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Initial Value Method Extended for General Singular Perturbation Problems

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Abstract In this paper the initial value method is extended for solving singularly perturbed two point boundary value problems with internal and terminal layers. It is distinguished by the following fact: The given singularly perturbed boundary value problem is replaced by two first order initial value problems. These first order problems are solved using Runge Kutta method. Model example for each is solved to demonstrate the applicability of the method.

Keywords: *singular perturbations, internal layer, two layers, initial value method*

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1. Introduction

Singular perturbation problems are of common occurrence in many branches of applied mathematics, such as fluid dynamics, elasticity, chemical reactor theory, aerodynamics, plasma dynamics, magneto-hydrodynamics and other domains of the world of fluid motion. A few notable examples are boundary layer problems, WKB problems, the modeling of steady and unsteady viscous flow problems with large Reynolds number, convective heat transport problems with large peclet numbers etc. It is well known fact that the solutions of these problems exhibit a multi scale character. That is, there is a thin layer(s) where the solution varies rapidly (non-uniformly), while away from the layer the solution behaves regularly (uniformly) and varies slowly. Therefore, the numerical treatment for singularly perturbed boundary value problems gives major computational difficulties.

A wide variety of papers and books have been published in the recent years, describing various methods for solving singular perturbation problems, among these we mention Awoke [1], Bender and Orzag [2], Kadalbajoo and Reddy [3,4], Hemker and Miller [5], Kevorkian and Cole [6], Nayfeh [7], O'Malley [8], Y.N.Reddy [9] and Van Dyke [10].

In this paper the initial value method is extended for solving singularly perturbed two point boundary value problems with internal and terminal layers. It is distinguished by the following fact: The given singularly perturbed boundary value problem is replaced by two first order initial value problems. These first order problems are solved using Runge Kutta method. Model example for each is solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

2. Initial Value Method

To describe the initial value method, we consider a class of singularly perturbed two point boundary value problem of the form:

$$\varepsilon y^{n} + a(x)y'(x) + b(x)y(x) = f(x) \quad x \in [0,1] \tag{1}$$

$$with\ y(0) = \alpha \tag{2a}$$

$$and\ y(1) = \beta \tag{2b}$$

where ε is a small positive parameter $0 < \varepsilon << 1$, α, β are given constants, $a(x), b(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[0,1]$ where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$. We now present an initial value method as follows:

Step 1: We obtain the reduced problem of (1) by taking ε= 0.

$$a(x)y'(x) + b(x)y(x) = f(x) \quad x \in [0,1] \tag{3}$$

$$y(1) = \beta \tag{3a}$$

Let its solution be $y_0(x) = y_0$.

Step 2: Integrate equation (1)

$$\varepsilon y'(x) + \int a(x)y'(x)dx + \int b(x)y(x)dx = \int f(x)dx$$

$$\varepsilon y'(x) + a(x)y(x) + R = K \tag{4}$$

where K is a constant and

$$R = \int b(x)y(x)dx - \int a'(x)y(x)dx - \int f(x)dx$$

$$R' = b(x)y(x) - a'(x)y(x) - f(x) \quad (5)$$

Step 3: If $R_0 = R_0(x)$ satisfies (5) we can approximate the solution of (1) – (2) by solving the initial value problem

$$\varepsilon y'(x) + a(x)y(x) + R_0 = K \quad x \in [0,1] \quad (6)$$

$$y(0) = \alpha \quad (6a)$$

with a suitable choice of R_0 and K

Step 4: To determine one solution R_0 of (5) we note that y_0 satisfies (3). If we require R_0 *=0 we will be able to find* R_0 *by solving the initial value problem*

$$R_0' = b(x)y_0(x) - a'(x)y_0(x) - f(x) \quad (7)$$

$$with \;\; R_0(1) = 0 \quad (7a)$$

Step 5: To find an approximate value for $'K'$ we impose the condition that the reduced equation of (4) satisfy the boundary condition $y(1) = \beta$

$$a(x)y(x) + R_0 = K \;\; with \;\; y(1) = \beta$$
$$K = R_0(1) + a(1)y(1)$$
$$K = a(1)\beta \;\; \text{since} \;\; R_0(1) = 0$$

Remark: We have taken $R_0(1) = 0$ and found $'K'$. In fact for any finite value of $R_0(1)$, K will change accordingly and there will be no effect to our problem.

