American Journal of Numerical Analysis, 2014, Vol. 2, No. 5, 136-143
Available online at http://pubs.sciepub.com/ajna2/5/1
© Science and Education Publishing
DOI:10.12691/ajna-2-5-1

Fitted Second Order Scheme for Singularly Perturbed Differential-difference Equations

Lakshmi Sirisha, Y.N. Reddy*

Department of Mathematics, National Institute of Technology, WARANGAL, India
*Corresponding author: ynreddy_nitw@yahoo.com

Received August 24, 2014; Revised September 06, 2014; Accepted September 11, 2014

Abstract In this paper, we present a fitted second order stable central finite difference scheme for solving singularly perturbed differential-difference equations (with delay and advanced parameter). First, the given second order differential difference equation is replaced by an asymptotically equivalent second order singularly perturbation problem. Then, a fitting factor is introduced into the second order stable central difference scheme and determined its value from the theory of singular perturbations. Discrete Invariant Imbedding Algorithm is used to solve the resulting tri-diagonal system. The error analysis and convergence of the scheme are also discussed. To validate the applicability of the method, several model examples have been solved by taking different values for the delay parameter δ , advanced parameter η and the perturbation parameter ε .

Keywords: *differential- difference equations, central differences, boundary layer*

Cite This Article: Lakshmi Sirisha, and Y.N. Reddy, "Fitted Second Order Scheme for Singularly Perturbed Differential-difference Equations." *American Journal of Numerical Analysis*, vol. 2, no. 5 (2014): 136-143. doi: 10.12691/ajna-2-5-1.

1. Introduction

Singularly Perturbed Differential-difference equations (SPDDEs), also called as a class of functional differential equations, are mathematical models of a number of real phenomenon. Their applications permeate all branches of contemporary sciences such as engineering, physics, economics, biomechanics, and evolutionary biology [1]. The study of bistable devices [2], the description of human pupil-light reflex[3], the first exit time problem in the modeling of the activation of neuronal variability [4], the study of a variety of models for physiological processes or diseases [5,6] and the study of dynamic systems with time delays, which arise in neural networks[7] are all examples involving differential-difference equations. This motivates the investigation on SPDDEs and as a result, a series of papers have been devoted to these type of boundary-value problems with small shifts, the negative shift and positive shift(often referred as "delay" and "advanced" parameters respectively). For example, Lange and Miura [8,9,10] published a series of papers on numerical treatment of second order singularly perturbed differential difference equations with small shifts and studied the boundary and interior layer phenomenon, rapid oscillations resonance behavior and also turning point behavior. Tain [11] extended the concept of singular perturbation theory for ordinary differential equations to delay differential equations, applying Malley-Hoppensteadt technique to obtain approximate solutions. Kadalbajoo and Sharma [12] constructed an ε-uniform fitted mesh method for solving singularly perturbed differential-difference equations with mixed type of shifts. The same authors [13] described a numerical approach based on finite difference method to solve a mathematical model arising from neuronal variability. Patidar and Sharma [14] approximated the term containing delay by Taylor series expansion and then applied an ε-uniformly convergent non-standard finite difference methods to SPDDEs with small delay. Ramos [15] proposed a variety of exponential methods based on piecewise analytical solutions of advection-reaction-diffusion operators for the numerical solution of linear ordinary differential-difference equations with small delay. Kumar and Sharma [16] presented a numerical technique to approximate the solution of boundary value problems for SPDDEs with delay as well as advanced. Prathima and Sharma [17] presented a numerical method to solve boundary value problems for SPDDEs with an isolated turning point at the left end of the boundary.

In this paper, we present a fitted second order stable central finite difference scheme for solving singularly perturbed differential-difference equations (with delay and advanced parameter). First, the given second order differential difference equation is replaced by an asymptotically equivalent second order singularly perturbation problem. Then, a fitting factor is introduced into the second order stable central difference scheme and determined its value from the theory of singular perturbations. Discrete Invariant Imbedding Algorithm is used to solve the resulting tri-diagonal system. The error analysis and convergence of the scheme are also discussed. To validate the applicability of the method, several model examples have been solved by taking different values for the delay parameter δ , advanced parameter η and the perturbation parameter ε .

2. Description of the Method

2.1. Left End Boundary Layer Problems

Consider singularly perturbed differential equation with small delay as well as advance parameter of the form:

$$\varepsilon y''(x)+a(x)y'(x)+b(x)y(x-\delta)+c(x)y(x)+ d(x)y(x+\eta)=f(x) \quad (1)$$

$\forall\, x\in(0,1)$ and subject to the interval and boundary conditions

$$y(x)=\varphi(x)\,,\quad \text{on}\quad -\delta\le x\le 0 \quad (2)$$

$$y(x)=\gamma(x),\quad on\quad 1\le x\le 1+\eta \quad (3)$$

where $a(x), b(x), c(x), d(x), \varphi(x)$ and $\gamma(x)$ are bounded and continuously differentiable functions on (0, 1), $0<\varepsilon<<1$ is the singular perturbation parameter; and $0<\delta=o(\varepsilon)$ and $0<\eta=o(\varepsilon)$ are the delay and the advance parameters respectively. In general, the solution of (1)-(3) exhibits boundary layer behavior at one end of the interval [0,1] depending on the sign.

