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Optimal Quadrature Formulas with Polynomial Weight in Sobolev Space

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Abstract In this paper we construct the optimal quadrature formula with polynomial weight in the Sobolev space $L_2^{(m)}(0,1)$. Using S.L. Sobolev's method we obtain new optimal quadrature formula of such type and give explicit expressions for the corresponding optimal coefficients. Also, we include a few numerical examples in order to illustrate the application of the obtained optimal quadrature formula.

Keywords: *optimal quadrature formulas, error functional, extremal function, sobolev space, optimal coefficients*

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1. Introduction and Statement of the Problem

We consider the following quadrature formula

$$\int_0^1 x^\alpha \varphi(x)dx \cong \sum_{\gamma=0}^{N} C_\gamma \varphi(x_\gamma), \qquad (1.1)$$

with the error functional given by

$$\ell(x) = x^\alpha \varepsilon_{[0,1]}(x) - \sum_{\gamma=0}^{N} C_\gamma \delta(x-x_\gamma), \qquad (1.2)$$

in the space $L_2^{(m)}(0,1)$, where C_γ and $x_\gamma (\in[0,1])$ are coefficients and nodes of the formula (1.1), respectively, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $\alpha \geq 0$, $\delta(x)$ is Dirac's delta-function, function φ belongs to the space $L_2^{(m)}(0,1)$. $L_2^{(m)}(0,1)$ is the Sobolev space of functions with square integrable *m-th* generalized derivative. In this space the norm of a function is defined by the formula

$$\left\| \varphi \mid L_2^{(m)}(0,1) \right\| = \left\{ \int_0^1 \left(\varphi^{(m)}(x) \right)^2 dx \right\}^{1/2}. \qquad (1.3)$$

The difference

$$(\ell,\varphi) = \int_0^1 \varphi(x)dx - \sum_{\gamma=0}^{N} C_\gamma \varphi(x_\gamma) = \int_{\mathbb{R}} \ell(x)\varphi(x)dx \quad (1.4)$$

is called *the error* of the quadrature formula (1.1). The error (1.4) of the formula (1.1) is a linear functional in $L_2^{(m)*}(0,1)$, where $L_2^{(m)*}(0,1)$ is the conjugate space to $L_2^{(m)}(0,1)$ space.

For the error functional (1.4) to be defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [32])

$$\left(\ell(x), x^\mu\right) = 0, \ \mu = 0,1,...,m-1. \qquad (1.5)$$

By the Cauchy - Schwarz inequality

$$|(\ell,\varphi)| \leq \| \varphi | L_2^{(m)} \| \cdot \| \ell | L_2^{(m)*}(0,1) \|$$

the error (1.4) of the formula (1.1) is estimated with the help of the norm

$$\left\| \ell | L_2^{(m)*}(0,1) \right\| = \sup_{\left\| \varphi | L_2^{(m)}(0,1) \right\| = 1} \left| (\ell,\varphi) \right|$$

of the error functional (1.2). Consequently, estimation of the error of the quadrature formula (1.1) on functions of the space $L_2^{(m)}(0,1)$ is reduced to finding the norm of the error functional ℓ in the conjugate space $L_2^{(m)*}(0,1)$.

A minimization of the norm of the error functional ℓ with respect to the coefficients C_γ, when the nodes are fixed, is called as *Sard's problem*. The obtained formula is called *the optimal quadrature formula in the sense of Sard*. This problem was first investigated by A.Sard [17].

There are several methods of construction of optimal quadrature formulas in the sense of Sard [2,34]. In the space $L_2^{(m)}(a,b)$, based on these methods, Sard's problem was investigated by many authors (see, for example, [1-10,13-16,19,20,22-29,31-36] and references therein).

In the present paper we give the sulution of Sard's problem for the formula (1.1) in the space $L_2^{(m)}(0,1)$. Namely, we find the coefficients C_γ such that

$$\| \ell | L_2^{(m)*}(0,1) \| = \inf_{C_\beta} \| \ell | L_2^{(m)*}(0,1) \|. \qquad (1.6)$$

Thus, in order to construct an optimal quadrature formula of the form (1.1) in the space $L_2^{(m)}(0,1)$, we need conse-quently to solve the following two problems:

Problem 1. *Calculate the norm of the error functional* ℓ *for the given quadrature formula (1.1) in* $L_2^{(m)*}(0,1)$ *space.*

Problem 2. *Find the values of the coefficients* C_β *such that the equality (1.6) be satisfied with fixed nodes* x_β.

In order to solve Problem 1, i.e., to calculate the norm of the error functional (1.2) in the space $L_2^{(m)*}(0,1)$, we use a concept of the extremal function for a given functional. The function ψ_ℓ is called the *extremal* for the functional ℓ (cf. [25]) if the following equality is fulfilled

$$(\ell, \psi_\ell) = \left\| \ell | L_2^{(m)*}(0,1) \right\| \cdot \left\| \psi_\ell | L_2^{(m)}(0,1) \right\|. \qquad (1.7)$$

For the extremal function ψ_ℓ of the error function $\ell \in L_2^{(m)*}(0,1)$ the following result was proved by S.L.Sobolev [32].

