American Journal of Numerical Analysis, 2014, Vol. 2, No. 6, 177-183
Available online at http://pubs.sciepub.com/ajna/2/6/2

DOI:10.12691/ajna-2-6-2

Numerical Solution of Singularly Perturbed Two-Point Singular Boundary Value Problems Using Differential Quadrature Method

H.S. Prasad[1], Y.N. Reddy[2,*]

[1]Department of Mathematics, National Institute of Technology, Jamshedpur, INDIA
[2]Department Mathematics, National Institute of Technology, Warangal, INDIA
*Corresponding author: ynreddy_nitw@yahoo.com

Received August 06, 2014; Revised December 18, 2014; Accepted December 25, 2014

Abstract This paper presents the application of Differential Quadrature Method (DQM) for finding the numerical solution of singularly perturbed two point singular boundary value problems. The DQM is an efficient discretization technique in solving initial and/or boundary value problems accurately using a considerably small number of grid points. This method is based on the approximation of the derivatives of the unknown functions involved in the differential equations at the mess point of the solution domain. To demonstrate the applicability of the method, we have solved model example problems and presented the computational results. The computed results have been compared with the exact solution to show the accuracy and efficiency of the method.

Keywords: *singular boundary value problem, singularly perturbations, singular point, boundary layer, differential quadrature method*

Cite This Article: H.S. Prasad, and Y.N. Reddy, "Numerical Solution of Singularly Perturbed Two-Point Singular Boundary Value Problems Using Differential Quadrature Method." *American Journal of Numerical Analysis*, vol. 2, no. 6 (2014): 177-183. doi: 10.12691/ajna-2-6-2.

1. Introduction

Singularly perturbed singular boundary value problems arise in many branches of applied mathematics such as, fluid dynamics, quantum mechanics, chemical reactor theory, elasticity, aerodynamics, and the other domain of the great world of fluid motion. It is well known fact that the solution of these problems exhibits a multi scale character, that is, there are thin transition layer(s) where the solution varies rapidly, while away from the layers(s) the solution behaves regularly and varies slowly. The numerical treatment of singularly perturbed singular boundary value problems present some major computational difficulties due to the boundary layer behaviour of the solution and the presence of singularity. In general, classical numerical methods fail to give reliable results for these problems. In recent years, a good number of special purpose methods have been proposed to provide accurate numerical solutions [7,8,10,11,12,18]. In [7], the authors Kadalbajoo and Aggarwal presented a Fitted mesh B-spline method for the solution of a class of singular singularly perturbed boundary value problems. In [8], the author Li described a computational method for solving singularly perturbed two-point singular boundary value problem in which exact solution is represented in the form of series in reproducing kernel space. In [10], the authors Mohanty and Arora proposed a family of non-uniform mesh tension spline methods for the solution of singularly perturbed two-point singular boundary value problems with significant first derivatives. In [11], the authors Mohanty and Evans suggested a Convergent spline in tension methods for the solution of singularly perturbed two-point singular boundary value problems. In [12], the authors Mohanty and Jha presented a class of variable mesh spline in compression methods for singularly perturbed two point singular boundary value problems. In [18] the authors Rashidinia et. al. presented a numerical technique for a class of singularly perturbed two point singular boundary value problems on an uniform mess using Polynomial cubic spline. For a good discussion on singularly perturbed problems one may refer to the books and high level monographs: Bender and Orszag [3], Nayfeh [13], O'Malley [14], Farrell et. al. [6], Roos et. al. [19] and Miller et. al. [9].

In this paper, we have presented the Differential Quadrature Method (DQM) for finding the numerical solution of singularly perturbed two-point singular boundary value problems of the form:

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x); 0 \le x \le 1 \quad (1)$$

subject to the boundary conditions

$$y(0) = \alpha \text{ and } y(1) = \beta \quad (2)$$

where ε is a small parameter $0 < \varepsilon << 1$; α, β are given constants. The coefficient functions failed to be analytic at $x = 0$. We know that, if a function is analytic at a point $x = x_0$, then the point x_0 is said to be an ordinary point.