By using Taylor series expansion in the neighborhood of the point *x*, we have

$$y(x-\delta)\approx y(x)-\delta y'(x) \quad (4)$$

$$y(x+\eta)\approx y(x)+\eta y'(x) \quad (5)$$

Using equations (4) and (5) in (1) we get an asymptotically equivalent singularly perturbed boundary value problem of the form:

$$\varepsilon y''(x)+p(x)y'(x)+q(x)y(x)=f(x) \quad (6)$$

$$y(0)=\varphi(0)=\varphi_0 \quad (7)$$

$$y(1)=\gamma(1)=\gamma_1 \quad (8)$$

where

$$p(x)=a(x)+d(x)\eta-b(x)\delta \quad (9)$$

and

$$q(x)=b(x)+c(x)+d(x) \quad (10)$$

The transition from Eq.(1) to Eq.(6) is admitted, because of the condition that $0<\delta<<1$ and $0<\eta<<1$ are sufficiently small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Elsgolt's and Norkin [18]. Thus, the solution of Eq. (6) will provide a good approximation to the solution of Eq. (1). Further, we assume that

$$p(x)=a(x)+d(x)\eta-b(x)\delta\ge M>0$$

$q(x)=a(x)+c(x)+d(x)\le 0$, throughout the interval [0, 1], where M is some constant. Under these assumptions, (6) has a unique solution $y(x)$ which exhibits a boundary layer of width O(ε) on the left side $(x=0)$ of the underlying interval.

From the theory of singular perturbations, it is known that the solution of (6)-(8) is of the form [cf. O' Malley [19] pp. 22-26]

$$y(x) = y_0(x)+\frac{p(0)}{p(x)}(\alpha-y_0(0))e^{-\int_0^x\left(\frac{p(x)}{\varepsilon}-\frac{q(x)}{p(x)}\right)dx}+o(\varepsilon) \quad (11)$$

where $y_0(x)$ is the solution of the reduced problem

$$p(x)y_0'(x)+q(x)y_0(x)=f(x),\ y_0(1)=\beta$$

By taking Taylor series expansion for $p(x)$ and $q(x)$ about the point '0' and restricting to their first terms, (6) becomes

$$y(x)=y_0(x)+(\alpha-y_0(0))e^{-\int_0^x\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)dx}+o(\varepsilon) \quad (12)$$

Now we divide the interval [0,1] into N equal subintervals of mesh size $h=\frac{1}{N}$ so that $x_i=ih,\quad i=0,1,2...N$. From (12) we have

$$y(x_i)=y_0(x_i)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)x_i}+o(\varepsilon)$$

i.e.,

$$y(ih)=y_0(ih)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)ih}+o(\varepsilon)$$

Therefore

$$\lim_{h\to 0} y(ih)=y_0(0)+(\alpha-y_0(0))e^{-\left(\frac{p(0)}{\varepsilon}-\frac{q(0)}{p(0)}\right)i\rho}+o(\varepsilon) \quad (13)$$

where $\rho=h/\varepsilon$

Eq.(6) at x= x_i ,we have:

$$\varepsilon y''(x_i)+p(x_i)y'(x_i)+q(x_i)y(x_i)=f(x_i) \quad (14)$$

and using the Second order central differences:

$$y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}-\frac{h^2}{12}y_i^{(4)}(\xi_1)+R_1 \quad (15)$$

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h}-\frac{h^2}{6}y_i'''(\xi_2)+R_2 \quad (16)$$

Here $R_1=-\frac{2h^4y^{(6)}(\xi_1)}{6!}$ and

$$R_2=-\frac{h^4y^{(5)}(\xi_2)}{5!}$$

for $\xi_1,\xi_2\in[x_{i-1},x_i]$.

Now, from (16) and (15) in (14) we have:

$$\frac{\varepsilon}{h^2}[y_{i+1}-2y_i+y_{i-1}]+\frac{p_i}{2h}[y_{i+1}-y_{i-1}]-\frac{h^2p_i}{6}y_i'''+ q_iy_i=f_i+R \quad (17)$$

where

$$R = R_1 + R_2 - \frac{\varepsilon^2 h^2 y^{(4)}(\xi_1)}{12}$$

$$p(x_i) = p_i,\ \ q(x_i) = q_i,$$
$$f(x_i) = f_i,\ \ y(x_i) = y_i.$$

From (6) we have

$$\varepsilon y''(x) = f_i - p_i y_i' - q_i y_i \tag{18}$$

Differentiating both sides of Eq.(18) and substituting into (17) we have :

$$\frac{\varepsilon}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right] + \frac{p_i}{2h}\left[y_{i+1} - y_{i-1}\right] - \frac{h^2 p_i}{6}\left(p_i y_i'' + \left(p_i' + q_i\right) y_i' + q_i' y_i\right) + q_i y_i = f_i + \frac{h^2 p_i}{6\varepsilon} f_i' + R$$