Theorem 1. *The extremal function* ψ_ℓ *of the error functional* $\ell(x)$ *has the following form*

$$\psi_\ell(x) = (G * \ell)(x) + P_{m-1}(x),$$

where $P_{m-1}(x)$ *is a polynomial of degree* $m-1$, *and*

$$G(x) = \frac{x^{2m-1} sign(x)}{2 \cdot (2m-1)!} \qquad (1.8)$$

is the solution of the equation

$$G^{(2m)}(x) = \delta(x).$$

The symbol * is the convolution of functions *f* and *g* is defined by the formula

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) dy = \int_{\mathbb{R}} f(y) g(x-y) dy.$$

Now, using Theorem 1, we immediately obtain a representation of the norm of the error functional

$$\left\| \ell | L_2^{(m)*}(0,1) \right\|^2 = (-1)^m \left[\sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta C_\gamma \cdot \frac{(h\beta - h\gamma)^{2m-1}}{2 \cdot (2m-1)!} \right.$$
$$- \sum_{\beta=0}^{N} C_\beta \sum_{\gamma=0}^{2m-1} \frac{(-1)^\gamma}{\gamma!(2m-1-\gamma)!} \cdot \left(\frac{(h\beta)^{2m+\alpha}}{\alpha+\gamma+1} + \frac{(h\beta)^\gamma - (h\beta)^{2m+\alpha}}{\alpha - \gamma + 2m} \right) \qquad (1.9)$$
$$\left. + \sum_{\gamma=0}^{2m-1} \frac{(-1)^\gamma}{\gamma!(2m-1-\gamma)!} \cdot \left(\frac{1}{(\alpha+\gamma+1)(2\alpha+2m+1)} \right) \right].$$

Thus, Problem 1 is solved.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 explicit formulas for the coefficients of the optimal quadrature formula of the form (1.1) are found; finally, In Section 4 some numerical results are presented.

2. Preliminaries

In the present section we give some definitions and known results that we need to prove the main results.

Below mainly we use the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given in [32,33]. For completeness we give some definitions about functions of discrete argument.

Assume that the nodes x_β are equal spaced, i.e., $x_\beta = h\beta$, $h = 1/N$, $N = 1, 2, \ldots$.

Definition 1. The function $\varphi(h\beta)$ is a *function of discrete argument* if it is given on some set of integer values of β.

Definition 2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is given by

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 3. *The convolution of two functions* $\varphi(h\beta)$ *and* $\psi(h\beta)$ *is the inner product*

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)]$$
$$= \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

The Euler-Frobenius polynomials $E_k(x)$, $k = 1, 2, \ldots$ are defined by the following formula [26]

$$E_k(x) = \frac{(1-x)^{k+2}}{x} \left(x \frac{d}{dx} \right)^k \frac{x}{(1-x)^2}, \qquad (2.1)$$

$E_0(x) = 1$.

The following theorem is true

Theorem 2. *(Lemma 3 of [18]). Polynomial* $Q_k(x)$ *which is defined by the formula*

$$Q_k(x) = (x-1)^{k+1} \sum_{i=0}^{k+1} \frac{\Delta^i 0^{k+1}}{(x-1)^i}$$

is the Euler-Frobenius polynomial (2.1) of degree k, i.e. $Q_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i} (-1)^{i-l} C_i^l l^k$.

The following formula is valid [30]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q} \right)^i \Delta^i 0^k -$$
$$- \frac{q^n}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q} \right)^i \Delta^i \gamma^k |_{\gamma=n}, \qquad (2.2)$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k, q is the ratio of a geometric progression. When $|q|<1$ from (2.2) we have

$$\sum_{\gamma=0}^{\infty} q^{\gamma} \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q} \right)^i \Delta^i 0^k. \qquad (2.3)$$

In our computations we need the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m} which satisfies the following equality

$$hD_m(h\beta) * G(h\beta) = \delta(h\beta), \qquad (2.4)$$

where $G(h\beta) = \frac{|h\beta|^{2m-1}}{2(2m-1)!}$ is the discrete argument function corresponding to the function $G(x)$ defined by (1.8), $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e. $\delta(h\beta)$ is the discrete delta-function.

It should be noted that the operator $D_m(h\beta)$ was firstly introduced and investigated by S.L. Sobolev [32].

In [21] the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m}, which satisfies equation (2.4), is constructed and the following theorem is proved.

Theorem 3. *The discrete analogue of the differential operator* d^{2m}/dx^{2m} *has the form*

$$D_m(h\beta) = p \begin{cases} \sum_{k=1}^{m-1} A_k q_k^{|\beta|-1} & for \quad |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} A_k & for \quad |\beta| = 1, \\ C + \sum_{k=1}^{m-1} \frac{A_k}{q_k} & for \quad \beta = 0, \end{cases} \qquad (2.5)$$

where

$$p = \frac{(2m-1)!}{h^{2m}}, \quad A_k = \frac{(1-q_k)^{2m+1}}{E_{2m-1}(q_k)}, \quad C = -2^{2m-1}, \qquad (2.6)$$

$E_{2m-1}(q)$ is the Euler-Frobenius polynomial of degree $2m-1$, q_k are the roots of the Euler-Frobenius polynomial $E_{2m-2}(q)$, $|q_k|<1$, h is a small positive parameter.

Furthermore several properties of the discrete argument function $D_m(h\beta)$ were proved in [21]. Here we give the following property of $D_m(h\beta)$ which we need in our computations.