The point $x = x_0$ is a singular point if the functions fail to be analytic at x_0. Such problems are called singularly perturbed singular boundary value problems. We have employed the natural cubic spline interpolation technique to interpolate the solution values at uniform points. The DQM is a simple and efficient numerical technique, which approximates the derivative with respect to a coordinate direction at a grid point by a weighted linear sum of all the functional values in that direction. To the best of the authors knowledge, the Differential Quadrature Method, where approximation of the derivatives have been based on a polynomial of high degree, has not been implemented for the singularly perturbed two-point singular boundary value problems. This paper is organized as follows: Section 2 presents the description of the Differential Quadrature Method, including the formula for finding the weighting coefficients for any order derivative discretization and selection of sampling points. Section 3 presents the basic key procedure to solve differential equation with boundary conditions. The solution procedure by DQM in detail, are given in the Section 4. In the Section 5, we have considered three example problems and presented the computational results, show the accuracy and efficiency of the method. The conclusions are presented in section 6. The paper ends with the references.

2. Description of the Differential Quadrature Method

The Differential Quadrature Method (DQM) was introduced by Bellman et al. [1,2] in the early 1970s and, since then, the technique has successfully employed in finding the solutions of many problems in applied and physical sciences [4,5,15,20,21,22]. The basic idea of differential quadrature method is that the derivative of a function with respect to a space variable at a given point is approximated as a weighted linear sum of the functional values at all discrete points in the domain of that variable.

In order to show the mathematical representation of the method, we consider a one dimensional field variable $f(x)$ prescribed in a field domain $a = x_1 \le x \le x_N = b$. Let $f_i = f(x_i)$ be the function values specified in a finite set of N discrete points $x_i, (i = 1,2,......,N)$ of the field domain. Next, consider the value of the function derivative $d^m f / dx^m$ at some discrete points x_i, and let it be expressed as a linearly weighted some of the function values.

$$f^{(m)}(x_i) = \frac{d^m f(x_i)}{dx^m} = \sum_{J=1}^{N} A_{ij}^{(m)} f_j, (i = 1,2,.......,N) \quad (3)$$

where $A_{ij}^{(m)}$ are the weighting coefficients of the m^{th}-order derivative of the function associated with points x_i. Equation (3) the quadrature rule for a derivative is the essential basis of the Differential Quadrature Method. Thus using equation (3) for various order derivatives, one may write a given differential equation at each point of its solution domain and obtain the quadrature analog of the differential equation as a set of algebraic equations in terms of the N function values. These equations may be solved, in conjunction with the quadrature analog of the boundary conditions, to obtain the unknown function values provided that the weighting coefficients are known a priori. The key to DQM is the determination of weighting coefficients for any order derivative discretization. The weighting coefficients may be determined by some appropriate functional approximations; and the approximate functions are referred to as test functions. The primary requirements for the choices of the test functions are of differentiability and smoothness. That is, the test function of the differential equation must be differentiable at least up to the n^{th} derivative (here n is the highest order of the differential equation) and sufficiently smooth to be satisfied the condition of the differentiability. Bellman et al. [2] proposed two approaches to compute the weighting coefficients. The first approach solves an algebraic equation system and the second approach uses a simple algebraic formulation, but with the coordinates of grid points chosen as the roots of the shifted Legendre polynomial. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus it is very difficult to compute the weighting coefficients for a large number of grid points. To improve the Bellman's approaches in computing the weighting coefficients, many attempts have been made by researchers. One of the most useful approaches is the one introduced by Quan and Chang [16,17]. After that, Shu's (Shu [21]) general approach which is based on the high order polynomial approximation and linear vector space analysis, was made available in the literature. This generalized approach computes the weighting coefficients of the first order derivative by a simple algebraic formulation without any restriction on choice of grid points, and the weighting coefficients of second and higher order derivatives by a recurrence relationship.

