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Numerical Simulation of Time-fractional Fourth Order Differential Equations via Homotopy Analysis Fractional Sumudu Transform Method

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Abstract The work provides an incipient analytical technique called the Homotopy Analysis Fractional Sumudu Transform Method (HAFSTM) for solving time-fractional fourth order differential equations with variable coefficients. The HAFSTM is the cumulation of the homotopy analysis method (HAM) and sumudu transform method (STM). The numerical simulation of the proposed method has the sundry applications, it can solve linear and nonlinear boundary value quandaries without utilizing Adomian polynomial, and He's polynomial, which can be considered a clear advantage of this incipient algorithm. The solutions obtained by proposing technique are very lucid and less computationally implementable.

Keywords: *Homotopy Analysis Method, Fractional Sumudu Transform Method, Fractional Partial Differential equation, variable coefficients, boundary value problem*

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1. Introduction

Sumudu integral transform was introduced by Watugala [1] to facilitate the process of solving differential and integral equations in the time domain, and for the utilization in sundry applications of system engineering and applied physics. Albeit the mathematical properties of the Sumudu transform have been explored in some details [2-9], to the best of our cognizance, no systematic derivation of the Sumudu transform is available in the open literature.

The fractional calculus deals with arbitrary orders of derivatives and integrals of applied mathematics. In the last decade, the fractional calculus has found applications in numerous ostensibly diverse fields of science and engineering. Fractional differential equations are increasingly used to model quandaries in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes.

In sundry fields of science and engineering, it is consequential to obtain exact or numerical simulation of the nonlinear partial differential equations. In science and engineering, to find the exact and numerical solution of nonlinear equations is still challenging, therefore it's required lucid methods for finding the exact and approximate solutions. Sundry potent mathematical methods such as Adomian decomposition method (ADM) [10,11,12,13,14], homotopy perturbation method (HPM) [15-20], homotopy analysis method (HAM) [21,22,23,24,25], variational iteration method (VIM) [26-32], Laplace decomposition method (LDM) [33,34,35], homotopy perturbation transform method (HPTM) [36], homotopy perturbation sumudu transform method (HPSTM) [37] and homotopy analysis transform method (HATM) [38,39,40] have been proposed to obtain exact and approximate analytical solutions of nonlinear equations.

Inspired by all these perpetual research, we propose HAFSTM for the solution of fourth order differential equations with variable coefficient.

2. Sumudu Transform

In early 90's, Watugala [1] introduced an incipient integral transform, designated the sumudu transform and applied it to the solution of mundane differential equation in control engineering quandaries. The sumudu transform is defined over the set of functions

$$A=\left\{ f(t) \,\middle|\, \begin{array}{l} \exists M,\tau_1,\tau_2>0, \left|f(t)\right|<Me^{\frac{|t|}{\tau_j}}, \\ if\ t\in(-1)^j\times[0,\infty) \end{array} \right\},$$

by the following formula

$$\bar{f}(u)=\mathbb{S}\left[f(t)\right]=\int_0^\infty f(ut)e^{-t}\,dt, u\in(-\tau_1,\tau_2).$$

Weerakoon [2,3,4] was established the properties of mentioned transform. Further fundamental properties of this transform were also established by Asiru [4]. In the same way, this transform was applied to the unidimensional neutron convey equation in [5] by Kadem. In fact, it was shown that there is a vigorous relationship between Sumudu and other integral transforms refer to Kılıçman et al. [6]. In particular the relation between Sumudu transform and Laplace transforms was proved in Kılıçman and Gadain [7]. Further, in Eltayeb et al. [9], the Sumudu transform was elongated to the distributions and some of their properties were furthermore studied in Kılıc¸man and Eltayeb [9]. Recently, this transform is applied to solve the system of differential equations [41,42,43].

3. Basic Definition of Fractional Calculus

Definition 3.1 A real function $f(t), t>0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(>\mu)$, such that $f(t)=t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu, m \in N$.

Definition 3.2 The Riemann Liouville Fractional integral operator of order $\alpha \geq 0$, of a function $f(t) \in C_\mu$, and $\mu \geq -1$ is defined as [44,45]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha>0,\ x>0$$

and

$$J^0 f(t) = f(t).$$

For the Riemann – Liouville fractional integral, we have

$$J^\alpha t^y = \frac{\Gamma(y+1)}{\Gamma(y+\alpha+1)} t^{\alpha+y}.$$

Definition 3.3 The fractional derivative of $f(t)$ in the Caputo sense is defined as [46]

$$D_t^\alpha f(t) = \begin{cases} J^{m-\alpha} D^n f(t), \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau, \end{cases}$$

where $m-1<\alpha \leq m,\ m \in N,\ t>0$.

For the Riemann – Liouville fractional derivative, we obtain the following relation.

$$J_t^\alpha \left[D_t^\alpha f(t) \right] = u^{-\alpha} \mathcal{S}\left[f(t) \right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}\left(0^+\right),$$

where $m-1<\alpha \leq m$.

4. Solution by Homotopy Analysis Fractional Sumudu Transform Method

To illustrate the rudimental conception of the HAFSTM for the fractional partial differential equation, we consider the following fractional partial differential equation as

$$D_t^{n\alpha} U(x,t) + R[x] U(x,t) + N[x] U(x,t) = G(x,t); \quad (4.1)$$
$$t>0, x \in R, n-1<\alpha \leq n,$$

where $D_t^{n\alpha} = \dfrac{\partial^{n\alpha}}{\partial x^{n\alpha}}$, $R[x]$ is the linear operation in x, $N[x]$ is the general nonlinear operation in x and $G(x,t)$ is a continuous function.

All initial and boundary conditions are ignored for minimalism, which can be treated in a homogeneous way now applying the Sumudu transform first on both sides of the equation (4.1), we get

$$\begin{aligned} &\mathcal{S}\left[\mathrm{D}_{\mathrm{t}}^{\mathrm{n\alpha}} \mathrm{u}(\mathrm{x,t}) \right] + \mathcal{S}\left[\mathrm{R}[\mathrm{x}] U(\mathrm{x,t}) \right] \\ &+ \mathcal{S}\left[N[\mathrm{x}] U(\mathrm{x,t}) \right] = \mathcal{S}\left[G(\mathrm{x,t}) \right]; \\ &t>0, \mathrm{x} \in \mathbb{R}, n-1<\alpha \leq n, \end{aligned} \quad (4.2)$$