Theorem 4. *The discrete argument function* $D_m(h\beta)$ *and the monomials* $(h\beta)^k$ *are related to each other as follows*

$$\sum_{\beta=-\infty}^{\infty} D_m(h\beta)(h\beta)^k = \begin{cases} 0 & when \quad 0 \leq k \leq 2m-1, \\ (2m)! & when \quad k = 2m, \\ 0 & when \quad 2m+1 \leq k \leq 4m-1, \\ \frac{h^{2m}(4m)!B_{2m}}{(2m)!} & when \quad k = 4m, \end{cases}$$

where B_{2m} *is the Bernoulli number.*

3. The Optimal Coefficients of the Quadrature Formula (1.1)

Let the nodes x_γ of the quadrature formula (1.1) be fixed. The error functional (1.2) satisfies the conditions (1.5). Norm of the error functional ℓ is a multidimensional function of the coefficients C_γ, $(\gamma = 0,1,...,N)$. For finding its minimum under the conditions (1.5), we apply the Lagrange method.

Namely, we consider the function

$$\Phi(C,\lambda) = \|\ell_N\|^2 - 2 \cdot (-1)^m \cdot \sum_{\tau=0}^{m-1} \lambda_\tau \left(\ell_N, x^\tau \right),$$

where $C = (C_0, C_1, ..., C_N)^T$ and $\lambda = (\lambda_0, \lambda_1, ..., \lambda_{m-1})^T$, it's partial derivatives by C_γ $\left(\gamma = \overline{0,N}\right)$ and λ_τ $\left(\tau = \overline{0,m-1}\right)$ equating to zero, so that we obtain the following system of linear equations

$$G(h\beta) * C_\beta + P_{m-1}(h\beta) = f(h\beta), \; \beta = 0,1,...,N, \qquad (3.1)$$

$$C_\beta = 0, \quad h\beta \in [0,N],$$

$$\sum_{\beta=0}^{N} C_\beta \cdot (h\beta)^\mu = g_\mu, \; \mu = 0,1,...,m-1, \qquad (3.2)$$

where $P_{m-1}(x)$ is a polynomial of degree $m-1$, and $G(x)$ is defined by (1.8), $g_\mu = \frac{1}{\alpha+\mu+1}$,

$$f(h\beta) = \frac{1 - 2 \cdot (h\beta)^{2 \cdot m + \alpha}}{2 \cdot (2 \cdot m - 1)! \cdot (2 \cdot m + \alpha)} + \sum_{l=1}^{2 \cdot m - 1} \frac{(-1)^l \cdot \left((h\beta)^l - 2 \cdot (h\beta)^{2 \cdot m + \alpha} \right)}{2 \cdot l! \cdot (2 \cdot m - 1 - l)! \cdot (2 \cdot m + \alpha - l)}. \qquad (3.3)$$

Further we investigate Problem 2. Instead of C_β we introduce the following functions

$$v(h\beta) = G_m(h\beta) * C_\beta, \qquad (3.4)$$
$$u(h\beta) = v(h\beta) + P_{m-1}(h\beta).$$

In such statement it is necessary to express the coefficients C_β by the function $u(h\beta)$.

Then taking into account (2.4), (3.4) and Theorems 3, 4, for the coefficients we have

$$C_\beta = hD_m(h\beta) * u(h\beta). \qquad (3.5)$$

Thus, if we find the function $u(h\beta)$, then the coefficients C_β will be found from equality (3.5).

To calculate the convolution (3.5) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From equality (3.1) we get that $u(h\beta) = f(h\beta)$ when $h\beta \in [0,1]$. Now we need to find the representation of the function $u(h\beta)$ when $\beta<0$ and $\beta>N$.

Since $C_\beta=0$ when $h\beta\notin[0,1]$ then

$$C_\beta=hD_m(h\beta)*u(h\beta)=0,\quad h\beta\notin[0,1].$$

Now we calculate the convolution $v(h\beta)=G_m(h\beta)*C_\beta$ when $\beta\le 0$ and $\beta\ge N$.

Suppose $\beta\le 0$ then taking into account equalities (3.2) we have

$$\begin{aligned}
v(h\beta)&=\sum_{\gamma=-\infty}^{\infty}C_\gamma G_m(h\beta-h\gamma)\\
&=\sum_{\gamma=0}^{N}C_\gamma\frac{|h\beta-h\gamma|^{2m-1}}{2(2m-1)!}\\
&=-\sum_{\gamma=0}^{N}C_\gamma\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}\\
&=-\sum_{\gamma=0}^{N}C_\gamma\sum_{k=0}^{2m-1}\frac{(h\beta)^{2m-1-k}(-1)^k(h\gamma)^k}{2\cdot k!\cdot(2m-1-k)!}\\
&=-\sum_{k=0}^{m-1}\frac{(h\beta)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^k\\
&\quad-\sum_{k=m}^{2m-1}\frac{(h\beta)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^k\\
&=-\sum_{k=0}^{m-1}\frac{(h\beta)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!}\cdot g_k-Q_{m-1}(h\beta)\\
&=-R(h\beta)-Q_{m-1}(h\beta).
\end{aligned}$$

Thus, when $\beta\le 0$, we get

$$v(h\beta)=-R(h\beta)-Q_{m-1}(h\beta), \tag{3.6}$$

where

$$R(h\beta)=\sum_{k=0}^{m-1}\frac{(h\beta)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!}\cdot g_k,$$

is the polynomial of degree $2m-1$ and

$$Q_{m-1}(h\beta)=\sum_{k=m}^{2m-1}\frac{(h\beta)^{2m-1-k}(-1)^k}{2\cdot k!\cdot(2m-1-k)!}\sum_{\gamma=0}^{N}C_\gamma(h\gamma)^k$$

is a unknown polynomial of degree $m-1$ of $(h\beta)$.