Now, approximating the converted error term, which has the stabilizing effect, in Eq.(18) by using the central difference formula for y_i' and y_i'' from Eqs.(15) and (16), we obtain the Second order stable central difference scheme:

$$\left(\varepsilon + \frac{h^2 p_i^2}{6\varepsilon}\right)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(p_i + \frac{h^2 p_i}{6\varepsilon}(p_i' + q_i)\right)\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) \left(q_i + \frac{h^2 p_i q_i'}{6\varepsilon}\right) y_i = f_i + \frac{h^2 p_i}{6\varepsilon} f' + \tau_i \tag{19}$$

where

$$\tau_i = -\left(\frac{h^2\varepsilon}{12} + \frac{p_i^2 h^4}{12\varepsilon}\right) y_i^{(4)} - \frac{p_i(p_i' + q_i)h^4}{36\varepsilon} y_i''' + \varepsilon R_2 - p_i R_1$$

is the local truncation error.

Now introducing a fitting factor σ into Eq.(19) we obtain

$$\sigma\left(\varepsilon + \frac{h^2 p_i^2}{6\varepsilon}\right)\left(\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right) + \left(p_i + \frac{h^2 p_i}{6\varepsilon}(p_i' + q_i)\right)\left(\frac{y_{i+1} - y_{i-1}}{2h}\right)\left(q_i + \frac{h^2 p_i q_i'}{6\varepsilon}\right) y_i = f_i + \frac{h^2 p_i}{6\varepsilon} f' + \tau_i \tag{20}$$

$y_0 = \phi(0)$, $y_N = \gamma_1$ which is to be determined in such a way that the solution of (20) with boundary conditions converges uniformly to the solution of (6)-(8) which is in turn a good approximation to the solution of (1)-(3).

Multiplying (20) by h and taking limits as $h \to 0$ we obtain σ

$$\lim_{h\to 0}\frac{\sigma}{\rho}\left(1 + \frac{\rho^2 p^2(ih)}{6}\right)\left(y(ih+h) - 2y(ih) + y(ih-h)\right) + \lim_{h\to 0}\frac{1}{2} p(ih)\left(1 + \frac{\rho h}{6}\right)\left(p'(ih) + q(ih)\right)\left(y(ih+h) - y(ih-h)\right) = 0 \tag{21}$$

where $\rho = h/\varepsilon$ and $f_i + \frac{h^2 p_i}{6\varepsilon} f_i' - \left(q_i + \frac{h^2 p_i q_i'}{6\varepsilon}\right) y_i$ is bounded.

By substituting (13) in (21) we get the fitting factor as:

$$\sigma = \left(\frac{3\rho p(0)}{6 + \rho^2 p^2(0)}\right)\coth\left[\left(\frac{p^2(0) - \varepsilon q(0)}{p(0)}\right)\frac{\rho}{2}\right] \tag{22}$$

Finally, making use of Eq. (20) and Eq. (22), we get the three term recurrence relation of form:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; i = 1,\ 2,\ ...,\ N-1 \tag{23}$$

where

$$E_i = \frac{\sigma}{h^2}\left(\varepsilon + \frac{h^2 p_i^2}{6\varepsilon}\right) - \frac{p_i}{2h}\left(1 + \frac{h^2}{6\varepsilon}(p_i' + q_i)\right)$$

$$F_i = \frac{2\sigma}{h^2}\left(\varepsilon + \frac{h^2 p_i^2}{6\varepsilon}\right) - \left(q_i + \frac{h^2 p_i^2 q_i'}{6\varepsilon}\right)$$

$$G_i = \frac{\varepsilon}{h^2}\left(\varepsilon + \frac{h^2 p_i^2}{6\varepsilon}\right) + \frac{p_i}{2h}\left(1 + \frac{h^2}{6\varepsilon}(p_i' + q_i)\right)$$

$$H_i = f_i + \frac{h^2 p_i f_i'}{6\varepsilon^2}$$

This gives us the tri-diagonal system which can be solved easily by Discrete Invariant Imbedding Algorithm described in the next section.

2.2. Discrete Invariant Imbedding Algorithm

A brief description for solving the tri-diagonal system using Discrete Invariant Imbedding, also called Thomas algorithm is presented as follows:

Consider the scheme:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i; i = 1,\ 2,\ ...,\ N-1 \tag{24}$$

subject to the boundary conditions

$$y_0 = y(0) = \varphi_0; \tag{25}$$

$$y_N = y(1) = \gamma_1. \tag{26}$$

We set

$$y_i = W_i y_{i+1} + T_i \quad for\ \ i = N-1,\ N-2,\ ...,\ 2,\ 1. \tag{27}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ which are to be determined.