Using the differentiation property of the Sumudu transform

$$\begin{aligned} &\frac{\mathcal{S}\left[U(\mathrm{x,t}) \right]}{u^\alpha} - \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} + \mathcal{S}\left[\mathrm{R}[\mathrm{x}] U(\mathrm{x,t}) \right] \\ &+ \mathcal{S}\left[N[\mathrm{x}] U(\mathrm{x,t}) \right] - \mathcal{S}\left[G(\mathrm{x,t}) \right] = 0, \\ &\mathcal{S}\left[U(\mathrm{x,t}) \right] - u^\alpha \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} \\ &+ u^\alpha \mathcal{S} \begin{bmatrix} \mathrm{R}[\mathrm{x}] U(\mathrm{x,t}) \\ + N[\mathrm{x}] U(\mathrm{x,t}) - G(\mathrm{x,t}) \end{bmatrix} = 0. \end{aligned} \quad (4.3)$$

we define nonlinear operator as

$$\begin{aligned} N\left[\phi(\mathrm{x,t;q}) \right] = \mathcal{S}\left[\phi(\mathrm{x,t;q}) \right] - u^\alpha \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{u^{(\alpha-k)}} \\ + u^\alpha \mathcal{S} \begin{bmatrix} \mathrm{R}[\mathrm{x}] \phi(\mathrm{x,t;q}) \\ + N[\mathrm{x}] \phi(\mathrm{x,t;q}) - G(\mathrm{x,t;q}) \end{bmatrix} \end{aligned} \quad (4.4)$$

where $q \in [0,1]$ be an embedding parameter and $\phi(\mathrm{x,t;q})$ is a real function of $\mathrm{x,t}$ and q. we construct a homotopy as follow:

$$\begin{aligned} &(1-\mathrm{q}) \mathcal{S}\left[\phi(\mathrm{x,t;q}) - \mathrm{U}_0(\mathrm{x,t}) \right] \\ &= \hbar \mathrm{q} \mathrm{H}(\mathrm{x,t}) N\left[\phi(\mathrm{x,t;q}) \right] \end{aligned} \quad (4.5)$$

where $\hbar$ is a nonzero auxiliary parameter and $\mathrm{H}(\mathrm{x,t}) \neq 0$. An auxiliary function $\mathrm{U}_0(\mathrm{x,t})$ is an initial guess of $\mathrm{U}(\mathrm{x,t})$ and $\phi(\mathrm{x,t;q})$ is an unknown function. It is important that one has great freedom to choose auxiliary parameter in HAFSTM. Obviously, when $q=0$ and $q=1$ it holds

$$\phi(\mathrm{x,t;0}) = U_0(\mathrm{x,t}), \qquad \phi(\mathrm{x,t;1}) = U(\mathrm{x,t}) \quad (4.6)$$

Thus, as q increases from 0 to 1,the solution varies from initial guess $U_0(\mathrm{x,t})$ to the solution $U(\mathrm{x,t})$. Now, expanding $\phi(\mathrm{x,t;q})$ on Taylor's series with respect to q, we get

$$\phi(x,t;q) = U_0(x,t) + \sum_{m=1}^{\infty} q^m U_m(x,t) \qquad (4.7)$$

where

$$U_m(x,t) = \frac{1}{\lfloor m} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0} \qquad (4.8)$$

The convergence of the series solution (4.7) is controlled by $\hbar$. If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are properly chosen, the series (4.7) converges at $q = 1$. Hence we obtain

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t) \qquad (4.9)$$

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial guess $U_0(x,t)$ and the exact solution $U(x,t)$ by means of the terms $U_m(x,t)(m = 1,2,3,...)$, which are still to be determined.

Define the vectors

$$\vec{U} = \{U_0(x,t), U_1(x,t), U_2(x,t), ..., U_m(x,t)\}. \qquad (4.10)$$

Differentiating the zero order deformation Eq. (4.5) m times with respect to embedding parameter q and then setting $q = 0$, and finally dividing them by $m!$, we obtain the m^{th} order deformation equation as follows:

$$\begin{aligned} &\mathbb{S}\left[U_m(x,t) - \chi_m U_{m-1}(x,t)\right] \\ &= \hbar H(x,t) R_m\left(\vec{U}_{m-1}, x, t\right). \end{aligned} \qquad (4.11)$$

Operating the inverse Sumudu transform of both sides, we get

$$\begin{aligned} &U_m(x,t) = \chi_m U_{m-1}(x,t) \\ &+\hbar \mathbb{S}^{-1}\left[H(x,t) R_m\left(\vec{U}_{m-1}, x, t\right)\right], \end{aligned} \qquad (4.12)$$

where

$$R_m\left(\vec{U}_{m-1}, x, t\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\phi(x,t;q)}{\partial q^{m-1}}\bigg|_{q=0} \qquad (4.13)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1 & m > 1. \end{cases}$$

In this way, it is easy to obtain $U_m(x,t)$ for $m \ge 1$, at M^{th} order, we have

$$U(x,t) = \sum_{m=0}^{M} U_m(x,t), \qquad (4.14)$$

where $M \to \infty$, we obtain an accurate approximation of the original equation (4.1).

5. Numerical Examples

In this section, we apply the HAFSTM developed in Section 4 to solve one and two dimensional initial boundary value quandaries with variable coefficients. The methods may additionally be applicable for higher dimensional spaces. Numerical results reveal that the HAFSTM is facile to implement and reduce the computational work to a tangible level while still maintaining a higher caliber of precision. All the results for the following three applications are calculated by utilizing the symbolic calculus software MATHEMATICA 8.0.

Example 5.1: Consider the following problem of one-dimensional time-fractional fourth-order PDE [47]

$$\begin{aligned} &D_t^{\alpha} U(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U(x,t)}{\partial x^4} = 0, \\ &\frac{1}{2} < x < 1;\ t > 0,\ 1 < \alpha \le 2, \end{aligned} \qquad (5.1)$$

subject to the initial and boundary conditions

$$\left.\begin{aligned} &U(x,0) = 0, \frac{\partial U(x,t)}{\partial t} = 1 + \frac{x^5}{120}; \\ &U\left(\frac{1}{2}, t\right) = \left(1 + \frac{0.5^5}{120}\right)\sin(t,\alpha); \\ &\frac{\partial^2 U}{\partial x^2}\left(\frac{1}{2}, t\right) = \frac{1}{6}\frac{1}{2^3}\sin(t,\alpha); \\ &U(1,t) = \frac{121}{120}\sin(t,\alpha); \\ &\frac{\partial^2 U}{\partial x^2}(1,t) = \frac{1}{6}\sin(t,\alpha); \end{aligned}\right\} \qquad (5.2)$$

where the function $\sin(t,\alpha)$ is defined as

$$\sin(t,\alpha) = \sum_{i=0}^{\infty} \frac{(-1)^i t^{i\alpha+1}}{\Gamma(i\alpha+2)}. \qquad (5.3)$$

We start with initial condition

$$U_0(x,t) = U(x,0) + tU_t(x,0) = \left(1 + \frac{x^5}{120}\right)t. \qquad (5.4)$$

Operating the Sumudu transform of both sides in Eq. (5.1) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$\mathbb{S}\left[D_t^{\alpha} U(x,t)\right] + \mathbb{S}\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U(x,t)}{\partial x^4}\right] = 0, \qquad (5.5)$$

or

$$\mathbb{S}\left[U(x,t)\right] + u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 U(x,t)}{\partial x^4}\right] = 0. \qquad (5.6)$$