Similarly, in the case $\beta\ge N$ for the convolution $v(h\beta)=G_m(h\beta)*C_\beta$ we obtain

$$v(h\beta)=R(h\beta)+Q_{m-1}(h\beta). \tag{3.7}$$

We denote

$$P^-_{m-1}(h\beta)=P_{m-1}(h\beta)-Q_{m-1}(h\beta), \tag{3.8}$$

$$P^+_{m-1}(h\beta)=P_{m-1}(h\beta)+Q_{m-1}(h\beta), \tag{3.9}$$

where

$$P^-_{m-1}(h\beta)=\sum\nolimits_{\tau=0}^{m-1}p^-_\tau\cdot(h\beta)^\tau,$$

$$P^+_{m-1}(h\beta)=\sum\nolimits_{\tau=0}^{m-1}p^+_\tau\cdot(h\beta)^\tau.$$

Taking into account (3.4), (3.6) and (3.7) we get the following problem

Problem 3. *Find the solution of the equation*

$$D_m(h\beta)*u(h\beta)=0,\quad h\beta\notin[0,1] \tag{3.10}$$

having the form:

$$u(h\beta)=\begin{cases}R(h\beta)+P^-_{m-1}(h\beta), & \beta\le 0,\\ f(h\beta), & 0\le\beta\le N,\\ R(h\beta)+P^+_{m-1}(h\beta), & \beta\ge N.\end{cases} \tag{3.11}$$

Here $P^-_{m-1}(h\beta)$ *and* $P^+_{m-1}(h\beta)$ *are unknown polynomials of degree* $m-1$ *with respect to* $h\beta$.

If we find $P^-_{m-1}(h\beta)$, $P^+_{m-1}(h\beta)$ then from (3.8), (3.9) we have

$$P_{m-1}(h\beta)=\frac{1}{2}\left(P^+_{m-1}(h\beta)+P^-_{m-1}(h\beta)\right), \tag{3.12}$$

$$Q_{m-1}(h\beta)=\frac{1}{2}\left(P^+_{m-1}(h\beta)-P^-_{m-1}(h\beta)\right).$$

Unknowns $P^-_{m-1}(h\beta)$, $P^+_{m-1}(h\beta)$ can be found from equation (3.10), using the function $D_m(h\beta)$ defined by (2.5). Then we obtain explicit form of the function $u(h\beta)$ and from (3.5) we find the coefficients C_β. Furthermore from (3.12) we get $P_{m-1}(h\beta)$.

Thus **Problem 3** and respectively **Problem 2** will be solved.

The main result of the present is the following theorem.

Theorem 5. *Coefficients of the optimal quadrature formula (1.1), with equally spaced nodes in the space* $L_2^{(m)}(0,1)$, *have the following form*

$$\begin{aligned}C_0=\frac{(2m-1)!}{h^{2m-1}}\Bigg[&\sum_{k=1}^{m-1}\frac{A_k}{q_k}\cdot\left(M_k+q_k^N\cdot N_k+\sum_{\gamma=0}^{N}q_k^\gamma\cdot f(h\gamma)\right)\\
&+\sum_{t=0}^{m-1}p_t^-\cdot(-h)^t-R(-h)+f(h)+C\cdot f(0)\Bigg]\end{aligned} \tag{3.13}$$

$$\begin{aligned}C_\beta=\frac{(2m-1)!}{h^{2m-1}}\Bigg[&\sum_{k=1}^{m-1}\frac{A_k}{q_k}\cdot\Big(q_k^\beta\cdot M_k+q_k^{N-\beta}\cdot N_k+\\
&+\sum_{\gamma=0}^{N}q_k^{|\beta-\gamma|}\cdot f(h\gamma)+f(h\cdot(\beta-1))+C\cdot f(h\beta)+\\
&+f(h\cdot(\beta+1))\Bigg],\quad \beta=1,2,\ldots,N-1,\end{aligned} \tag{3.14}$$

$$\begin{aligned}C_N=\frac{(2m-1)!}{h^{2m-1}}\Bigg[&\sum_{k=1}^{m-1}\frac{A_k}{q_k}\cdot\Big(q_k^N\cdot M_k+N_k+\\
&+q_k^N\cdot\sum_{\gamma=0}^{N}q_k^{-\gamma}\cdot f(h\gamma)\Big)+\sum_{t=0}^{m-1}p_t^+\cdot(1+h)^t+\\
&+R(1+h)+f(1-h)+C\cdot f(1)\Bigg]\end{aligned} \tag{3.15}$$

$$p_\tau=\frac{1}{2}\left(p^+_\tau+p^-_\tau\right),\ \tau=0,1,\ldots,m-1,$$

where

$$M_k = \sum_{t=1}^{m-1}\left[p_t^-(-h)^t \sum_{i=1}^{t} \frac{q_k^i \Delta^i 0^t}{(1-q_k)^{i+1}} - \right. \tag{3.16}$$
$$\left. -\frac{(-1)^t \cdot (-h)^{2m-1-t}}{2\cdot t!(2m-1-t)!} \cdot g_t \times \sum_{i=1}^{2m-1-t} \frac{q_k^i \Delta^i 0^{2m-1-t}}{(1-q_k)^{i+1}} \right] + \frac{p_0^- q_k}{1-q_k},$$