From (27), we have:

$$y_{i-1} = W_{i-1} y_i + T_{i-1} \tag{28}$$

By substituting (28) in (24) and comparing with (27) we get the recurrence relations:

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) \quad (29)$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right) \quad (30)$$

To solve these recurrence relations for $i = 1, 2, ..., N-1$, we need the initial conditions for W_0 and T_0. For this we take $y_0 = \varphi_0 = W_0 y_1 + T_0$. We choose $W_0 = 0$ so that the value of $T_0 = \varphi_0$. With these initial values, we compute W_i and T_i for $i = 1, 2, ...N-1$ from (29) and (30) in forward process, and then obtain y_i in the backward process from (26) and (27).

3. Stability and Convergence Analysis

Writing the tri-diagonal system (17) in matrix-vector form, we get

$$AY = C \quad (31)$$

where, $A = \left(m_{ij}\right)$, $1 \le i, j \le N$-1 is a tri-diagonal matrix of order N-1, with

$$m_{ii+1} = -\sigma\varepsilon - \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^2 - \frac{h^3}{12\varepsilon} p_i(p_i' + q_i)$$

$$m_{ii} = 2\sigma\varepsilon + \frac{h^2\sigma}{3\varepsilon} p_i^2 - h^2 q_i - \frac{h^4}{6\varepsilon} p_i q_i'$$

$$m_{ii-1} = -\sigma\varepsilon + \frac{hp_i}{2} - \frac{h^2\sigma}{6\varepsilon} p_i^2 + \frac{h^3}{12\varepsilon} p_i(p_i' + q_i)$$

and $C = \left(d_i\right)$ is a column vector with $d_i = -h^2 f_i - h^4 \dfrac{p_i f_i'}{6\varepsilon}$, where $i = 1,2,...,N$-1 with local truncation error

$$T_i(h) = \frac{h^4}{12} K + o(h^6) \quad (32)$$

where $K = \left(\varepsilon - \dfrac{h^2 p_i^2}{\varepsilon}\right) y_i^{(4)}$

$A = \left(m_{ij}\right)$, $1 \le i, j \le N$-1 is a tri-diagonal matrix of order N-1, with

We also have

$$A\,\overline{Y} - T(h) = C \quad (33)$$

where $\overline{Y} = \left(\overline{y}_0, \overline{y}_1, \overline{y}_2, ..., \overline{y}_N\right)^t$ denotes the actual solution and $T(h) = \left(T_1(h), T_2(h), ..., T_N(h)\right)^t$ is the local truncation error.

From (31) and (33), we get

$$A\left(\overline{Y} - Y\right) = T(h) \quad (34)$$

Thus, we obtain the error equation

$$AE = T(h) \quad (35)$$

where $E = \overline{Y} - Y = \left(e_0, e_1, e_2, ...e_N\right)^t$.

Let S_i be the sum of elements of the i^{th} row of A, then we have

$$S_i = \sum_{j=1}^{N-1} m_{1j} = \sigma\varepsilon - \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^2 - h^2 q_i - \frac{h^3}{12\varepsilon} p_i(p_i' + q_i) + O\left(h^4\right) \quad \text{for } i = 1$$

$$S_i = \sum_{j=1}^{N-1} m_{ij} = -h^2 q_i) + O\left(h^4\right) \quad \text{for } i = 2,3,..N\text{-}2$$

$$S_i = \sum_{j=1}^{N-1} m_{N-1j} = \sigma\varepsilon + \frac{hp_i}{2} + \frac{h^2\sigma}{6\varepsilon} p_i^2 - h^2 q_i + \frac{h^3}{12\varepsilon} p_i(p_i' + q_i) + O\left(h^4\right) \quad \text{for } i = N\text{-}1$$

Since $0 < \varepsilon << 1$ and $\delta = o\left(\varepsilon\right)$, for sufficiently small h the matrix A is irreducible and monotone (Mohanty and Jha [20]). Then it follows that A^{-1} exists and its elements are non-negative.

Hence, from (35) we get

$$E = A^{-1} T(h) \quad (36)$$

and

$$\|E\| \le \left\|A^{-1}\right\| . \left\|T(h)\right\| \quad (37)$$

Let $\overline{m}_{k,i}$ be the $(k,i)^{th}$ element of A^{-1}. Since $\overline{m}_{k,i} \ge 0$, from the theory of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i}\, S_i = 1, \quad k = 1,2,...\, N\text{-}1 \quad (38)$$

Therefore,

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1 \le i \le N-1} S_i} = \frac{1}{h^2 |q_i|} \quad (39)$$

We define $\left\|A^{-1}\right\| = \max\limits_{1 \le k \le N-1} \sum\limits_{i=1}^{N-1} \left|\overline{m}_{k,i}\right|$

and $\left\|T(h)\right\| = \max\limits_{1 \le i \le N-1} \left|T_i(h)\right|$.

From (32), (36), (37) and (39), we obtain

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{k,i}\, T_i(h), \quad j = 1,2,3..N\text{-}1$$

which implies

$$e_j \le \frac{h^2 k}{|q_i|}, \quad j = 1,2...N\text{-}1 \quad (40)$$

where K is a constant.