The nonlinear operator is

$$\begin{aligned} &N\left[\phi(x,t;q)\right] = \mathbb{S}\left[\phi(x,t;q)\right] \\ &+u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 \phi(x,t;q)}{\partial x^4}\right] \end{aligned} \qquad (5.7)$$

and thus

$$R_m\left(\vec{U}_{m-1}\right) = \mathbb{S}\left[U_{m-1}(x,t)\right] + u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x}+\frac{x^4}{120}\right)\frac{\partial^4 U_{m-1}(x,t)}{\partial x^4}\right]. \quad (5.8)$$

The $m^{th}-$ order deformation equation is given by

$$\mathbb{S}\left[U_m(x,t)-\chi_m U_{m-1}(x,t)\right] = \hbar H(x,t) R_m\left(\vec{U}_{m-1}(x,t)\right).$$

Applying the inverse Sumudu transform, we have

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \mathbb{S}^{-1}\left[\hbar H(x,t) R_m\left(\vec{U}_{m-1}(x,t)\right)\right]. \quad (5.9)$$

On solving above equation from $m=1,2,\ldots,$ we get

$$U_1(x,t) = \hbar\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad (5.10)$$

$$U_2(x,t) = \hbar\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \hbar^2\left(1+\frac{x^5}{120}\right)\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right], \quad (5.11)$$

$$U_3(x,t) = \hbar\left(1+\frac{x^5}{120}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2\hbar^2\left(1+\frac{x^5}{120}\right)\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + 2\hbar^3\left[\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right], \quad (5.12)$$

and so on, in the same manner the rest of the components of the series $m \geq 4$ can be obtained.

Finally, the solution of (5.1) is given as

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t). \quad (5.13)$$

According to Liao [24], the accuracy and convergence of the HAM series solution depends on the careful selection of the auxiliary parameter $\hbar$. Here we choose $\hbar=-1,$ then

$$U(x,t) = \left(\begin{array}{l} t-\dfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}+\dfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ -\dfrac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\ldots+(-1)^n\dfrac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}-\ldots \end{array}\right). \quad (5.14)$$

For $\hbar=-1,$ the above expressions are exactly the same as those given by the Decomposition method by N. A. Khan [47].

Nevertheless, most of the results given by the ADM, LDM, HPM and HPTM converge to the corresponding numerical solutions in a rather small region. Except, different from these methods, the HAFSTM provides us through a simple way to alter and manage the convergence region of solution series by choosing a proper value for the auxiliary parameter $\hbar$. So, the valid region for $\hbar$, where the series converge is the horizontal segment of each $\hbar$ curve. When we choose $\alpha=2,$ then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty} U_m(x,t)$ converges to the exact solution $U(x,t)=\sin t\left(1+\frac{x^5}{120}\right)$ obtained by Wazwaz [45,46].

Example 5.2: Consider the following problems of one-dimensional time-fractional fourth-order PDE [47]

$$D_t^{\alpha}U(x,t)+\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 U(x,t)}{\partial x^4}=0; \quad (5.15)$$
$$0<x<1,\ t>0, 1<\alpha\leq 2,$$

subject to the initial and boundary conditions

$$\left.\begin{array}{l} U(x,0)=x-\sin x; \\ \dfrac{\partial U}{\partial t}(x,0)=-x+\sin x; \\ U(0,t)=0; \\ \dfrac{\partial^2 U}{\partial x^2}(0,t)=0; \\ U(1,t)=Exp(t,\alpha)(1-\sin 1); \\ \dfrac{\partial^2 U}{\partial x^2}(1,t)=Exp(t,\alpha)\sin 1; \end{array}\right\} \quad (5.16)$$

where the function $Exp(t,\alpha)=(-1)^i\dfrac{t^{i\alpha/2}}{\Gamma\left(\frac{\alpha}{2}+1\right)}.$

We start with initial condition

$$U_0(x,t)=U(x,0)+tU_t(x,0)=(1-t)(x-\sin x).$$

Operating the Sumudu transform on both sides in (5.15) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$\mathbb{S}\left[D_t^{\alpha}U(x,t)\right]+\mathbb{S}\left[\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 U(x,t)}{\partial x^4}\right]=0. \quad (5.17)$$

Applying the initial and boundary conditions of (5.15) from (5.16) to (5.17), we obtain

$$\mathbb{S}\left[U(x,t)\right]-(1-\chi_m)(1-u)(x-\sin x) + u^{\alpha}\mathbb{S}\left[\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 U(x,t)}{\partial x^4}\right]=0.$$

The nonlinear operator is

$$N\left[\phi(x,t;q)\right]=\mathbb{S}\left[\phi(x,t;q)\right]-(1-u)(x-\sin x) + u^{\alpha}\mathbb{S}\left[\left(\frac{x}{\sin x}-1\right)\frac{\partial^4\phi(x,t;q)}{\partial x^4}\right] \quad (5.18)$$

and thus

$$R_m\left[U_{m-1}(x,t)\right]=\mathcal{S}\left[U_{m-1}(x,t)\right]-(1-u)(x-\sin x)$$
$$+u^{\alpha}\mathcal{S}\left[\left(\frac{x}{\sin x}-1\right)\frac{\partial^4 U_{m-1}(x,t)}{\partial x^4}\right]. \tag{5.19}$$

The m^{th} - order deformation equation

$$\mathcal{S}\left[U_m(x,t)-\chi_m U_{m-1}(x,t)\right]$$
$$=\hbar H(x,t)R_m\left(\vec{U}_{m-1}(x,t)\right).$$

Applying the inverse Sumudu transform,

$$U_m(x,t)=\chi_m U_{m-1}(x,t)$$
$$+\mathcal{S}^{-1}\left[\hbar H(x,t)R_m\left(\vec{U}_{m-1}(x,t)\right)\right] \tag{5.20}$$

On solving above equation from $m=1,2,\ldots,$ we get

$$U_1(x,t)=\hbar(x-\sin x)\left[\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right], \tag{5.21}$$

$$U_2(x,t)$$
$$=\hbar(x-\sin x)\left[\begin{array}{l}\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\\ +\hbar\left(\begin{array}{l}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{array}\right)\end{array}\right], \tag{5.22}$$

$$U_3(x,t)$$
$$=\hbar(x-\sin x)\left[\begin{array}{l}\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\\ +\hbar\left(\begin{array}{l}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{array}\right)\end{array}\right]$$
$$+\hbar(x-\sin x)\left[\hbar^2\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\right.$$
$$+\hbar^3\left(\begin{array}{l}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\ -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\end{array}\right)$$
$$-\hbar^2\left(\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)$$
$$\left.+\hbar^3\left(\begin{array}{l}\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}-\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ -\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\end{array}\right)\right], \tag{5.23}$$

and so on, in the same manner the rest of the components of the series $m\geq 4$ can be obtained.