In the DQM, It is supposed that the solution of a one–dimensional differential equation is approximated by a $N-$terms high degree polynomial:

$$f(x) = \sum_{k=1}^{N} c_k . x^{k-1} \quad (4)$$

where c_k is a constant. The generalized approach uses two sets of base polynomials (test functions) to determine the weighting coefficients (Shu [21]). The first set of base polynomials is chosen as the Lagrange interpolated polynomials, which are written as

$$r_k(x) = \frac{M(x)}{(x - x_k)\ M^{(1)}(x_k)}; \ k = 1, 2, ..., N \quad (5)$$

Where $M(x) = (x - x_1).(x - x_2).......(x - x_N)$

And $M^{(1)}(x_k) = \prod_{j=1, j \ne k}^{N} (x_k - x_j)$ being the first derivative of $M(x)$ at x_k. Here $x_1, x_2, ..., x_N$ are the coordinates of the grid points, can be chosen arbitrarily but distinct. The polynomials

$$r_k(x) = x^{k-1}, \ k = 1, 2, ..., N \quad (6)$$

are taken as the second set of base polynomials.For simplicity, by setting

$$M(x) = N(x, x_k).(x - x_k),$$
$$k = 1, 2, ..., N$$

with $N(x_i, x_j) = M^{(1)}(x_i).\delta_{ij}, (i, j = 1, 2, ..., N)$ where δ_{ij} is the Kronecker operator, the equation (5) is simplified as:

$$r_k(x) = \frac{N(x, x_k)}{M^{(1)}(x_k)};\ k = 1, 2, ..., N \quad (7)$$

Substituting equation (7) into the equation (3) for $m = 1$ and using equation (6), Shu [21] obtained the following weighting coefficients of the first order derivative discretization.

$$\begin{aligned} A_{ij}^{(1)} &= \frac{M^{(1)}(x_i)}{(x_i - x_j) M^{(1)}(x_j)}, (i, j = 1, 2, ...N; i \neq j) \\ A_{ii}^{(1)} &= -\sum_{j=1; i \neq j}^{N} A_{ij}^{(1)}, \quad (i = 1, 2, ...N) \end{aligned} \quad (8)$$

The Shu's (Shu [21]) recurrence formulation for determination of weighting coefficients for higher order derivatives discretization are given as

$$\begin{aligned} A_{ij}^{(m)} &= m[A_{ii}^{(m-1)} A_{ij}^{(1)} - \frac{A_{ij}^{(m-1)}}{(x_i - x_j)}], \\ &(i, j = 1, 2, ...N; i \neq j; 2 \leq m \leq N-1) \\ A_{ii}^{(m)} &= -\sum_{j=1; i \neq j}^{N} A_{ij}^{(m)}, (i = 1, 2, ...N;\ m \geq 2) \end{aligned} \quad (9)$$

Obviously, equations (8) and (9) offer an easy way of computing the weighting coefficients for any order derivative discretization. These explicit formulae's merit is that highly accurate weighting coefficients may be determined for any number of arbitrarily spaced sampling points.

2.1. Choice of Sampling Points

A convenient and natural choice for the sampling points is that of the equally spaced points. But the Differential Quadrature solutions usually deliver more accurate results with unequally spaced sampling points. A rational basis for the sampling points is provided by the zeros of the orthogonal polynomials. A well accepted kind of sampling points in the DQM is the so called Gauss-Lobatto-Chebyshev sampling points. For a domain specified by $a \leq x \leq b$ and discretized by a set of unequally spaced points (non-uniform grid), then the coordinate of any point i can be evaluated by:

$$x_i = a + \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right)(b - a) \quad (9a)$$

3. Application to Differential Equation

The basic key procedure in the DQM is to approximate the derivatives in a differential equation by equation (3). Substituting the equation (3) into the governing equations and equating both sides of the governing equations, we obtain simultaneous equations which can be solved by use of Gauss elimination or other methods. That is, DQM is composed of the following procedure:

a) The function to be determined is replaced by a group of function values at a group of selected sampling points. Gauss-Lobatto-Chebyshev sampling points: (9a) are strongly recommended for numerical stability.
b) Approximate derivatives in a differential equation by these N unknown function values.
c) Form a system of linear equations and
d) Solving the system of linear equation yields the desired unknowns.

The proper implementation of boundary condition is very important for the accurate numerical solution of differential equation. Essential and natural boundary condition can be approximated by DQM. Using the technique in solving differential equation, the governing equations are actually satisfied at each sampling point of the domain, so one has one equation for each point, for each unknown. In the resulting system of algebraic equation from the DQM, each boundary condition replaces the corresponding field equation. This procedure is straightforward when there is one boundary condition at each boundary and when we have distributed the sampling points so that there is one point at each boundary.