Therefore, using the definitions and Eq.(40)

$$\|E\| = o\left(h^2\right)$$

Hence, our method gives a quadratic order convergence for uniform mesh.

4. Numerical Examples

To demonstrate the applicability of the method we have applied it to three boundary value problems of the type given by equations (1)-(3). The approximate solution is compared with exact solution. The exact solution of such boundary value problems having constant coefficients (i.e. $a(x) = a,\ b(x) = b,\ c(x) = c,\ \ d(x) = d,\ f(x) = f,$ $\varphi(x) = \varphi$ and $\gamma(x) = \gamma$ are constants) is given by:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f / c \tag{41}$$

where

$$c_1 = \frac{\left[-f + \gamma c + e^{m_2}(f - \phi c)\right]}{\left[(\exp(m_1) - \exp(m_2))c\right]},$$

$$c_2 = \frac{\left[f - \gamma c + e^{m_1}(-f + \phi c)\right]}{\left[(\exp(m_1) - \exp(m_2))c\right]}$$

$$m_1 = \frac{\left[-(a - \alpha\delta + \beta\eta) + \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}\right]}{2\varepsilon} \tag{42}$$

$$m_2 = \frac{\left[-(a - \alpha\delta + \beta\eta) - \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}\right]}{2\varepsilon}$$

$$c = (\alpha + \beta + \omega)$$

Example 4.1: Consider the model boundary value problem given by equations (1)-(3) with

$$\text{a}(x)=1,\ \text{b}(x) = 2,\ \ \text{c}(x) = 0,\ \ \text{d}(x) = -3,$$
$$f(x) = 0,\ \ \varphi(x) = 1,\ \ \gamma(x) = 1.$$

Table 1. Numerical solution of Example 1 for $\varepsilon = 0.01$ and $\eta = 0.005$

x	$\delta = 0.001$		$\delta = 0.005$		$\delta = 0.009$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.4605948	0.4620298	0.4593751	0.4608593	0.4581675	0.4597025
0.04	0.3967701	0.3969577	0.3942621	0.3944705	0.3917449	0.3919752
0.06	0.3951776	0.3950116	0.3923544	0.3921947	0.3895112	0.3893581
0.08	0.4018464	0.4016090	0.3989663	0.3987310	0.3960627	0.3958297
0.10	0.4097315	0.4094820	0.4068468	0.4065983	0.4039374	0.4036901
0.20	0.4524103	0.4521629	0.4495757	0.4493289	0.4467144	0.4464683
0.40	0.5516319	0.5514056	0.5490377	0.5488116	0.5464148	0.5461890
0.60	0.6726145	0.6724306	0.6705041	0.6703200	0.6683670	0.6681828
0.80	0.8201308	0.8200186	0.8188431	0.8187307	0.8175371	0.8174245
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

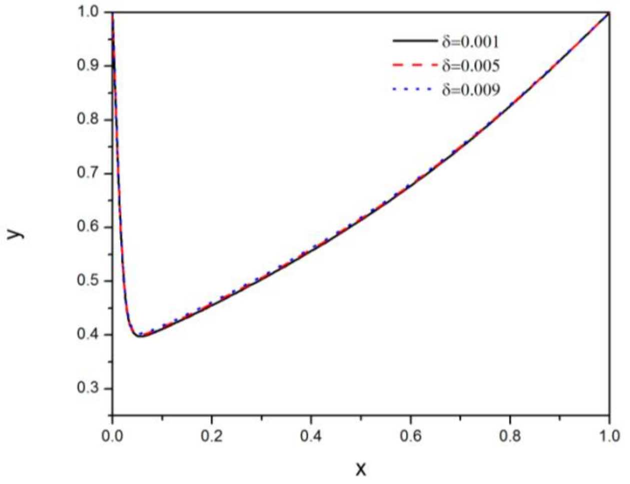
Table 2. Numerical solution of Example 1 for $\varepsilon = 0.001$ and $\eta = 0.0005$

x	$\delta = 0.0001$		$\delta = 0.0005$		$\delta = 0.0009$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3887036	0.3883906	0.3884525	0.3881405	0.3882013	0.3878903
0.04	0.3853804	0.3848477	0.3850905	0.3845578	0.3848004	0.3842677
0.06	0.3929207	0.3923808	0.3926298	0.3920899	0.3923387	0.3917987
0.08	0.4008049	0.4002657	0.4005145	0.3999752	0.4002238	0.3996845
0.10	0.4088508	0.4083127	0.4085610	0.4080228	0.4082709	0.4077327
0.20	0.4515692	0.4510409	0.4512847	0.4507562	0.4509999	0.4504713
0.40	0.5508625	0.5503791	0.5506022	0.5501186	0.5503415	0.5498577
0.60	0.6719890	0.6715958	0.6717772	0.6713838	0.6715652	0.6711716
0.80	0.8197493	0.8195094	0.8196202	0.8193801	0.8194908	0.8192506
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