Finally, the solution of Eq. (5.15) is given as

$$U(x,t)=U_0(x,t)+\sum_{m=1}^{\infty}U_m(x,t). \tag{5.24}$$

According to Liao [24], the accuracy and convergence of the HAM series solution depends on the careful selection of the auxiliary parameter $\hbar$, here, we choose $\hbar=-1,$ then

$$U(x,t)=\left(\begin{array}{l}1-t+\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\ +\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\ +\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}+\ldots\end{array}\right)(x-\sin x). \tag{5.25}$$

For $\hbar=-1,$ the above expressions are exactly the same as those given by the Decomposition method by N. A. Khan [47].

When we choose $\alpha=2,$ then clearly, we can conclude that the obtained solution $\sum_{m=0}^{\infty}U_m(x,t)$ converge to the exact solution $U(x,t)=e^{-t}\sin t$ obtained by Wazwaz [48,49], and Biazar and Ghavini [50].

Example 5.3: Consider the following problem of two-dimensional time-fractional fourth-order PDE [47]

$$D_t^{\alpha}U(x,y,t)+2\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4 U(x,y,t)}{\partial x^4}$$
$$+2\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4 U(x,y,t)}{\partial x^4}=0, \tag{5.26}$$
$$\frac{1}{2}<x,y<1,\ t>0,1<\alpha\leq 2,$$

subject to the initial and boundary conditions:

$$\left.\begin{array}{l}U(x,y,0)=0;\\ \frac{\partial U}{\partial t}(x,y,0)=2+\frac{x^6}{6!}+\frac{y^6}{6!};\\ U\left(\frac{1}{2},y,t\right)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin(t,\alpha);\\ U(1,y,t)=\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin(t,\alpha);\\ \frac{\partial^2 U}{\partial x^2}\left(\frac{1}{2},y,t\right)=\frac{0.5^4}{24}\sin(t,\alpha);\\ U(1,y,t)=\frac{1}{24}\sin(t,\alpha).\end{array}\right\} \tag{5.27}$$

we start with initial condition

$$U_0(x,y,t)=U(x,y,0)+tU_t(x,y,0)$$
$$=t\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right).$$

Operating the Sumudu transform on both sides in (5.26) and after using the differentiation property of Sumudu transform for fractional derivative, we get

$$\begin{aligned}&\mathbb{S}\left[D_t^{\alpha}U(x,y,t)\right]\\&+2\mathbb{S}\left[\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4U(x,y,t)}{\partial x^4}\right]\\&+2\mathbb{S}\left[\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4U(x,y,t)}{\partial y^4}\right]=0.\end{aligned}$$

Applying the initial and boundary conditions of equation (5.26) from equation (5.27), we obtain

$$\begin{aligned}&\mathbb{S}\left[U(x,y,t)\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4U(x,y,t)}{\partial x^4}\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4U(x,y,t)}{\partial y^4}\right]=0.\end{aligned}\tag{5.28}$$

The nonlinear operator is

$$\begin{aligned}&N\left[\phi(x,y,t;q)\right]=\mathbb{S}\left[\phi(x,y,t;q)\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4\phi(x,y,t;q)}{\partial x^4}\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4\phi(x,y,t;q)}{\partial y^4}\right],\end{aligned}\tag{5.29}$$

and thus

$$\begin{aligned}&R_m\left[U_{m-1}(x,y,t)\right]\\&=\mathbb{S}\left[U_{m-1}(x,y,t)\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{x^2}+\frac{x^4}{6!}\right)\frac{\partial^4U_{m-1}(x,y,t)}{\partial x^4}\right]\\&+2u^{\alpha}\mathbb{S}\left[\left(\frac{1}{y^2}+\frac{y^4}{6!}\right)\frac{\partial^4U_{m-1}(x,y,t)}{\partial y^4}\right].\end{aligned}\tag{5.30}$$

The m^{th} - order deformation equation

$$\begin{aligned}&\mathbb{S}\begin{bmatrix}U_m(x,y,t)\\-\chi_mU_{m-1}(x,y,t)\end{bmatrix}\\&=\hbar H(x,t)R_m\left(\vec{U}_{m-1}(x,y,t)\right).\end{aligned}$$

Applying the inverse Sumudu transform,

$$\begin{aligned}&U_m(x,t)=\chi_mU_{m-1}(x,t)\\&+\mathbb{S}^{-1}\left[\hbar H(x,t)R_m\left(\vec{U}_{m-1}(x,t)\right)\right].\end{aligned}\tag{5.31}$$

On solving above equation from $m=1,2,\ldots$ we get

$$U_1(x,y,t)=\hbar\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\tag{5.32}$$

$$\begin{aligned}&U_2(x,y,t)\\&=\hbar\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^2\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\&+\hbar^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},\end{aligned}\tag{5.33}$$

$$\begin{aligned}U_3(x,y,t)=&\hbar\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^2\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\&+\hbar^2\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^4\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\&+\hbar^2\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^3\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&+\hbar^4\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)},\end{aligned}\tag{5.34}$$

and so on, in the same manner the rest of the components of the series $m\geq 4$ can be obtained.

Finally, the solution of Eq. (5.26) is given as

$$U(x,t)=U_0(x,t)+\sum_{m=1}^{\infty}U_m(x,t).\tag{5.35}$$

The accuracy and convergence of the HAM series solution depend on the careful selection of the auxiliary parameter $\hbar$, here, we choose $\hbar=-1$, them

$$U(x,y,t)=\begin{pmatrix}t-\dfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\+\dfrac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\\-\dfrac{t^{3\alpha+1}}{\Gamma(3\alpha+1)}+\ldots\end{pmatrix}\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right).\tag{5.36}$$

For $\hbar=-1$, the above expressions are exactly the same as those given by the Decomposition method by N. A. Khan [47].

6. Numerical Results and Discussion

Example 1

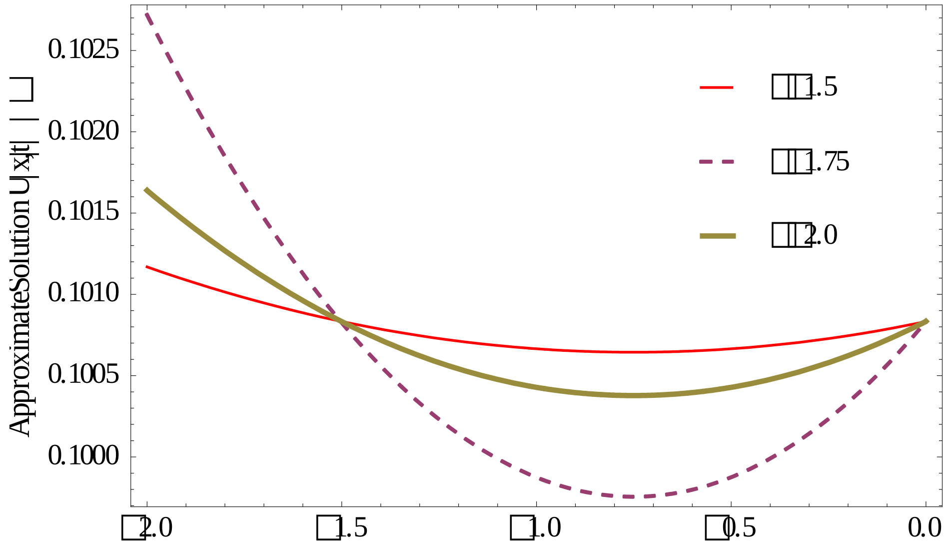

Figure 1. Plot of $U(x,t)$ w.r. to ħ at $t=0.1$ and $x=1$.