$$N_k = \frac{1}{1-q_k} \cdot \sum_{t=0}^{m-1}\left[p_t^+ \sum_{j=1}^{t} C_t^j h^j \sum_{i=0}^{j} \left(\frac{q_k}{1-q_k}\right)^i \Delta^i 0^j + \right.$$
$$+ p_t^+ q_k + \frac{(-1)^t}{2\cdot t!(2m-1-t)!} \cdot g_t \times \tag{3.17}$$
$$\left. \times \left(q_k + \sum_{j=1}^{2m-1-t} \frac{(2m-1-t)! h^j}{l! \cdot (2m-1-t-j)!} \cdot \sum_{i=0}^{j} \left(\frac{q_k}{1-q_k}\right)^i \Delta^i 0^j \right) \right]$$

and p, C, A_k are defined by (2.6), q_k are the roots of the Euler-Frobenius polynomial $E_{2m-2}(q)$, $|q_k|<1$, $\Delta^i 0^\tau = \sum_{l=1}^{i}(-1)^{i-l} C_i^l l^\tau$ and $p_\tau^-, p_\tau^+ \ (\tau = 0,1,...,m-1)$ are defined from the system (3.18)-(3.19), (3.21)-(3.22).

Proof. First we find the expressions for p_0^- and p_0^+. When $\beta = 0$ and $\beta = N$ from (3.11) for p_0^- and p_0^+ we get

$$p_0^- = f(0), \tag{3.18}$$

$$p_0^+ = f(1) - R(1) - \sum_{\tau=1}^{m-1} p_\tau^+. \tag{3.19}$$

Now we have $2m-2$ unknowns p_τ^-, p_τ^+, $\tau = 1,2,...,m-1$.

From equation (3.10), by choosing $\beta = -1,-2,...,-(m-1)$ and $\beta = N+1, N+2,..., N+m-1$, we are able to find p_τ^- and $p_\tau^+ \left(\tau = \overline{1, m-1}\right)$.

Taking into account (3.11), from (3.10) we get the following system

$$\sum_{\tau=1}^{m-1} p_\tau^- \sum_{\gamma=1}^{\infty} D_m(h\beta + h\gamma)(-h\gamma)^\tau + \sum_{\tau=1}^{m-1} p_\tau^+ \sum_{\gamma=1}^{\infty} D_m(h(N+\gamma)$$
$$-h\beta) \sum_{j=1}^{\tau} C_\tau^j (h\gamma)^j = -\sum_{\gamma=0}^{N} D_m(h\beta - h\gamma) f(h\gamma) \tag{3.20}$$
$$- f(0) \sum_{\gamma=1}^{\infty} D_m(h\beta + h\gamma) - f(1) \sum_{\gamma=1}^{\infty} D_m(h(N+\gamma) - h\beta),$$

where $\beta = -1,-2,...,-(m-1)$ and $\beta = N+1, \ N+2,..., N+m-1$.

First we consider the cases $\beta = -1,-2,...,-(m-1)$. From (3.20) replacing β by $-\beta$ and using (2.5) and (2.3), after some calculations for $\beta = 1,2,...,m-1$, we get the following system of $m-1$ linear equations

$$\sum_{\tau=1}^{m-1} p_\tau^- B_{\beta\tau}^- + \sum_{\tau=1}^{m-1} p_\tau^+ B_{\beta\tau}^+ = T_\beta, \ \ \beta = 1,2,...,m-1, \tag{3.21}$$

where

$$B_{\beta\tau}^- = (-h)^\tau \left[\begin{array}{l} \sum_{k=1}^{m-1} \frac{A_k}{q_k} \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|} \gamma^\tau + (\beta-1)^\tau \\ \qquad + C\beta^\tau + (\beta+1)^\tau \end{array} \right],$$

$$B_{\beta\tau}^+ = \sum_{k=1}^{m-1} A_k q_k^{N+\beta-1} \sum_{j=1}^{\tau} C_\tau^j h^j \sum_{i=1}^{j} \frac{q_k^i \Delta^i 0^j}{(1-q_k)^{i+1}},$$

$$T_\beta = -\sum_{k=1}^{m-1} \frac{A_k}{q_k} [q_k^\beta (\sum_{\gamma=0}^{N} q_k^\gamma f(h\gamma) + q_k^N \sum_{\gamma=1}^{\infty} q_k^\gamma R(h\gamma+1)$$
$$+ (f(1) - R(1)) \frac{q_k^{N+1}}{1-q_k}) + \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|} (f(0) - R(-h\gamma))]$$
$$-(2+C) \cdot f(0) + R(-h(\beta-1))$$
$$+C \cdot R(-h\beta) + R(-h(\beta+1)).$$

Here $\beta = 1,2,...,m-1$ and $\tau = 1,2,...,m-1$.

Now we consider the cases $\beta = N+1, N+2,..., N+m-1$.