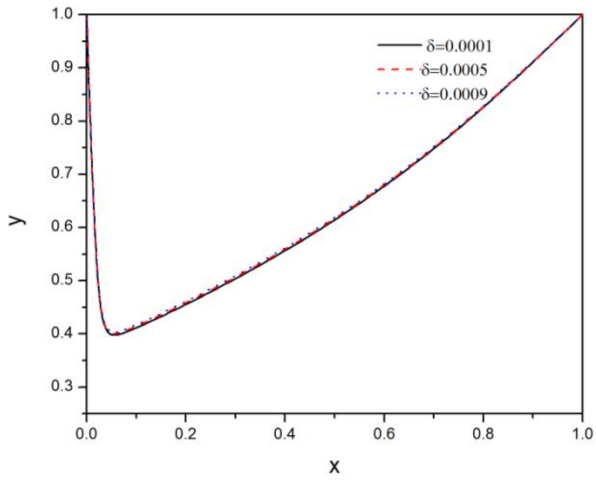
The numerical results are given in Table 1, Table 2 for ε =0.01, 0.001, η =0.005 and 0.0005 respectively.

Example 4.2. Consider the model boundary value problem given by equations (1)-(3) with

$$\text{a}(x)=1,\ \text{b}(x) = 0,\ \ \text{c}(x) = 2,\ \ \text{d}(x) = -3,$$
$$f(x) = 0,\ \ \varphi(x) = 1,\ \ \gamma(x) = 0.$$



Graph 1. Example-1 for ε = 0.01 and η=0.005



Graph 2. Example-1 for ε = 0.001 and η=0.0005

Table 3. Numerical solution of Example 2 for ε = 0.01 **and** δ = 0.005

x	η = 0.001		η = 0.005		η = 0.009	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.4612091	0.4626200	0.4624460	0.4638099	0.4636938	0.4650122
0.04	0.3980205	0.3981981	0.4005142	0.4006724	0.4029980	0.4031376
0.06	0.3965817	0.3964126	0.3993748	0.3992000	0.4021478	0.4019676
0.08	0.4032776	0.4030393	0.4061226	0.4058824	0.4089445	0.4087026
0.10	0.4111647	0.4109146	0.4140128	0.4137618	0.4168369	0.4165849
0.20	0.4538177	0.4535700	0.4566129	0.4563646	0.4593822	0.4591334
0.40	0.5529185	0.5526921	0.5554707	0.5552441	0.5579954	0.5577687
0.60	0.6736599	0.6734760	0.6757314	0.6755476	0.6777774	0.6775938
0.80	0.8207679	0.8206558	0.8220288	0.8219170	0.8232723	0.8231608
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

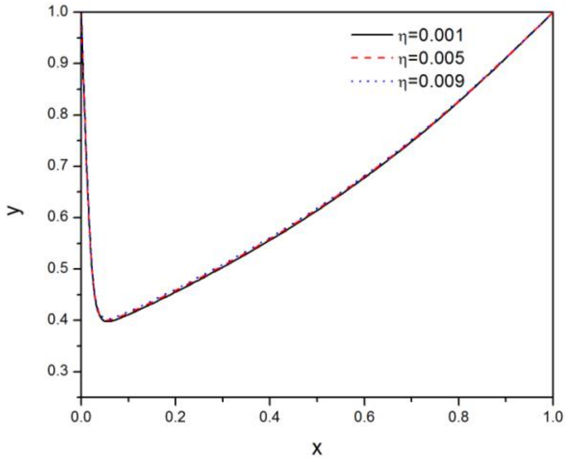
Table 4. Numerical solution of Example 2 for ε = 0.001 **and** δ = 0.0005

x	η = 0.0001		η = 0.0005		η = 0.0009	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.3888291	0.3885156	0.3890801	0.3887656	0.3893309	0.3890155
0.04	0.3855253	0.3849925	0.3858149	0.3852821	0.3861042	0.3855714
0.06	0.3930660	0.3925262	0.3933565	0.3928167	0.3936468	0.3931070
0.08	0.4009500	0.4004108	0.4012401	0.4007010	0.4015299	0.4009908
0.10	0.4089956	0.4084576	0.4092851	0.4087471	0.4095743	0.4090363
0.20	0.4517114	0.4511831	0.4519955	0.4514674	0.4522794	0.4517513
0.40	0.5509926	0.5505093	0.5512525	0.5507694	0.5515122	0.5510292
0.60	0.6720948	0.6717016	0.6723061	0.6719132	0.6725172	0.6721245
0.80	0.8198138	0.8195740	0.8199427	0.8197031	0.8200714	0.8198320
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

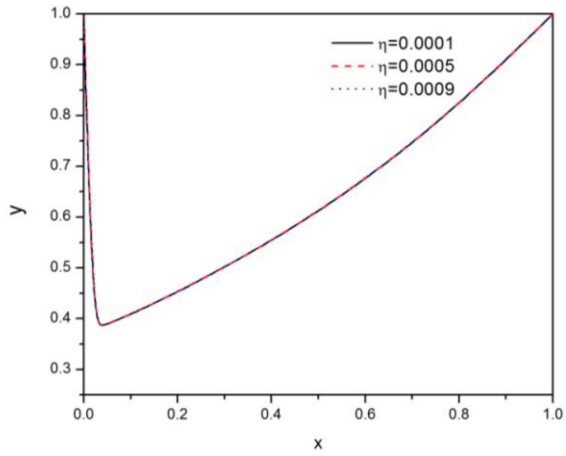
The numerical results are given in Table 3, Table 4 for ε =0.01, 0.001, δ =0.005 and 0.0005 respectively.