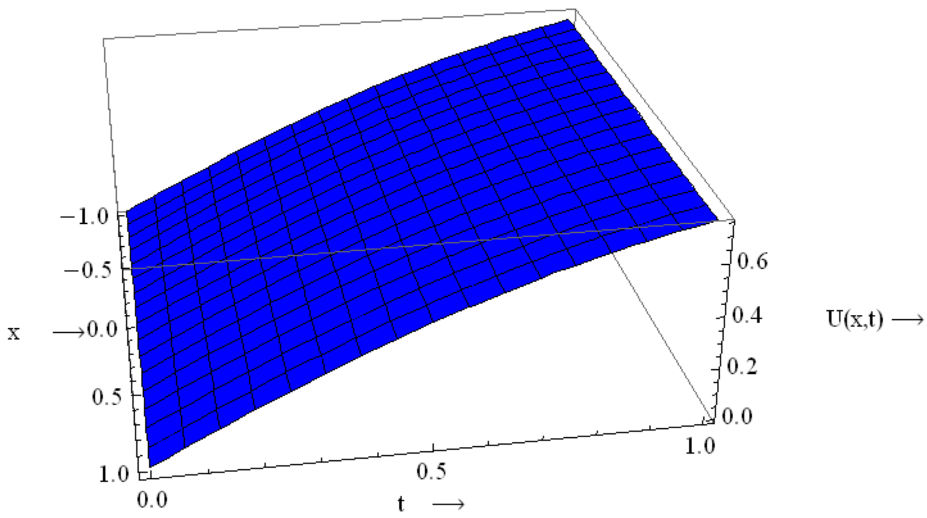

Figure 2. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.5$.

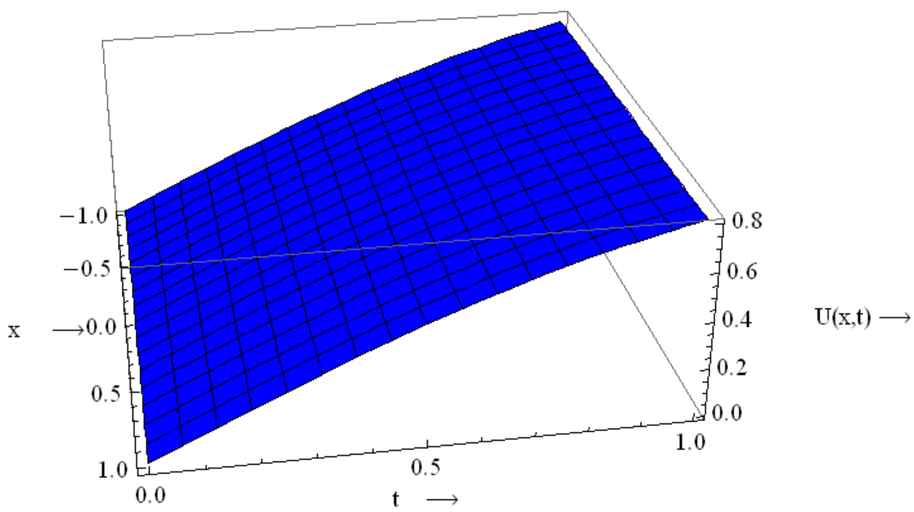

Figure 3. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.75$.

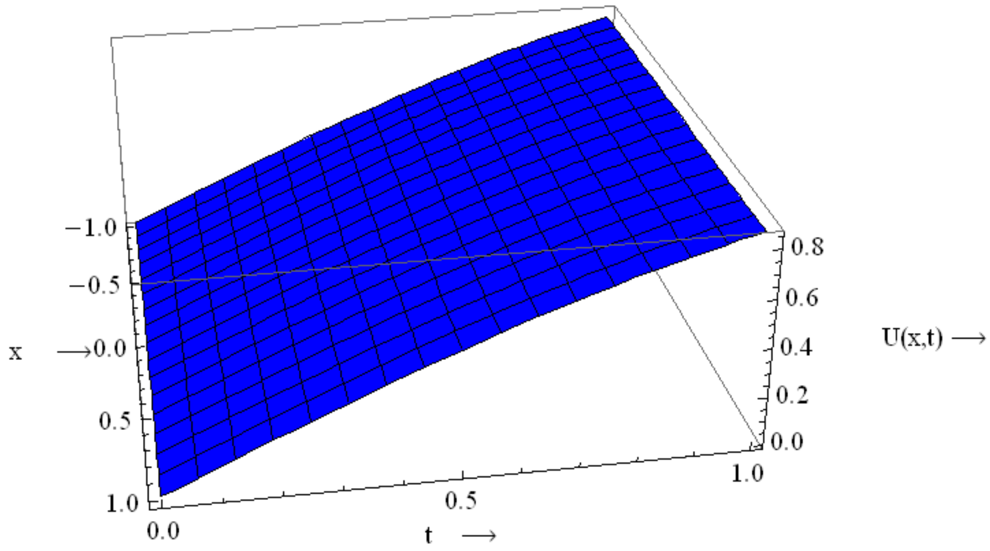

Figure 4. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 2.0$.

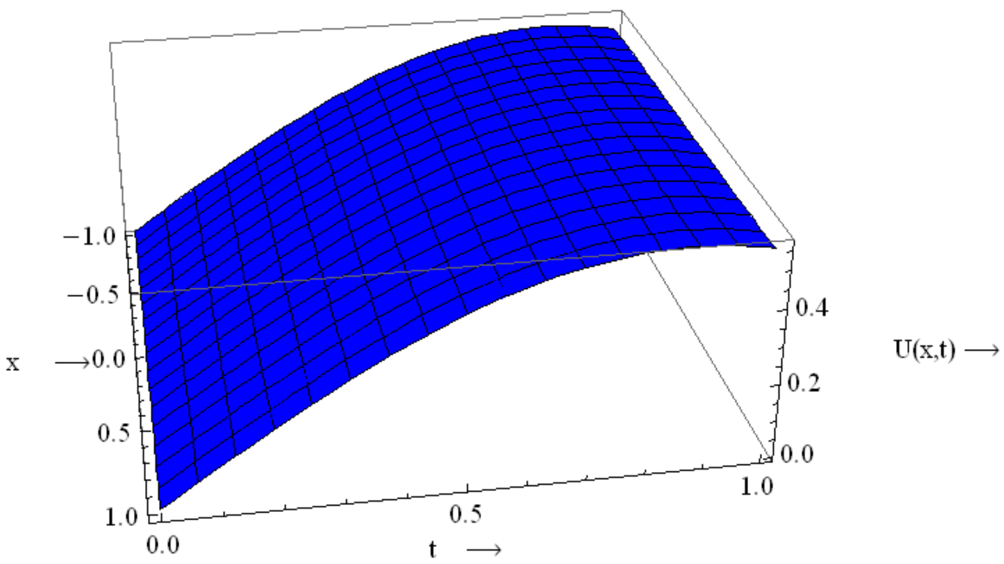

Figure 5. Plot of Exact Solution of $U(x,t)$ w.r.t x and t.

