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# A Priori Error Analysis and Numerical Simulation of Fully Discrete $H^1$-Galerkin Mixed Element Method for Nonlinear Pseudo-Hyperbolic Equation

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**Abstract** In this article, a fully discrete two-step $H^1$-Galerkin mixed method is presented for nonlinear pseudo-hyperbolic equation. The spatial direction and time direction are approximated by $H^1$-Galerkin mixed method and two-step difference method, respectively. Some a priori error results are analyzed for the scalar unknown function $u$ and its flux $\mathbf{q} = a(x)\nabla u + a(x)\nabla u_t$ . Moreover, a numerical test is made to verify our theoretical error analysis.

***Keywords:*** *two-step discrete method, $H^1$-Galerkin mixed method, nonlinear pseudo-hyperbolic equation, a priori error analysis, fully discrete scheme*

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## 1. Introduction

In this article, we consider the following nonlinear pseudo-hyperbolic equation [1-9] with initial and boundary conditions

$$\begin{aligned} & u_{tt} + u_t - \nabla\cdot(a(x)\nabla u + a(x)\nabla u_t) \\ & = f(u), (x,t) \in \Omega\times(0,T], \\ & u(x,t) = 0, (x,t) \in \partial\Omega\times[0,T], \\ & u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{aligned} \tag{1}$$

where $\Omega$ is a bounded domain in $R^n (n = 1,2,3)$ , which boundary $\partial\Omega$ is smooth. The initial values $u_0(x)$ and $u_1(x)$ are known functions, the coefficient $a(x)$ is a bounded and smooth function. $f(u)$ is a nonlinear bounded function of $u$ with $f(0) = 0$ and a bounded derivative $f_u(u)$.

As we know, the pseudo-hyperbolic equations, which describe many physical phenomena, such as heat and mass transfer and nerve conduction, are a kind of important hyperbolic wave equations. In view of the importance of the pseudo-hyperbolic equations, a lot of numerical methods have been studied, such as splitting positive definite mixed element method [1,2], least-squares mixed element method [3,4], $H^1$-Galerkin mixed element methods [5,6,7,8], mixed covolume method [9] and so forth.

Pani [10] presented and analyzed an $H^1$-Galerkin mixed method for solving the linear parabolic equation with initial and boundary condition and derived some semi-discrete and fully discrete a priori error results in detail. This method holds some good features: free of LBB condition, the freedom of approximation space's selection and the better order of convergence for the flux in $L^2$-norm. Based on the above value points, the $H^1$-Galerkin mixed element method for many evolution equations (pseudo-hyperbolic equations [5,6], integro-differential equation [11], hyperbolic equation [12], Sobolev equation [13], RLW equation [14,15], Schrodinger equation [16], Burgers' equation [20]) were formulated. At the same time, some good numerical methods (Nonconforming $H^1$-Galerkin mixed method [7,17], Splitting $H^1$-Galerkin mixed method [8], $H^1$-Galerkin expanded mixed methods [18,19]) based on the $H^1$-Galerkin mixed element method were developed by many authors. Liu and Li [5], Zhou [6] gave the theoretical analysis of a priori errors for $H^1$-Galerkin mixed method, respectively. Zhang et al. [7] studied a semi-discrete nonconforming $H^1$-Galerkin mixed scheme. Liu et al. [8] proposed a new splitting $H^1$-Galerkin mixed method, then analyzed a Crank-Nicolson fully discrete a priori error results. But in [5,6,7,8], the considered pseudo-hyperbolic equation is only a linear problem.

In this article, our aim is to consider a nonlinear pseudo-hyperbolic problem and to propose a fully discrete two-step difference method combining with an $H^1$-Galerkin mixed element method. Compared to backward Euler discrete method [5,6] based on the linear problem, our method get some optimal second-order results of convergence in time direction for nonlinear pseudo-hyperbolic equation (1). As far as we know, the study of the $H^1$-Galerkin mixed method based on two-step difference scheme is fairly limited in the literatures, so our study is meaningful. We will give some a priori error analysis in detail and make some numerical simulation.