From (3.20) replacing β by $N+\beta$ and using (2.5) and (2.3), after some calculations for $\beta = 1,2,...,m-1$ we get the following system of $m-1$ linear equations

$$\sum_{\tau=1}^{m-1} p_\tau^- A_{\beta\tau}^- + \sum_{\tau=1}^{m-1} p_\tau^+ A_{\beta\tau}^+ = S_\beta, \ \ \beta = 1,2,...,m-1, \tag{3.22}$$

where

$$A_{\beta\tau}^- = (-h)^\tau \sum_{k=1}^{m-1} A_k q_k^{N+\beta-1} \sum_{i=1}^{\tau} \frac{q_k^i \Delta^i 0^\tau}{(1-q_k)^{i+1}},$$

$$A_{\beta\tau}^+ = \sum_{j=1}^{\tau} C_\tau^j h^j \left[\begin{array}{l} \sum_{k=1}^{m-1} \frac{A_k}{q_k} \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|} \gamma^j + (\beta-1)^j \\ \qquad + C\beta^j + (\beta+1)^j \end{array} \right],$$

$$S_\beta = -\sum_{k=1}^{m-1} \frac{A_k}{q_k} [q_k^{N+\beta} \left(\begin{array}{l} \sum_{\gamma=0}^{N} q_k^{-\gamma} f(h\gamma) - \sum_{\gamma=1}^{\infty} q_k^\gamma R(-h\gamma) \\ + \frac{q_k}{1-q_k} \cdot f(0) \end{array} \right)$$
$$+ \sum_{\gamma=1}^{\infty} q_k^{|\beta-\gamma|} \left(f(1) - R(1) + R(h\gamma+1) \right)]$$
$$-(2+C) \cdot (f(1) - R(1)) + R(h(\beta-1)+1)$$
$$+C \cdot R(h\beta+1) + R(h(\beta+1)+1).$$

Here $\beta = 1,2,...,m-1$ and $\tau = 1,2,...,m-1$.

Thus for the unknowns p_τ^-, p_τ^+, $\tau = 1,2,...,m-1$ we have obtaint the system (3.21), (3.22) of $2m-2$ linear equations. Since our problem has a unique solution, the main matrix of this system is non singular. Unknowns p_τ^-, $p_\tau^+ \ (\tau = 1,2,...,m-1)$ can be found from system (3.21), (3.22). Then taking into account (3.12), using (3.18) and (3.19) we have

$$p_\tau = \frac{1}{2}\left(p_\tau^+ + p_\tau^-\right), \ \tau = 0,1,...,m-1.$$

Now we find the coefficients C_β, $\beta = 0,1,...,N$.

From (3.5), taking into account (2.5), we deduce

$$C_\beta = h\left[\begin{array}{l}\sum_{\gamma=0}^{N} D_m(h\beta - h\gamma) f(h\gamma) \\ +\sum_{\gamma=1}^{\infty} D_m(h\beta + h\gamma)\sum_{\tau=0}^{m-1} p_\tau^-(-h\gamma)^\tau \\ +\sum_{\gamma=1}^{\infty} D_m(h(N+\gamma)-h\beta)\sum_{\tau=0}^{m-1} p_\tau^+(1+h\gamma)^\tau\end{array}\right],$$

$\beta = 0,1,...,N.$

From here, using (2.5) and formula (2.3), taking into account (3.16) and (3.17), after some calculations we arrive at the expressions of the coefficients C_β, $\beta = 0,1,...,N$ which are given in the assertion of the theorem.

Theorem 5 is proved.

To point out the applicability of the formulas obtained above, we will focus on the particular cases $m=1$ and $m=2$.

The case $m=1$.

In this case Problem 2 is as follows.

Problem 4. *Find coefficients of the optimal quadrature formula (1.1) in the space* $L_2^{(1)}(0,1)$.

The solution of Problem 4 is the coefficients C_γ, $\gamma = 0,1,...,N$ and p_0. They satisfy thefollowing system

$$\sum_{\gamma=0}^{N} C_\gamma \frac{|h\beta - h\gamma|}{2} + p_0 = f(h\beta),\ \beta = 0,1,...,N,$$

$$\sum_{\gamma=0}^{N} C_\gamma = \frac{1}{\alpha+1} = g_0,$$

where

$$f(h\beta) = \frac{1-2\cdot(h\beta)^{2+\alpha}}{2\cdot(2+\alpha)} - \frac{(h\beta)-2\cdot(h\beta)^{2+\alpha}}{2\cdot(1+\alpha)}$$

##

So, as a direct consequence of Theorem 5, we arrive at the following result

Corollary 1. *The coefficients of the optimal quadrature formula (1.1) in the space* $L_2^{(1)}(0,1)$ *have the following form*

$$C_0 = \frac{1}{h}\left[\frac{h}{2}\cdot g_0 + f(h) - f(0)\right],$$

$$C_\beta = \frac{1}{h}\left[\begin{array}{l}(f(h\beta - h) - 2f(h\beta) \\ + f(h\beta + h))\end{array}\right],\ \beta = 1,2,...,N-1,$$

$$C_N = \frac{1}{h}\left[f(1-h) - f(1) + \frac{h}{2}\cdot g_0\right],$$

$$p_0 = \frac{1}{2}\left[f(0) + f(1) - \frac{1}{2}\cdot g_0\right].$$

The case $m=2$. Then from Problem 2 we have

Problem 5. *Find coefficients of the optimal quadrature formula (1.1) in the space* $L_2^{(2)}(0,1)$.

The solution of Problem 5, the coefficients C_γ, $\gamma = 0,1,...,N$, p_1 and p_0 which satisfy the system

$$\sum_{\gamma=0}^{N} C_\gamma \frac{|h\beta - h\gamma|^3}{12} + p_1\cdot(h\beta) + p_0 = f(h\beta),$$

$\beta = 0,1,...,N,$

$$\sum_{\gamma=0}^{N} C_\gamma = \frac{1}{\alpha+1} = g_0,$$

$$\sum_{\gamma=0}^{N} C_\gamma\cdot(h\gamma) = \frac{1}{\alpha+2} = g_1,$$

where

$$f(h\beta) = \frac{1}{12}\cdot\left(\frac{1}{\alpha+4} - \frac{3\cdot(h\beta)}{\alpha+3} + \frac{3\cdot(h\beta)^2}{\alpha+2} - \frac{(h\beta)^3}{\alpha+1}\right) + \frac{(h\beta)^{\alpha+4}}{(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)}.$$

In this case we have the following result as an immediate corollary of Theorem 5.