Example 2

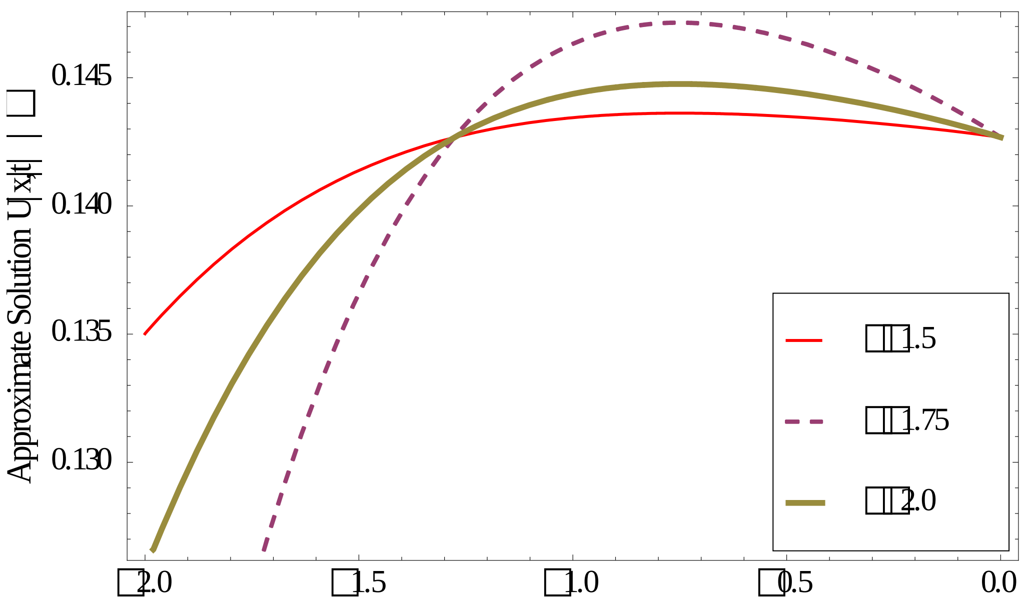

Figure 6. Plot of $U(x,t)$ w.r. to ħ at $t = 0.1$ and $x = 1$.

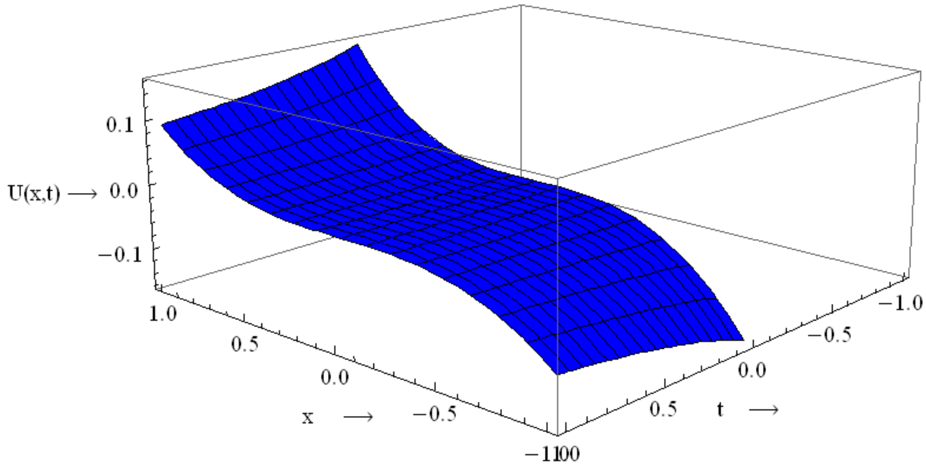

Figure 7. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.5$.

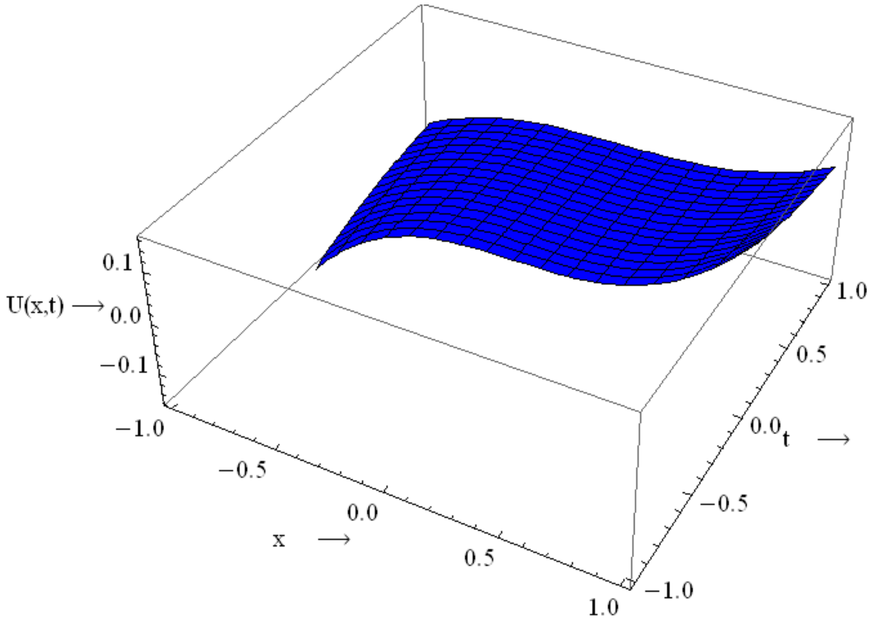

Figure 8. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.75$.

