0.02	$7.445582792251253\times 10^1$	$6.917257226959492\times 10^1$	5.283255652917606
0.03	$7.985820414885036\times 10^1$	$7.179984339365856\times 10^1$	8.058360755191794
0.04	$8.538831165004036\times 10^1$	$7.445583541242368\times 10^1$	$1.093247623761669\times 10^1$
0.05	$9.105494800453650\times 10^1$	$7.714161075811147\times 10^1$	$1.391333724642504\times 10^1$
0.06	$9.686710856193670\times 10^1$	$7.985824377667102\times 10^1$	$1.700886478526568\times 10^1$
0.07	$1.028339980088341\times 10^{+2}$	$8.260682115753203\times 10^1$	$2.022717685130204\times 10^1$
0.08	$1.089650421718045\times 10^{+2}$	$8.538844236829502\times 10^1$	$2.357659980350952\times 10^1$
0.09	$1.152699000704785\times 10^{+2}$	$8.820422009453307\times 10^1$	$2.706567997594539\times 10^1$
0.1	$1.217584762339314\times 10^{+2}$	$9.105528068488088\times 10^1$	$3.070319554905052\times 10^1$

The results of $s_0^{(6)}(x)$ and $y^{(6)}(x)$ and the errors e

x	$s_0^{(6)}(x)$	$y^{(6)}(x)$	e
0	256	256	0
0.01	$2.613261910230608\times 10^{+2}$	$2.58611372367695\times 10^{+2}$	2.714818646291362
0.02	$2.670705351694838\times 10^{+2}$	$2.61321927506872\times 10^{+2}$	5.744342418796551
0.03	$2.732421598839359\times 10^{+2}$	$2.641455470861010\times 10^{+2}$	9.096612797834984
0.04	$2.798508298569252\times 10^{+2}$	$2.67075631623367\times 10^{+2}$	$1.278026669458841\times 10^1$
0.05	$2.869095947696252\times 10^{+2}$	$2.701024110248258\times 10^{+2}$	$1.680454845214137\times 10^1$
0.06	$2.944216260260077\times 10^{+2}$	$2.732423034531382\times 10^2$	$2.117932257286948\times 10^1$
0.07	$3.024065836326062\times 10^{+2}$	$2.764914964461112\times 10^2$	$2.591508718649500\times 10^1$
0.08	$3.108742779982446\times 10^{+2}$	$2.798512897242651\times 10^2$	$3.102298827397953\times 10^1$
0.09	$3.198378619130031\times 10^{+2}$	$2.833230272497090\times 10^2$	$3.651483466329411\times 10^1$
0.1	$3.293112115777390\times 10^{+2}$	$2.869080977637435\times 10^2$	$4.240311381399560\times 10^1$

The results of $s_0^{(7)}(x)$ and $y^{(7)}(x)$ and the errors e

x	$s_0^{(7)}(x)$	$y^{(7)}(x)$	e
0	256	256	0
0.01	$2.76690287289077\times 10^2$	$2.662918843870095\times 10^2$	$1.039840290229819\times 10^1$
0.02	$2.978233116900501\times 10^2$	$2.766902890783797\times 10^2$	$2.113302261167042\times 10^1$
0.03	$3.194328165954015\times 10^2$	$2.871993735746343\times 10^2$	$3.223344302076718\times 10^1$
0.04	$3.415532466001615\times 10^2$	$2.978233416496947\times 10^2$	$4.372990495046678\times 10^1$
0.05	$3.642197920181460\times 10^2$	$3.085664430324459\times 10^2$	$5.565334898570016\times 10^1$
0.06	$3.874684342477468\times 10^2$	$3.194329751066841\times 10^2$	$6.803545914106273\times 10^1$
0.07	$4.113359920353363\times 10^2$	$3.304272846301281\times 10^2$	$8.090870740520818\times 10^1$
0.08	$4.358601686872182\times 10^2$	$3.415537694731801\times 10^2$	$9.430639921403807\times 10^1$
0.09	$4.610796002819139\times 10^2$	$3.528168803781323\times 10^2$	$1.082627199037816\times 10^2$
0.1	$4.870339049357256\times 10^2$	$3.642211227395235\times 10^2$	$1.228127821962021\times 10^2$

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