Step 6: Thus the two Initial value problems are (7), (6) as given below:

$$R_0' = b(x)y_0(x) - a'(x)y_0(x) - f(x) \quad x \in [0,1]$$

with $R_0(1)$ = 0 and

$$\varepsilon y'(x) + a(x)y(x) + R_0 = K$$

with $y(0) = \alpha$.

Thus it is possible to approximate the solution of the given two-point boundary value problem by solving two initial value problems. It is interesting to note that the approximate solution $y(x)$ improves as the small parameter ε tends to zero.

3. Right End Boundary Layer

We now describe this new initial value method for the singularly perturbed two point boundary value problems with the right end boundary layer of the underlying interval. To be specific we consider a class of linear singularly perturbed two point boundary value problems of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \quad x \in [0,1] \quad (8)$$

$$with \; y(0) = \alpha \quad (9a)$$

$$and \; y(1) = \beta \quad (9b)$$

where ε is a small positive parameter $(0 < \varepsilon << 1)$, α, β are given constants $a(x), b(x), h(x)$ are assumed to be sufficiently continuously differentiable functions in $[0,1]$. Furthermore, we assume that $a(x) \le M < 0$ throughout the interval $[0,1]$ where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$. We have the following steps

Step 1: The reduced problem in this case would be

$$a(x)y'(x) + b(x)y(x) = h(x) \quad (10)$$

$$with \;\; y(0) = \alpha \quad (10a)$$

Let y_0 be the solution of the reduced problem. A similar approach as in case of left end boundary layer problems leads to the following:

Step 2: The two initial value problems are given by

$$R_0' = b(x)y_0(x) - a'(x)y_0(x) - f(x) \quad (11)$$

$$R_0(0) = 0 \quad (11a)$$

and

$$\varepsilon y'(x) + a(x)y(x) + R_0 = K \quad x \in [0,1] \quad (12)$$

$$y(1) = \beta \quad (12a)$$

To find an approximate value for K, we impose the condition that the reduced equation of (12) satisfy the boundary condition $y(0) = \alpha$. This gives us

$$K = \alpha(0)y(0) + R_0(0) = \alpha(0)\alpha$$

There exist several efficient methods for solving these initial value problems. For detailed discussion and numerical examples refer Awoke [1].

4. Internal Layer Problems

We will now extend the new initial value method to singular perturbation problems with an internal layer of the underlying interval. In this case $a(x)$ changes sign in the domain of interest. Without loss of generality we can take $\alpha(0)$=0 and the interval to be [-1,1].

To describe the method we shall consider a class of linear singularly perturbed two point boundary value problems of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \quad x \in [-1,1] \quad (13)$$

$$with \;\; y(-1) = \alpha \quad (14a)$$

$$and \;\; y(1) = \beta \quad (14b)$$

where ε is a small positive parameter $(0 < \varepsilon << 1), \alpha, \beta$ are given constants $a(x), b(x), f(x)$ are assumed to be sufficiently continuously differentiable functions in $[-1,1]$. Furthermore, we assume that $a(x) \le M < 0$ throughout the interval $[-1,0]$ where M is some negative constant and $a(x) \le M < 0$ in the interval [0,1] where M is some positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$. We now proceed as follows:

Step 1: We first find the approximate solution at $x = 0$. Without loss of generality we can take $a(0) = 0$. At $x = 0$ equation (13) becomes

$$\varepsilon y''(0) + b(0) y(0) = f(0) \quad (15)$$

The reduced problem of (15) gives us an approximation to $y(0)$.

$$\therefore y(0) = \frac{f(0)}{b(0)} = \gamma \quad (16)$$

Step 2: We now divide the interval [-1, 1] into two subintervals [-1, 0] and [0, 1] so that equation (13) has a right layer in [-1, 0] and a left layer in [0, 1].