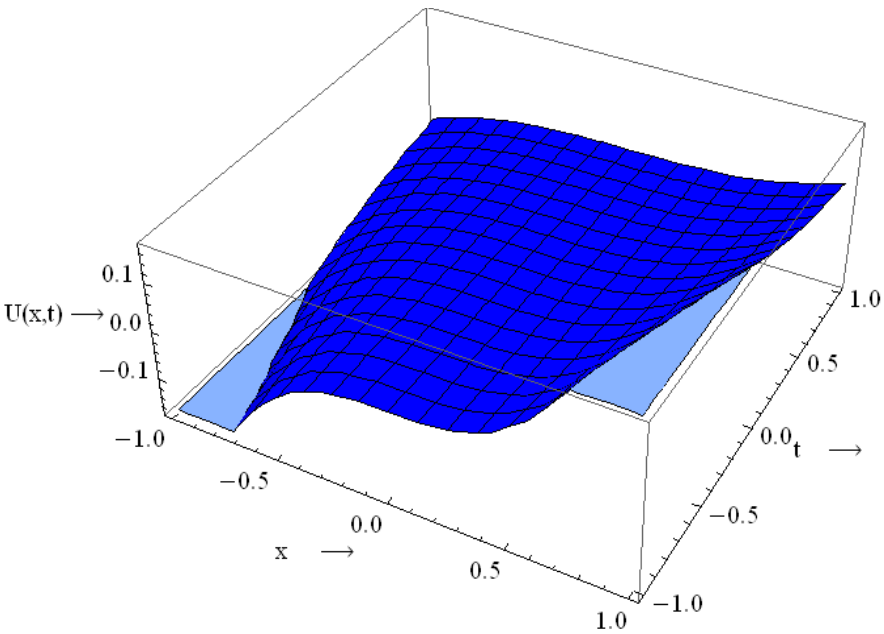

Figure 9. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 2.0$.

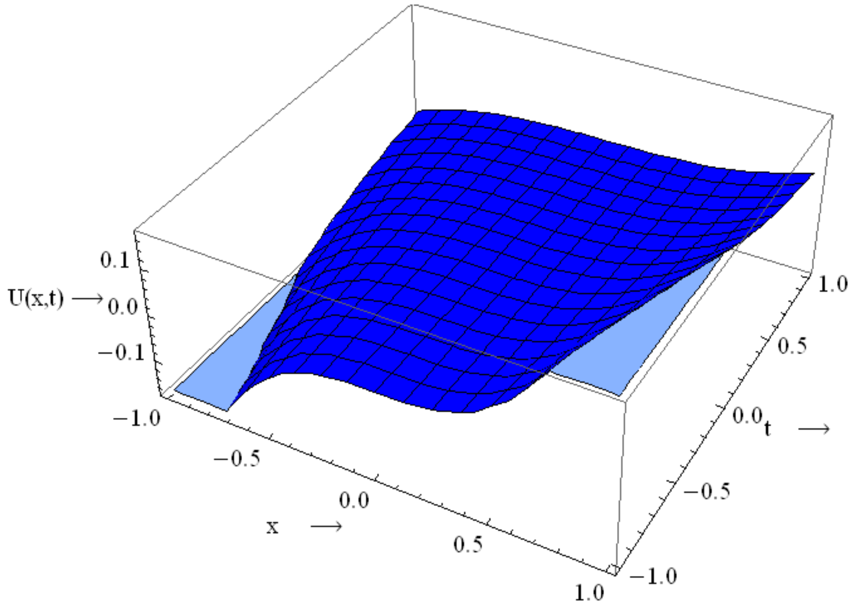

Figure 10. Plot of Exact Solution of $U(x,t)$ w.r.t x and t.

Example 3

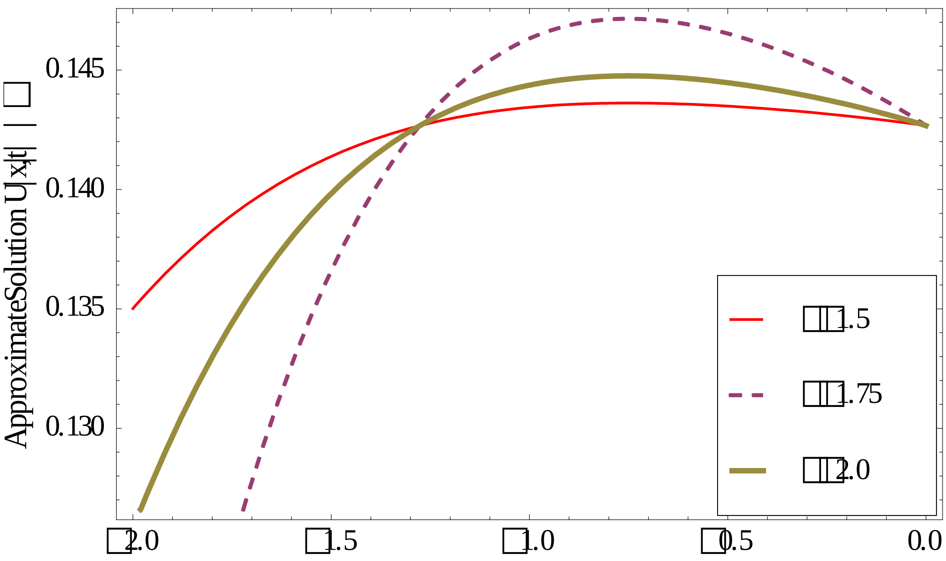

Figure 11. Plot of $U(x,t)$ w.r. to ħ at $t = 0.1$ and $x = 1$.

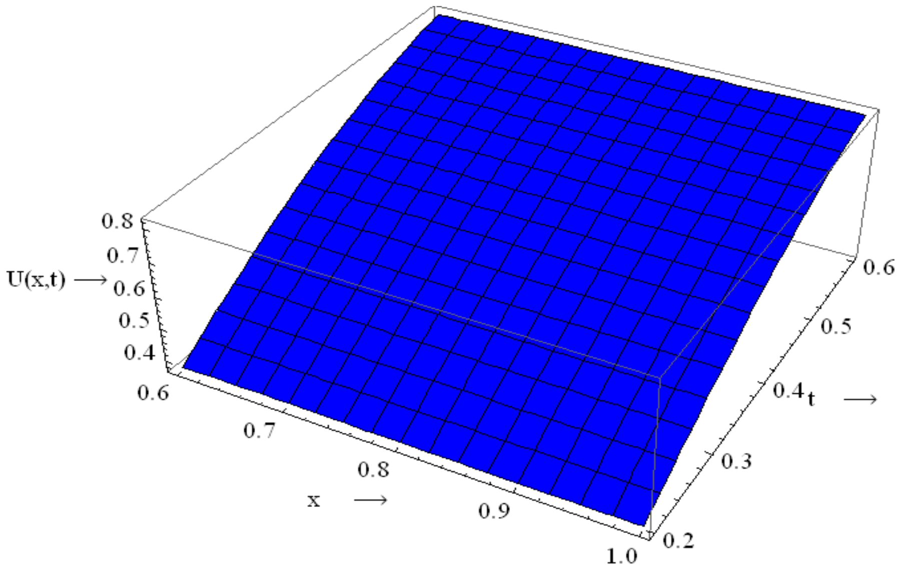

Figure 12. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.5$.