Corollary 2. *The coefficients of the optimal quadrature formula (1.1) in the space* $L_2^{(2)}(0,1)$ *have the form [11]*

$$C_0 = \frac{6}{h^3}\cdot\left[\begin{array}{l}p_0^- - 8\cdot f(0) + f(h) - p_1^-\cdot h \\ +\frac{1}{4}\cdot g_1\cdot h^2 + \frac{1}{12}\cdot g_0\cdot h^3\end{array}\right] + \frac{36\sqrt{3}}{h^3}\cdot\left[\sum_{\gamma=0}^{N} q_1^\gamma\cdot f(h\cdot\gamma) + M_1 + q_1^N\cdot N_1\right],$$

$$C_\beta = \frac{6}{h^3}\cdot\left[f(h\cdot(\beta-1)) - 8\cdot f(h\cdot\beta) + f(h\cdot(\beta+1))\right] + \frac{36\sqrt{3}}{h^3}\cdot\left[\sum_{\gamma=0}^{N} q_1^{|\beta-\gamma|}\cdot f(h\cdot\gamma) + q_1^\beta\cdot M_1 + q_1^{N-\beta}\cdot N_1\right],$$

$\beta = 1,2,...,N-1,$

$$C_N = \frac{6}{h^3}\cdot\left[\begin{array}{l}p_0^+ + p_1^+ - 8\cdot f(1) + f(1-h) \\ +\frac{1}{12}\cdot(g_0 - 3\cdot g_1) \\ +\frac{1}{4}\cdot(g_0 - 2\cdot g_1 + 4\cdot p_1^+)\cdot h \\ +\frac{1}{4}\cdot(g_0 - g_1)\cdot h^2 + \frac{1}{12}\cdot g_0\cdot h^3\end{array}\right] + \frac{36\sqrt{3}}{h^3}\cdot\left[\sum_{\gamma=0}^{N} q_1^{N-\gamma}\cdot f(h\cdot\gamma) + q_1^N M_1 + N_1\right],$$

$$p_\alpha = \frac{1}{2}(p_\alpha^+ + p_\alpha^-),\quad \alpha = 0,1,$$

where

$$M_1 = \frac{1}{6}\cdot(q_1 - 1)\cdot p_0^- + \frac{1}{6}\cdot p_1^-\cdot h - \frac{1}{72}\cdot(q_1+2)\cdot g_1\cdot h^2,$$

$$N_1 = \frac{1}{72}\cdot(q_1-1)\cdot(12\cdot(p_0^+ + p_1^+) + g_0 - 3\cdot g_1)$$
$$-\frac{1}{24}\cdot(4\cdot p_1^+ + g_0 - 2\cdot g_1)\cdot h$$
$$-\frac{1}{72}\cdot(q_1+2)\cdot(g_0-g_1)\cdot h^2,$$

$$p_0^- = f(0),\ p_0^+ = f(1) - \frac{1}{12}\cdot g_0 + \frac{1}{4}\cdot g_1 - p_1^+,$$

$$p_1^- = \frac{T_1 A_{11}^+ - S_1 B_{11}^+}{B_{11}^- A_{11}^+ - A_{11}^- B_{11}^+},\ p_1^+ = \frac{S_1 B_{11}^- - T_1 A_{11}^-}{B_{11}^- A_{11}^+ - A_{11}^- B_{11}^+},$$

$$B_{11}^- = h\cdot(2\cdot q_1+1),\ B_{11}^+ = -h\cdot(2\cdot q_1+1)\cdot q_1^N,$$

$$A_{11}^- = h\cdot(2\cdot q_1+1)\cdot q_1^N,\ A_{11}^+ = -h\cdot(2\cdot q_1+1),$$

$$T_1 = 6(2q_1+1)\sum_{\gamma=0}^{N} q_1^{\gamma} f(h\gamma) - 3\begin{pmatrix}(3q_1+1)f(1)q_1^N\\ +(q_1+1)f(0)\end{pmatrix}$$
$$-\frac{1}{4}(2q_1+1)q_1^N(g_0-2g_1)h + \frac{1}{4}\begin{pmatrix}(g_0-g_1)q_1^{N+1}\\ -(q_1+4)g_1+\frac{1}{2}\end{pmatrix}h^2$$

$$S_1 = 6(2q_1+1)\sum_{\gamma=0}^{N} q_1^{N-\gamma} f(h\gamma) - 3\begin{pmatrix}(3q_1+1)f(0)q_1^N\\ +(q_1+1)f(1)\end{pmatrix}$$
$$-\frac{1}{4}(2q_1+1)(g_0-2g_1)h - \frac{1}{4}\begin{pmatrix}(g_0-g_1)q_1^{N+1}\\ -(q_1+4)g_1+\frac{1}{2}\end{pmatrix}h^2.$$

The coefficients of the optimal quadrature formula (1.1) in the space $L_2^{(3)}(0,1)$ have given in [12].