Step 3: We now use our new initial value method for right end boundary layer as described in section (3), in the interval [-1,0]. The two initial value problems related to (13) are

$$R_0' = b(x) y_0(x) - a'(x) y_0(x) - f(x) \; with \; R_0(-1) = 0 \quad (17)$$

and

$$\varepsilon y'(x) + a(x) y(x) + R_0 = K \; with \; y(0) = \gamma \quad (18)$$

To find the value of K in (18) we impose the condition that the reduced problem of (18) satisfy the boundary condition at $x = -1$. This yields

$$K = R_0(-1) + a(-1) y(-1) = a(-1)\alpha$$

Step 4: We next use our new initial value method for left end boundary layer as described in section (2) in the interval [0,1]. The two initial value problems related to (13) are

$$R_0' = b(x) y_0(x) - a'(x) y_0(x) - f(x) \quad (19)$$

$$with \; R_0(-1) = 0$$

$$\varepsilon y'(x) + a(x) y(x) + R_0 = K \quad (20)$$

$$with \; y(0) = \gamma$$

To approximate K in (20), we impose the condition that the reduced problem of (20) satisfy the boundary condition at $x = 1$ i.e. $y(1) = \beta$

$$\therefore K = \alpha(1) y(1) + R_0(1) = a(1)\beta$$

Thus in a manner of speaking we have replaced the original second order problem (13-14) with two equivalent first order problems in each of the two subintervals. We solve these initial value problems to obtain solutions over the interval [-1,0] and [0,1] respectively. There now exists a number of efficient methods for the solution of these initial value problems. We use classical Runge Kutta method for our problem.

5. Numerical Example

To demonstrate the applicability of the method we solve one problem

Example 5.1: Consider the following SPP

$$\varepsilon y''(x) + x y'(x) - y(x) = 0; \; x \in [-1,1]$$
$$with \; y(-1) = 1 \; and \; y(1) = 2$$

For this example we have $a(x) = x$, $b(x) = -1$ and $f(x) = 0$. Further we have an internal layer of width $o\sqrt{\varepsilon}$ at $x = 0$ (for details, see O'Malley [[8], pp 68-172, eq8.1case (i)] and Kevorkian and Cole [[6], pp 41-43, eqs (2.3.76) and (2.3.77)]

Step 1: $y(0) = f(0)/b(0) = 0 = \gamma$

Step 2: In the interval [-1,0] we have a right layer. The problem is

$$\varepsilon y''(x) + x y'(x) - y(x) = 0; \; x \in [-1,0]$$

with y(-1)=1 and y(0)=0. The solution of the reduced problem

$$x y_0'(x) - y_0(x) = 0$$

with $y_0(-1) = 1$ is $y_0(x) = -x$ and $K = 0$.

The two initial value problems are

$$R_0' = 2x \; with \; R_0(-1) = 0$$

and $\varepsilon y'(x) + x y(x) + R_0 = -1$ with $y(0) = 0$.

Step 3: In the interval [0,1] we have a left layer. The problem is

$$\varepsilon y''(x) + x y'(x) - y(x) = 0 \; x \in [0,1]$$

with $y(0) = 0$ and $y(1) = 2$.

The solution of the reduced problem is $y_0(x) = 2x$ and $K = 2$.

The two initial value problems are

$$R_0' = -4x \; with \; R_0(1) = 0$$

and $\varepsilon y'(x) + x y(x) + R_0 = 2$ with $y(0) = 1$.

We solve the above equations using Runge Kutta method. The numerical results are presented in Table 1(a) and 1(b) for $\varepsilon = 10^{-3}$ *and* $\varepsilon = 10^{-4}$ respectively.