## 2. Two-Step Mixed Scheme and A Priori Error Estimates

### 2.1. Mixed Weak Formulation

By a similar selection to [5,6], we consider an auxiliary variable $\mathbf{q}=a(x)\nabla u+a(x)\nabla u_t$, then (1) can be split into a coupled system by:

$$\begin{aligned} &a(x)\nabla u+a(x)\nabla u_t=\mathbf{q},\\ &u_{tt}+u_t-\nabla\cdot\mathbf{q}=f(u). \end{aligned} \tag{2}$$

In (2), by making an inner product for $\nabla v$ ( $\forall\ v\in H_0^1$ ) and $\nabla\cdot\mathbf{w}$ ( $\forall\ \mathbf{w}\in\mathbf{H}(\mathbf{div};\Omega)$ ), respectively, we use Green's formula to obtain the mixed weak formulation: find $\{u,\mathbf{q}\}:[0,T]\to H_0^1\times\mathbf{H}(\mathbf{div};\Omega)$ such that:

$$\begin{cases} (a(x)\nabla u,\nabla v)+(a(x)\nabla u_t,\nabla v)\\ =(\mathbf{q},\nabla v),\forall v\in H_0^1,\\ (\alpha\mathbf{q_t},\mathbf{w})+(\nabla\cdot\mathbf{q},\nabla\cdot\mathbf{w})\\ =-(f(u),\nabla\cdot\mathbf{w}),\forall\mathbf{w}\in\mathbf{H}(\mathbf{div};\Omega). \end{cases} \tag{3}$$

where $\alpha(x)=\dfrac{1}{a(x)}$.

### 2.2. Fully Discrete Scheme and Some Important Lemmas

We give a partition $0=t_0<t_1<\cdots<t_N=T$ with mesh length $\Delta t=T/N$, ( $N$ is a positive integer) of the time interval $[0,T]$. Furthermore, we define $\phi^n=\phi(t_n)$ for a smooth function $\phi$ on $[0,T]$. In the following, we will get two-step time discrete scheme based on the above expression.

At $t=t_{n+1}$, we get an equivalent form of (3) as follows:

$$\begin{cases} (a(x)\nabla u^{n+1},\nabla v)+(a(x)\nabla u_t^{n+1},\nabla v)\\ =(\mathbf{q}^{n+1},\nabla v),\forall v\in H_0^1,\\ (\alpha\mathbf{q_t}^{n+1},w)+(\nabla\cdot\mathbf{q}^{n+1},\nabla\cdot\mathbf{w})\\ =-(f^{n+1}(u),\nabla\cdot\mathbf{w}),\forall\mathbf{w}\in\mathbf{H}(\mathbf{div};\Omega). \end{cases} \tag{4}$$

By two-step discrete method, (4) can be rewritten as:

$$\begin{cases} (a(x)\nabla u^{n+1},\nabla v)\\ +(a(x)\dfrac{\left(3\nabla u^{n+1}-4\nabla u^n+\nabla u^{n-1}\right)}{2\Delta t},\nabla v)\\ =(\mathbf{q}^{n+1},\nabla v)-(R_u^{n+1},\nabla v),\forall v\in H_0^1,\\ (\alpha\dfrac{3\mathbf{q}^{n+1}-4\mathbf{q}^n+\mathbf{q}^{n-1}}{2\Delta t},\mathbf{w})+(\nabla\cdot\mathbf{q}^{n+1},\nabla\cdot\mathbf{w})\\ =-(R_{\mathbf{q}}^{n+1},\mathbf{w})-(2f^n(u)-f^{n-1}(u),\nabla\cdot\mathbf{w})\\ -(R_f^{n+1},\mathbf{w}),\forall\mathbf{w}\in\mathbf{H}(\mathbf{div};\Omega). \end{cases} \tag{5}$$

where

$$\begin{aligned} R_u^{n+1}&=a(x)\nabla u_t^{n+1}-a(x)\frac{3\nabla u^{n+1}-4\nabla u^n+\nabla u^{n-1}}{2\Delta t}\\ R_{\mathbf{q}}^{n+1}&=\alpha\mathbf{q_t}^{n+1}-\alpha\frac{3\mathbf{q}^{n+1}-4\mathbf{q}^n+\mathbf{q}^{n-1}}{2\Delta t}\\ R_f^{n+1}&=f^{n+1}(u)-(2f^n(u)-f^{n-1}(u)) \end{aligned} \tag{6}$$