4. Application to Singular Singularly Perturbed Boundary Value Problems

To show the applicability of DQM, we consider the singularly perturbed two point singular boundary value problems of the form (1) subject to the boundary condition (2).

For finding the solution of the equation (1) with the boundary conditions (2) by DQM, we have followed the following procedure/steps:

i) Discritize the interval $[0,1]$, such that $0 = x_1 < x_2 < x_3 < .... < x_N = 1$ where, N is the number of sampling/grid points. Denote $y_i = y(x_i)$ and $f_i = f(x_i)$ etc.

ii) Apply the DQM to approximate the derivatives in the equation (1) with (2), that leads to the following discretized form of the equations:

$$\begin{aligned} &\varepsilon \sum_{j=1}^{N} A_{i,j}^{(2)} y_j + a_i \sum_{j=1}^{N} A_{i,j}^{(1)} y_j + b_i y_i - f_i = 0. \\ &i = 1, 2, ..., N \end{aligned} \quad (10)$$

with

$$y_1 = \alpha \ and \ y_N = \beta \quad (11)$$

iii) Apply the equation (10) at all interior points $x_i, (i = 2, 3, ..., N-1)$, that leads to a system of $(N-2)$ equations with N unknowns.

iv) Use the boundary values for y_1 and y_N from (11) in the obtained system of equations from step (iii) to get another system of $(N-2)$ equations with $(N-2)$ unknowns $(y_i, i = 2,3,........., N-1)$.

v) Solve the system of equations obtained in step (iv).

vi) Use the given boundary values to get the complete solution (x_i, y_i), $i = 1,3,........., N$.

We have applied the Gaussian elimination method and employed the double precision Fortran, to solve the obtained system of linear equations in the step (iv), for the unknowns $y_2, y_3,......., y_{N-1}$.

5. Numerical Illustrations

To demonstrate the applicability and efficiency of the **DQM** with Gaussian elimination and natural cubic spline interpolation polynomial, we have applied it to three singularly perturbed singular boundary value problems and computed the results for different values of N and ε. These examples have been chosen because they have been discussed in literature and because exact solutions are available for comparison.

Note that for the considered example problems, the DQM results in the tables are given in terms of Maximum Absolute Error (M.A.E.) at uniform grids $x_i = ih, (i = 0,1,2,......, K)$, with $h = 1/K$ for the considered interval [0,1], which have been interpolated through the use of natural cubic spline interpolation polynomial. For the derivation of this polynomial, we have used the DQM results $(x_i, y_i), i = 1,2,......, N$, where $y_i, (i = 1,2,......, N)$ are the value of y at non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, (i = 1,2,........, N)$ obtained from (10).

To show the accuracy and efficiency of the method with non-uniform grid points (Gauss-Lobatto-Chebyshev points) $x_i, (i = 1,2,........, N)$ obtained from (10), we have also given the computational results in terms of Maximum Absolute Error in the tables 5.1.3 and 5.2.3, for the example problems-5.1 and 5.2 respectively, for various values of N and small parameter: ε.

Example 5.1: Consider the following singularly perturbed singular boundary value problem from [8]:

$$\varepsilon\, y''(x) + \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = \frac{2}{x} - 2\varepsilon - 3;$$

$$0 < x \le 1$$

with $y(0) = 0$ and $y(1) = 0$.

For this example we have $a(x) = 1/x, b(x) = 1/x^2$ and $f(x) = \frac{2}{x} - 2\varepsilon - 3$.

The exact solution is given by: $y(x) = x - x^2$. The computational results are presented in the Tables 5.1.1, 5.1.2 and 5.1.3 in terms of Maximum Absolute Error (M.A.E.), for various values of N, K and small parameter: ε.

Example 5.2: Consider the following singularly perturbed singular boundary value problems from [8]:

$$\varepsilon\, y''(x) + \frac{1}{x \sin x} y'(x) + \frac{1}{x^2} y(x) = f(x), 0 < x \le 1$$

with $y(0) = 0$ and $y(1) = 0$, where $f(x) = \frac{\pi \cos(\pi x)\csc(x)}{x} - \varepsilon\pi^2 \sin(\pi x) + \frac{\sin(\pi x)}{x^2}$. For this example we have $a(x) = 1/x \sin x$, $b(x) = 1/x^2$.