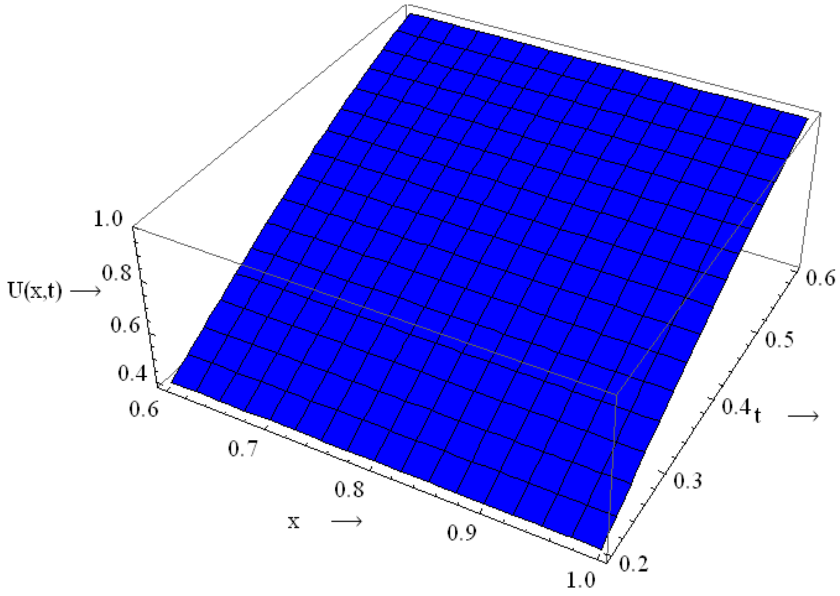

Figure 13. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 1.75$.

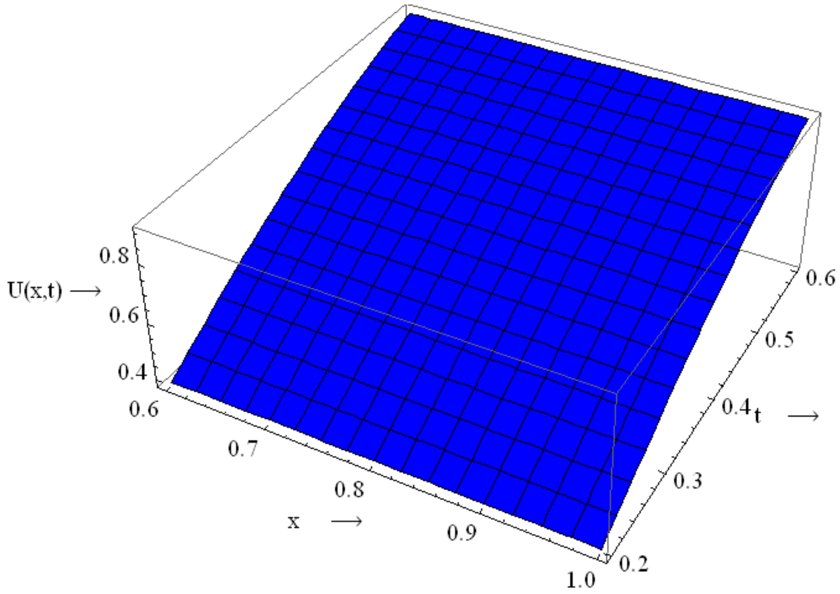

Figure 14. Plot of $U(x,t)$ w.r.t x and t at $\alpha = 2.0$.

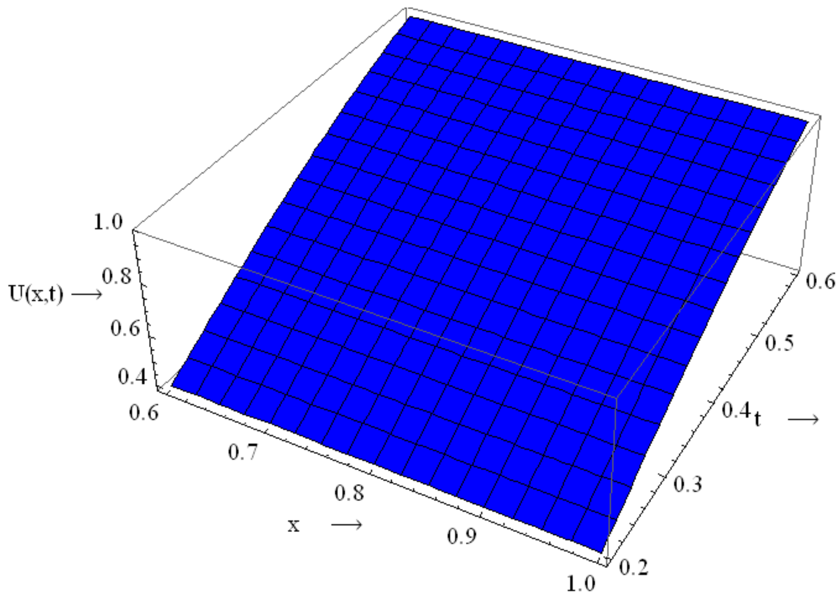

Figure 15. Plot of Exact Solution of $U(x,t)$ w.r.t x and t

Figure 1 Shows that the curve between approximate solution $U(x,t)$ and convergence control parameter $\hbar$ for the different values of fractional order α viz. $\alpha = 1.5, 1.75, 2.0$. The convergence lies between the range $-1 \le \hbar < 0$ at $t = 0.5$ and $x = 1$ for example 1.

Figure 2 shows that the three dimensional plot between $U(x,t)$ for independent variables x and t at $\alpha = 1.5$. Similarly Figure 3 and Figure 4 show the corresponding slight changes for different fractional Brownian motions of $\alpha = 1.75, 2.0$ respectively.

Figure 5 is plotted for the exact solution of $U(x,t)$ which is equal to the Figure 4.

In the subsequent manner the plot of example 2, $\hbar$ curve for $U(x,t)$ lies between $-1.3 \le \hbar < 0$ in Fig. 6. and Figure 7-Figure 9 are shown the plot of approximate solution correspond to two independent variables x and t verses $\alpha = 1.5, 1.75, 2.0$. Figure 10 is plotted for exact solution which shows the same plot as Figure 9.

Figure 11-Figure 15 show the evaluation results of the approximate analytical solution for the Example 3. These also figures show the behavior of the approximate solution obtained by the proposed method for different fractional Brownian motions $\alpha = 1.5, 1.75, 2.0$ and the convergence region for convergence control parameter $\hbar$ and approximate solutions.

Solution at integral value at $\alpha = 2.0$ in all above mentioned plots is shown the same as obtained by N.A. Khan et. al [44] , Wazwaz [45,46] and Biazar and Gazvini [47].

7. Conclusion

In science and engineering, the incipient modification of HAFSTM is potent implement to probe the solution of sundry linear and nonlinear quandaries. The main aim of this article is to provide the approximate solution and additionally analytic approximation utilizing the proposed method for fourth order boundary value quandaries. The analytical results have been given in terms of a potency series with facilely computed terms. The method surmounts the arduousness in different methods because it is efficient and lucid. Three examples were investigated to demonstrate the facileness and multifariousness of our incipient approach. The illustrative examples show that the method is facile to utilize and is an efficacious implement to solve fractional partial differential equations numerically.

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