Example 4.3. Consider the model boundary value problem given by equations (1)-(3) with

$$a(x)=1,\ b(x)=-2,\ c(x)=1,\ d(x)=-5,$$
$$f(x)=0,\ \varphi(x)=1,\ \gamma(x)=0.$$



Graph 3. Example-2 for $\varepsilon = 0.01$ and δ=0.005



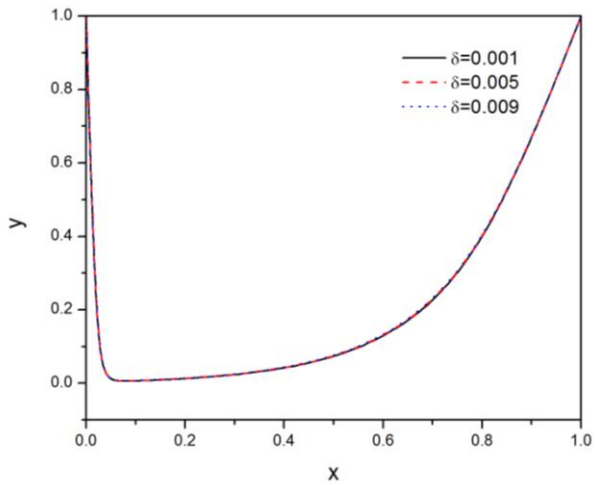
Graph 4. Example-2 for $\varepsilon = 0.001$ and δ=0.0005

Table 5. Numerical solution of Example 3 for $\varepsilon = 0.01$ and $\eta = 0.005$

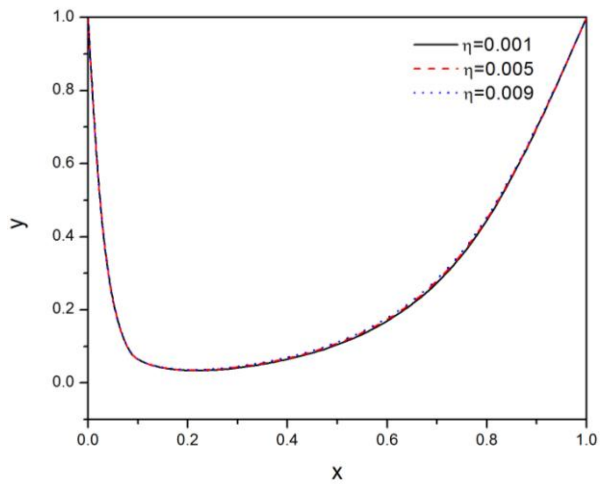
x	$\delta = 0.001$		$\delta = 0.005$		$\delta = 0.009$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.1095291	0.1227590	0.1081829	0.1211089	0.1068593	0.1194888
0.04	0.0156632	0.0186041	0.0155251	0.0183521	0.0153998	0.0181170
0.06	0.0062075	0.0066617	0.0063522	0.0067769	0.0065037	0.0069001
0.08	0.0057485	0.0057688	0.0059544	0.0059658	0.0061656	0.0061684
0.10	0.0063068	0.0062566	0.0065389	0.0064835	0.0067762	0.0067157
0.20	0.0110525	0.0109574	0.0114155	0.0113141	0.0117851	0.0116773
0.40	0.0340877	0.0338675	0.0349238	0.0346909	0.0357685	0.0355228
0.60	0.1051312	0.1046780	0.1068435	0.1063678	0.1085594	0.1080617
0.80	0.3242395	0.3235398	0.3268692	0.3261408	0.3294836	0.3287274
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 6. Numerical solution of Example 3 for $\varepsilon = 0.01$ and $\delta = 0.005$

x	$\eta = 0.001$		$\eta = 0.005$		$\eta = 0.009$	
	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.	Num. Sol	Exact Sol.
0.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.5361367	0.6021395	0.5341181	0.5984215	0.5320627	0.5947300
0.04	0.2900518	0.3649218	0.2880900	0.3606304	0.2861067	0.3564075
0.06	0.1597673	0.2237168	0.1584471	0.2200707	0.1571269	0.2165201
0.08	0.0910848	0.1399192	0.0904437	0.1372551	0.0898185	0.1346965
0.10	0.0552020	0.0904712	0.0551165	0.0887590	0.0550563	0.0871500
0.20	0.0230259	0.0298017	0.0241905	0.0306174	0.0253890	0.0314879
0.40	0.0556423	0.0611371	0.0580676	0.0633950	0.0605213	0.0656848
0.60	0.1457427	0.1551362	0.1499482	0.1589417	0.1541438	0.1627542
0.80	0.3817626	0.3938731	0.3872315	0.3986748	0.3926115	0.4034279
1.00	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000



Graph 5. Example-3 for $\varepsilon = 0.01$ and η=0.005



Graph 6. Example-3 for $\varepsilon = 0.001$ and δ=0.005

The numerical results are given in Table 5, Table 6 for ε =0.01 and δ =0.005 and η =0.005 respectively.