The exact solution is given by: $y(x) = \sin(\pi x)$. The computational results are presented in the Tables 5.2.1, 5.2.2 and 5.2.3 in terms of Maximum Absolute Error (M.A.E.), for various values of N, K and small parameter: ε.

Example 5.3: Finally, consider the following singularly perturbed singular boundary value problems from (Bender and Orszag [3], page-452-453):

$$\varepsilon\, y''(x) - \frac{1}{x} y'(x) - y(x) = 0 \qquad 0 \le x \le 1$$

with $y(0) = 1$ and $y(1) = 1$,

For this example, we have $a(x) = -1/x$, $b(x) = -1$ and $f(x) = 0$.

This problem has the boundary layer of thickness ε at $x = 1$.

We have taken the uniform approximation up to the order ε (Bender and Orszag [3], page-452-453) to the exact solution as the exact solution, which is given by:

$$y_{unif}(x) = e^{(-x^2/2)}[1 + \frac{1}{4}\varepsilon(x^2 - 1)^2]$$

$$+ (1 - e^{-1/2})[1 - \frac{1}{2}\varepsilon(X^2 - 4X)]e^{-X},$$

where $X = (1-x)/\varepsilon$.

The computational results are presented in the Tables 5.3.1 and 5.3.2, for different values of N, K and small parameter: ε.

6. Discussion and Conclusions

In this paper, the DQM has been applied to solve singularly perturbed two point singular boundary value problems. We have applied it to solve three example problems. These example problems have been chosen because they have been widely discussed in literature and because approximate solutions are available for comparison. The applications presented here showed that the DQM has the capability of solving such problems and of producing accurate results with little computational effort. It can be observed from the tables that the DQM approximates the exact solution very well with small number of sampling points. This shows the efficiency and accuracy of the present method. It has been observed that an increase in the number of grid points gives rise to an increase in the accuracy of the DQM solution, as in the most numerical techniques. However a small number of grid points in the DQM produces highly accurate results with the use of non-uniform grids which can be seen in the tables 5.1.3 and 5.2.3. The DQM with natural cubic spline interpolation technique provides a good alternative technique to the conventional ways of solving singularly perturbed two point singular boundary value problems.

Table 5.1.1. Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 10^{-3}$, $K = 10^3$ for example : 5.1

ε	$N = 16$	$N = 32$	$N = 82$	$N = 132$
10^{-2}	.273156E-02	.641325E-03	.940719E-04	.368158E-04
10^{-3}	.273156E-02	.641329E-03	.940049E-04	.359224E-04
10^{-4}	.273156E-02	.641330E-03	.940044E-04	.359398E-04
10^{-5}	.273156E-02	.641330E-03	.940043E-04	.359415E-04
10^{-6}	.273156E-02	.641330E-03	.940043E-04	.359417E-04

Table 5.1.2. Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 10^{-5}$, $K = 10^5$ for example: 5.1

ε	$N = 16$	$N = 32$	$N = 82$	$N = 132$
10^{-2}	.273156E-02	.641325E-03	.941051E-04	.368182E-04
10^{-3}	.273156E-02	.641329E-03	.940051E-04	.359229E-04
10^{-4}	.273156E-02	.641330E-03	.940045E-04	.359398E-04
10^{-5}	.273156E-02	.641330E-03	.940043E-04	.359415E-04
10^{-6}	.273156E-02	.641330E-03	.940043E-04	.359417E-04

Table 5.1.3. Maximum Absolute Error in the solution for non-uniform points $(x_i = 1,2,......,N)$ obtained from (10), for various values of small parameter ε , for example problem: 5.1

ε	$N = 16$	$N = 32$	$N = 82$	$N = 132$
10^{-2}	.293054E-07	.223483E-06	.586415E-05	.309757E-04
10^{-3}	.983532E-08	.999251E-08	.298624E-06	.306288E-06
10^{-4}	.855585E-08	.130485E-07	.299447E-06	.222475E-07
10^{-5}	.847788E-08	.129361E-07	.841157E-07	.171530E-07
10^{-6}	.847044E-08	.129258E-07	.796479E-07	.167155E-07