Table 1. (a): Computational results for Example 5.1 with $\varepsilon = 10^{-2}$ and $h = 0.01$

x	$y(x)$
-1.000	1.0000000
-0.500	0.4791165
-0.100	0.0220781
-0.080	0.0109326
-0.060	0.0041401
-0.040	0.0009350
-0.200	0.1327092
0.000	0.0000000
0.020	0.0005291
0.040	0.0041332
0.060	0.0134145
0.080	0.0301369
0.100	0.0550444
0.500	0.9581506
1.000	1.9797936

Table 1. (b): Computational results for Example 5.1 with $\varepsilon = 10^{-3}$ and $h = 0.01$

x	$y(x)$
-1.000	1.0000000
-0.500	0.4979919
-0.100	0.0884604
-0.080	0.0642969
-0.060	0.0384793
-0.040	0.0152044
-0.200	0.0021584
0.000	0.0000000
0.020	0.0049300
0.040	0.0316510
0.060	0.0777414
0.080	0.1287349
0.100	0.1768589
0.500	0.9959838
1.000	1.9979979

6. Two Boundary Layers Problems

The suggestions given for internal layer problems can be extended mutatis mutandis to problems with two boundary layers. To describe the method we shallconsider a class of linear singularly perturbed two point boundary value problems of the form

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \quad x \in [-1,1] \quad (21)$$

$$with\ y(-1) = \alpha \quad (22a)$$

$$and\ y(1) = \beta \quad (22b)$$

here ε is a small positive parameter $(0 < \varepsilon << 1), \alpha, \beta$ are given constants $a(x), b(x), f(x)$ are assumed to be sufficiently continuously differentiable functions in $[-1,1]$. Furthermore, we assume that $a(x) \geq M > 0$ throughout the interval $[-1,0]$ and $a(x) \leq M < 0$ in [0,1] where M is some negative constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = -1$ and 1. Without loss of generality $a(x) = 0$ at $x = 0$ since it changes sign in the domain of interest.

Step 1: We first find the approximate solution at $x = 0$ without loss of approximate we can take $a(0) = 0$

At $x = 0$ equation (21) becomes

$$\varepsilon y''(0) + b(0)y(0) = f(0) \quad (23)$$

The reduced problem of (23) gives us an approximation to $y(0) = 0$.

$$\therefore y(0) = \frac{f(0)}{b(0)} = \gamma \quad (24)$$

Step 2: We now divide the interval [-1, 1] into two subintervals [-1, 0] and [0, 1] so that equation (21) has a left layer in [-1, 0] and a right layer in [0, 1].

Step 3: We now use our new initial value method for left end boundary layer as described in section (2) in the interval [-1,0]. The two initial value problems related to (21) are

$$R_0' = b(x)y_0(x) - a'(x)y_0(x) - f(x) \quad x \in [-1,0] \quad (25)$$

with $R_0(0) = 0$ and

$$\varepsilon y'(x) = a(x)y(x) + R_0 = K \quad x \in [-1,0] \quad (26)$$

With $y(-1) = \alpha.$

To find the value of K in (26) we impose the condition that the reduced problem of (26) satisfy the boundary condition at $x = 0$ i.e. $y(0) = \gamma.$

$$\therefore K = R_0(0) + a(0)y(0) = a(0)\gamma$$

Step 4 ; We next use our new initial value method for right end boundary layer as described in section (3) in the interval [0,1]. The two initial value problems related to (21) are :

$$R_0' = b(x)y_0(x) - a'(x)y_0(x) - f(x) \, x \in [0,1] \quad (27)$$

with $R_0(0) = 0$ and

$$\varepsilon y'(x) = a(x)y(x) + R_0 = K \quad x \in [0,1] \quad (28)$$

with $y(1) = \beta.$

We obtain $K = a(0)y(0) + R_0(0) = a(0)\gamma.$

Thus in a manner of speaking we have replaced the original second order problem (21-22) with two equivalent first order problems in each of the two subintervals. We solve these initial value problems to obtain solutions over the intervals [-1,0] and [0,1] respectively. There now exist a number of efficient methods for the solution of initial value problems. We use classical Runge Kutta method for our problem.