5. Discussions and Conclusions

We have presented fitted second order stable central difference scheme for solving singularly perturbed differential difference equations with the delay and advance parameters whose solutions exhibit layer behavior on the left-end of the interval. To demonstrate the efficiency of the scheme we have implemented it on some model examples by taking different values of ε, δ and η, where the choices of delay parameter and advanced parameter are not unique, but can assume any number of values satisfying the condition $\delta(\varepsilon) = \tau\varepsilon = \eta(\varepsilon)$ and is not too large [10]. The numerical solution is compared with the exact solution. From the tables, it is observed that the present scheme approximates the exact solution very well even if. The error bound and convergence analysis are discussed and established that our scheme gives second order convergence.

References

[1] R. Bellman, and K. L. Cooke, *Differential-Difference Equations*. Academic Press, New York, 1963.

[2] M.W. Derstine, F.A.H.H.M. Gibbs, D.L. Kaplan, *Bifurcation gap in a hybrid optical system*, Phys. Rev. A, 26 1982, 3720-3722.

[3] A. Longtin, J. Milton, *Complex oscillations in the human pupil light reflex with mixed and delayed feedback*, Math. Biosci. 90 1988, 183-199.

[4] R. B. Stein, *A theoretical analysis of neuronal variability*, Biophys. J., 5 1965, 173-194.

[5] M.C. Mackey, G.L., *Oscillations and chaos in physiological control systems, Science*, 197, 1977, 287-289.

[6] M. Wazewska-Czyzewska, A. Lasota, *Mathematical models of the red cell system,* Mat. Stos. 6 1976, 25-40.

[7] H. C. Tuckwell and W. Richter, *Neuronal interspike time distributions and the estimation of neurophysiological and neuroanatomical parameters*, J. Theor. Biol., 71 1978, 167-183.

[8] C. G. Lange and R. M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations II. Rapid Oscillations and Resonances*, SIAM Journal on Applied Mathematics, 45 1985, 687-707.

[9] C. G. Lange and R. M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations III. Turning Point Problems*, SIAM Journal on Applied Mathematics, 45 1985, 708-734.

[10] C. G. Lange and R. M. Miura, *Singular Perturbation Analysis of Boundary-Value Problems for Differential-Difference Equations. v. small shifts with layer behavior*, SIAM Journal on Applied Mathematics, 54 1994, 249-272.

[11] H. Tain, *The exponential asymptotic stability for singularly perturbed delay differential equations with a bounded lag*, Journal of Math. Anal. Appl., 270 2002, 143-149.

[12] M. K. Kadalbajoo and K. K. Sharma, *ε -Uniform fitted mesh method for Singularly Perturbed Differential-Difference Equations: Mixed type of shifts with layer behaviour*, International Journal of Computation Mathematics, 81 2004, 49-62.

[13] M. K. Kadalbajoo and K. K. Sharma, *Numerical Treatment of mathematical model arising from a model of neuronal variability*, Journal of Math. Anal. Appl., 307 2005 606-627.

[14] K. C. Patidar and K. K. Sharma, *ε -Uniformly convergent non-standard finite difference methods for singularly perutrbed differential-difference equations with small delay*, Appl. Math. Comput. 175 2006, 864-890.

[15] J. I. Ramos, *Exponential methods for singularly perturbed ordinary differential difference equations*, Applied Mathematics and Computations, 182, 2006, 1528-1541.

[16] V. Kumar and K. K. Sharma, *A optimized B-Spline method for solving singularly perturbed differential difference equations with delay as well as advanced*, Neural, Parallel and Scientific Computations, 16 2008, 371-386.

[17] R. Pratima and K. K. Sharma, *Numerical method for singulary perturbed differential-difference equations with turning point*, International Journal of Pure and Applied Mathematics, 73 2011, 451-470.

[18] L. E. Els'golts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, Mathematics in Science and Engineering, 1973.

[19] O'Malley, R. E., *Introduction to singular perturbations,* Academic Press, London, 1974.

[20] R. K. Mohanty and N. Jha, *A class of variable mesh spline in compression methods for singularly perturbed two point singular boundary value problems*, Applied Mathematics and Computation, 168, 2005, 704-716.

[21] Joshua Y. Choo and David H. Schultz, *Stable High order methods for differential equations with small coefficients for the second order terms*, J. Math. Applied, 25 1993 105-123.