4. Numerical Results

It should be noted that constructed optimal quadrature formulas of the form (1.1) with the error functional (1.2), the coefficients which are determined by formulas (3.13)–(3.15) are exact for monomials $x^{\mu}, \alpha = 0,1,...,\mu-1$. This statement is also checked numerically.

Clearly, the optimal coefficients (3.13)–(3.15) depend only on the roots q_k (where $|q_k|<1$) of the Euler–Frobenius polynomial $E_{2m-2}(q)$, which is defined by formula (2.1). Therefore to obtain numerical values of the coefficients $C_{\gamma}, (\gamma = 0,1,...,N)$ it is sufficient to calculate the roots of the Euler–Frobenius polynomial $E_{2m-2}(q)$, whose absolute values are less than 1.

It should be noted that for $m = 2,3,...,7$ the Euler–Frobenius polynomials $E_{2m-2}(q)$ and their roots are given in [30].

Below we consider some particular cases.

We consider the case $m = 3,\ \alpha = 6$. In this case we obtain the optimal quadrature formulas of the form (1.1) in the space $L_2^{(3)}(0,1)$. Here we need the roots q_k of the Euler–Frobenius polynomial $E_4(q)$, in which $|q_k|<1$. From (2.1) we get

$$E_4(q) = q^4 + 26q^3 + 66q^2 + 26q + 1$$

and the roots q_k of this polynomial, whose absolute values less than 1 are

$$q_1 = \text{-}0.0043096288203264653822712376822 5501,$$
$$q_2 = \text{-}0.43057534709997379185143478349352011. \quad (4.1)$$

Now we give tables of values of the coefficients of optimal quadrature formulas of the form (1.1) for the cases $m = 3$ and $N = 10$. For $m = 3$ and $N = 10$ solving the system (3.21), (3.22) and using (4.1) from (3.13)–(3.15) we get the following optimal quadrature formula of the form (1.1) in the space $L_2^{(3)}(0,1)$

$$\int_0^1 x^6 \varphi(x)dx \cong \sum_{\gamma=0}^{10} C_{\gamma}\varphi(0.1\gamma), \quad (4.2)$$

The coefficients of the optimal formula (4.2) are presented in Table 1.

Table 1. The coefficients of the optimal quadrature formula (4.2)

C_0	=	-0.0000038538095770884243362060880000 0
C_1	=	0.0000224390606153564383130138508000 0
C_2	=	-0.0000632970355677873463607532048000 0
C_3	=	0.0002468240075601861219635140752000 0
C_4	=	0.0000002993254166783939169724860000 0
C_5	=	0.0025154026186269065613476418150000 0
C_6	=	0.0024519349914883178674648017696960 0
C_7	=	0.0168967966378197923188802812218000 0
C_8	=	0.0145210962830258857531943974815516 0
C_9	=	0.0750674918621832205411289027315220 0
C_{10}	=	0.0312020089155513888917344576939928 00

Using (1.9) we get the following estimation of the formula (4.2)

$$|(\ell,\varphi)| \le \left\| \varphi \mid L_2^{(3)}(0,1) \right\| \cdot 5.9\times10^{-6}$$

The case $m = 4,\ \alpha = 6$. In this case we obtain the optimal quadrature formulas of the form (1.1) in the space $L_2^{(4)}(0,1)$. We need the roots q_k of the Euler–Frobenius polynomial $E_6(q)$, in which $|q_k|<1$. From (2.1) we get

$$E_6(q) = q^6 + 120q^5 + 1191q^4$$
$$+ 2416q^3 + 1191q^2 + 120q + 1$$

and the roots q_k of this polynomial, whose absolute values less than 1 are

$$q_1 = \text{-}0.0091486948096082769285930216516 47$$
$$q_2 = \text{-}0.1225546151923266905152722643593 57 \quad (4.3)$$
$$q_3 = \text{-}0.5352804307964381655424037816816 46$$

For $m = 4$ and $N = 10$ solving the system (3.21), (3.22) and using (4.3) from (3.13)–(3.15) we get the following optimal quadrature formula of the form (1.1) in the space $L_2^{(4)}(0,1)$

$$\int_0^1 x^6 \varphi(x)\mathrm{d}x \cong \sum_{\gamma=0}^{10} C_\gamma \varphi(0.1\gamma), \qquad (4.4)$$

The coefficients of the optimal formula (4.4) are presented in Table 2.

Table 2. The coefficients of the optimal quadrature formula (4.4)

C_0 =	-0.00002388748176458733033859137474949
C_1 =	0.00015621039018418698391696852200730
C_2 =	-0.00050120466468228907462002155494944
C_3 =	0.00125269570715546376898977157881931
C_4 =	-0.00195632707228314965214474671410102
C_5 =	0.00606551868135686587141983759915262
C_6 =	-0.00372174035667546661697893129636807
C_7 =	0.02685006209065145386182090393959453
C_8 =	0.00277544265937620455331183026124118
C_9 =	0.08284143610769649242233504214176748
C_{10} =	0.02911893679612768235514507975472845

Using (1.9) we get the following estimation of the formula (4.4)

$$\left|(\ell,\varphi)\right| \le \left\|\varphi \mid L_2^{(4)}(0,1)\right\| \cdot 3.07\times 10^{-7}$$

Remark. It should be noted that when $\alpha = 0$ from Theorem 5 we get Theorem 2.1. of [13].

5. Conclusion

In the present work we constructed the optimal quadrature formulas with polynomial weight x^α by Sobolev method in the space $L_2^{(m)}(0,1)$.

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