Table 5.2.1. Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 10^{-3}$, $K = 10^3$ for example : 5.2

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	.316290E-02	.766618E-03	.187853E-03	.475286E-04
10^{-3}	.316305E-02	.766203E-03	.188402E-03	.466772E-04
10^{-4}	.316312E-02	.766780E-03	.188266E-03	.465538E-04
10^{-5}	.316313E-02	.766814E-03	.188630E-03	.467143E-04
10^{-6}	.316313E-02	.766817E-03	.188641E-03	.467704E-04

Table 5.2.2. Maximum Absolute Error in the solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ with $h = 10^{-5}$, $K = 10^5$ for example : 5.2

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	.316290E-02	.766618E-03	.187858E-03	.475510E-04
10^{-3}	.316305E-02	.766203E-03	.188402E-03	.466776E-04
10^{-4}	.316312E-02	.766780E-03	.188267E-03	.465759E-04
10^{-5}	.316313E-02	.766814E-03	.188630E-03	.467375E-04
10^{-6}	.316313E-02	.766817E-03	.188641E-03	.467934E-04

Table 5.2.3. Maximum Absolute Error in the solution for non-uniform points $(x_i , i = 1,2,......,N)$ obtained from (10), for various values of small parameter ε for example problem: 5.2

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	.790704E-06	.131778E-05	.367229E-05	.271647E-05
10^{-3}	.363292E-06	.225717E-05	.106315E-05	.566293E-06
10^{-4}	.141050E-06	.338272E-06	.142118E-05	.886341E-06
10^{-5}	.130930E-06	.230677E-06	.252196E-06	.410067E-06
10^{-6}	.129983E-06	.222497E-06	.220729E-06	.245180E-06

Table 5.3.1. DQM solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ **with** $h = 10^{-3}$, $K = 10^3$ **for example : 5.3**

x	Exact Solution-y(x)	DQM Solution-y(x) $N = 151, \varepsilon = 10^{-3}$ M.A.E:0.365370E-02	DQM Solution-y(x) $N = 251, \varepsilon = 10^{-3}$ M.A.E: 0.623504E-03
.000	1.0002500	1.0000000	1.0000000
.020	1.0000498	.9997994	.9997990
.040	.9994493	.9991988	.9991991
.060	.9984494	.9981977	.9981982
.080	.9970511	.9967988	.9968003
.200	.9804245	.9801736	.9801769
.400	.9232792	.9230397	.9230475
.600	.8353557	.8351411	.8351459
.800	.7261726	.7259901	.7259902
.900	.6669828	.6668165	.6668155
.925	.6519370	.6517747	.6517736
.950	.6368331	.6366746	.6366738
.975	.6216911	.6215365	.6215354
.992	.6115127	.6114187	.6113691
.994	.6111393	.6112517	.6110580
.996	.6161634	.6160913	.6160260
.998	.6611019	.6647556	.6613522
1.000	1.0000000	1.0000000	1.0000000

Table 5.3.2. DQM solution (computed from derived cubic spline interpolation polynomial) for uniform points: $x_i = ih, (i = 0,1,2,......,K)$ **with** $h = 10^{-3}$, $K = 10^3$ **for example :5.3**

x	Exact Solution-y(x)	DQM Solution-y(x) $N = 212, \varepsilon = 10^{-4}$ M.A.E: 0. 239965E-02	DQM Solution-y(x) $N = 312, \varepsilon = 10^{-4}$ M.A.E: 0. 254341E-04
.000	1.0000250	1.0000000	1.0000000
.030	.9995750	1.0010649	.9995534
.050	.9987756	.9994548	.9987554
.070	.9975777	.9976295	.9975610
.090	.9959827	.9982178	.9959593
.200	.9802213	.9808511	.9801967
.400	.9231326	.9230739	.9231101
.600	.8352788	.8374397	.8352600
.800	.7261514	.7275579	.7261384
.900	.6669774	.6670876	.6669649
.925	.6519340	.6521308	.6519161
.950	.6368318	.6381038	.6368168
.975	.6216908	.6220101	.6216768
.990	.6125958	.6132055	.6125853
.992	.6113828	.6132035	.6113714
.994	.6101698	.6111104	.6101554
.996	.6089568	.6097005	.6089434
.998	.6077437	.6094422	.6077285
.999	.6071550	.6083070	.6071558
1.000	1.0000000	1.0000000	1.0000000

References

[1] R. E. Bellman and J. Casti, *Differential Quadrature and Long-Term Integration*, Journal of Mathematical Analysis and Application, 34 (1971), 235-238.