7. Numerical Example

To demonstrate the applicability of the method we solve one problem

Example 7.1: Consider the following SPP

$$\varepsilon y'' - xy'(x) - y(x) = 0; \, x \in [-1,1]$$
$$y(-1) = 1, \ y(1) = 2.$$

For this example we have $a(x) = -x, b(x) = 1$ and $f(x) = 0.$ Further we have two boundary layers one at $x = -1$ and one at $x = 1$ (for details, see O'Malley [[8], pp168-173, eq 8.1 case (i)]

Step 1: $y(0) = \dfrac{f(0)}{b(0)} = 0 = \gamma$

Step 2: In the interval [-1,0] we have a left layer. The problem is

$$\varepsilon y''(x) - xy'(x) - y(x) = 0; \, x \in [-1,0]$$

with $y(-1) = 1\, and\ y(0) = 0.$

The solution of the reduced problem

$$-xy_0'(x) - y_0(x) = 0$$

with $y_0(0)=0$ is $y_0(x)=0$ and $K=0$.

The two initial value problems are:

$$R_0' = -y_0(x) + y_0(x) + 0 = 0$$

with $R_0(0)=0$ and

$$\varepsilon y'(x) - xy(x) + R_0 = K$$

with $y(-1)=1$.

Step 3: In the interval [0,1] we have a right layer the problem is

$$\varepsilon y''(x) - xy'(x) - y(x) = 0;\ x \in [0,1]$$

with $y(0)=0$ and $y(1)=2$.

The solution of the reduced problem is $y_0(x)=0$ with $K=0$.

The two initial value problems are

$$R_0' = -y_0(x) + y_0(x) + 0 = 0 \ \textit{with}\ R_0(0)=0$$

and $\varepsilon y'(x) - xy(x) + R_0 = K \ \textit{with}\ y(1)=2$.

We solve the above equations using Runge Kutta method. The numerical results are presented in Table 2(a) and 2(b) for $\varepsilon = 10^{-3}$ *and* 10^{-4} respectively.

Table 2. (a): Computational results for Example 7.1 with $\varepsilon = 10^{-2}$ **and** $h = 0.01$

x	$y(x)$
-1.000	1.0000000
-0.980	0.1421960
-0.960	0.0209600
-0.940	0.0032039
-0.920	0.0005080
-0.900	0.0000836
-0.700	0.0000000
-0.300	0.0000000
0.300	0.0000000
0.900	0.0001729
0.920	0.0010439
0.940	0.0065387
0.960	0.0424885
0.980	0.2863122
1.000	2.0000000

8. Discussion and Conclusions

We have extended the new initial value method for solving general singularly perturbed two-point boundary value problems with internal layer and two layers. In general the numerical solution of a boundary value problem will be more difficult than the numerical solution of the corresponding initial value problem. Hence we prefer to convert the given second order problem into first order problems. The solution of given singularly perturbed two-point boundary value problem is computed numerically by solving two initial value problems. We used classical fourth order Runge Kutta method. Numerical results are presented in tables. It is observed that the present method approximates the exact solution very well.

Table 2. (b): Computational results for Example 7.1 with $\varepsilon = 10^{-3}$ **and** $h = 0.01$

x	$y(x)$
-1.000	1.0000000
-0.980	0.0000000
-0.960	0.0000000
-0.940	0.0000000
-0.920	0.0000000
-0.900	0.0000000
-0.700	0.0000000
-0.300	0.0000000
0.300	0.0000000
0.900	0.0000000
0.920	0.0000000
0.940	0.0000000
0.960	0.0000000
0.980	0.0000000
1.000	2.0000000

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