[2] R. E. Bellman, B. G. Kashef and J. Casti, *Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations*, J. Comput. Phys., 10 (1972), 40-52.

[3] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York, 1978.

[4] F. Civan, and C. M. Sliepcevich, *Differential quadrature for multidimensional problems*, J. Math. Anal. Appl., 101 (1984), 423-443.

[5] H. Du, and M. K. Lin, *Application of differential quadrature to vibration analysis*, J. Sound Vib. 181 (1995), 279-293.

[6] P. A. Farrell, A. F. Hegarty, J.J.H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust Computational techniques for Boundary Layers*, Chapman & Hall/CRC Press, 2000.

[7] 7. M. K. Kadalbajoo, and V. K. Aggarwal, *Fitted mesh B-spline method for solving a class of singular singularly perturbed boundary value problems*, Int. J. Computer Mathematics, vol. (82), 1, (2005), 67-76.

[8] J. Li, *A Computational Method for Solving Singularly Perturbed Two-Point Singular Boundary Value Problem*, Int. Journal of Math. Analysis, Vol. 2, (2008), no. 22, 1089-1096.

[9] J. J. H. Miller, E. O. Riordan and G. I. Shishkin, *Fitted numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

[10] R. K. Mohanty and U. Arora, *A family of non-uniform mesh tension spline methods for singularly perturbed two-point singular boundary value problems with significant first derivatives*, App. Math. And Comput. Vol. (172), 1, (2006), 531-544.

[11] R. K. Mohanty, D. J. Evans and U. Arora, *Convergent spline in tension methods for singularly perturbed two-point singular boundary value problems*, Int. J. Comput. Math., 82 (1), (2005), 55-66.

[12] R. K. Mohanty and N. Jha, *A class of variable mesh spline in compression methods for singularly perturbed two point singular boundary value problems*.App. Math. And Comput. Vol. (168), 1, (2004), 704-716

[13] A. H. Nayfeh, *Perturbation Methods*, Wiley, New York, 1973,

[14] R. E. O'Malley, *Introduction to Singular Perturbations*, Academic Press, New York, 1974.

[15] H. S. Prasad, Y. N. Reddy, *Numerical Solution of Singularly Perturbed Differential-Difference Equations with Small Shifts of*

Mixed Type by Differential Quadrature Method, American Journal of Comput. and Appl. Mathematics, 2 (1), (2012), pp. 46-52.

[16] J. R. Quan, J.R. and C. T. Chang, *New insights in solving distributed system equations by the quadrature methods-I. Analysis*, Comput. Chem. Engrg, 13(1989), 779-788.

[17] J. R. Quan, and C. T. Chang, *New insights in solving distributed system equations by the quadrature methods-II. Application*, Comput. Chem. Engrg, 13, (1989), 1017-1024.

[18] J. Rashidinia, R. Mohammadi, M. Ghasemij, *Cubic Spline Solution of Singularly Perturbed boundary value problems with significant first derivatives*, App. Math. And Comput. Vol. 190, 2, (2007), 1762-1766.

[19] H. G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed differential equations,* Springer, Berlin, 1996.

[20] A. N. Sherbourne, M.D. Pandy, *Differential Quadrature Method in the buckling analysis of beams and composite plates.* Comput Struct, 40, (1991), pp 903-913.

[21] C. Shu, *Differential Quadrature and Its Application in Engineering.* Springer-Verlag, London, 2000.

[22] C. Shu, and B.E. Richards, *Application of Generalized differential quadrature to solve two-dimensional incompressible Navier-Stokes equations*, Int. J. Numer. Meth Fl, 15 (1992) 7, 791-798.