Based on the weak formulation (5), the fully discrete two-step mixed scheme is to find $\{U^{n+1},Q^{n+1}\}$ in $V_h\times\mathbf{W}_h$ such that

$$\begin{cases} (a(x)\nabla U^{n+1},\nabla v^h)\\ +(a(x)\dfrac{3\nabla U^{n+1}-4\nabla U^n+\nabla U^{n-1}}{2\Delta t},\nabla v^h)\\ =(Q^{n+1},\nabla v^h),\forall v^h\in V_h,\\ (\alpha\dfrac{3Q^{n+1}-4Q^n+Q^{n-1}}{2\Delta t},\mathbf{w}^h)\\ +(\nabla\cdot Q^{n+1},\nabla\cdot\mathbf{w}^h)\\ =-(2f^n(U)-f^{n-1}(U),\nabla\cdot\mathbf{w}^h),\forall\mathbf{w}\in\mathbf{W}_h. \end{cases} \tag{7}$$

We now introduce two important lemmas with $\eta=u-\tilde{u}^h$ and $\rho=q-\tilde{q}^h$ to analyze some a priori error results

**Lemma 1 [10,21]** Define a Ritz projection $\tilde{u}^h\in V_h$ of $u$ by:

$$(\nabla(u-\tilde{u}^h),\nabla v^h)=0,\forall v^h\in V_h. \tag{8}$$

and the estimates hold

$$\begin{aligned} &\|\eta\|_j+\|\eta_t\|_j\\ &\le Ch^{k+1-j}(\|u\|_{k+1}+\|u_t\|_{k+1}),\ j=0,1 \end{aligned} \tag{9}$$

**Lemma 2 [10]** Let $\tilde{\mathbf{q}}^h\in\mathbf{W}_h$ be the standard finite element interpolant of $\mathbf{q}$, then holds

$$\|\rho\|+h\|\rho\|_{\mathbf{H}(\mathbf{div};\Omega)}\le Ch^{r+1}\|\mathbf{q}\|_{r+1}. \tag{10}$$

For the need for error analysis, we decompose

$$\begin{aligned} &u(t_n)-U^n\\ &=(u(t_n)-\tilde{u}^h(t_n))+(\tilde{u}^h(t_n)-U^n)=\eta^n+\varsigma^n,\\ &\mathbf{q}(t_n)-Q^n\\ &=(\mathbf{q}(t_n)-\tilde{\mathbf{q}}^h(t_n))+(\tilde{\mathbf{q}}^h(t_n)-Q^n)=\rho^n+\xi^n. \end{aligned}$$

Combine (5) and (7) with (8) to get the error equations:

$$\begin{aligned} &(a(x)\nabla\varsigma^{n+1},\nabla v^h)\\ &+(a(x)\frac{3\nabla\varsigma^{n+1}-4\nabla\varsigma^n+\nabla\varsigma^{n-1}}{2\Delta t},\nabla v^h)\\ &=(\rho^{n+1}+\xi^{n+1},\nabla v^h)-(R_u^{n+1},\nabla v^h),\forall v^h\in V_h, \end{aligned} \tag{11}$$

and

$$(\alpha\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2\Delta t},\mathbf{w}^{h})+(\nabla\cdot\rho^{n+1}+\nabla\cdot\xi^{n+1},\nabla\cdot\mathbf{w}^{h})$$
$$=-(\alpha\frac{3\rho^{n+1}-4\rho^{n}+\rho^{n-1}}{2\Delta t},\mathbf{w}^{h}) \quad (12)$$
$$-(R_{\mathbf{q}}{}^{n+1},\mathbf{w}^{h})-(R_{f}{}^{n+1},\mathbf{w}^{h})-(2(f^{n}(u)-f^{n}(U))$$
$$-(f^{n-1}(u)-f^{n-1}(U)),\nabla\cdot\mathbf{w}^{h}),\forall\mathbf{w}\in\mathbf{W}_{h}.$$

**Lemma 3** [15] For $R_u{}^{n+1}$, $R_{\mathbf{q}}{}^{n+1}$ and $R_f{}^{n+1}$, the following errors hold:

$$\| R_u{}^{n+1}\|\le C\Delta t^2\|\nabla u_{ttt}\|_{L^\infty(L^2)}, \quad (13)$$

$$\| R_{\mathbf{q}}{}^{n+1}\|\le C\Delta t^2\|\nabla \mathbf{q}_{ttt}\|_{L^\infty(L^2)}, \quad (14)$$

$$\| R_f{}^{n+1}\|\le C\Delta t^2\| f_{tt}(u)\|_{L^\infty(L^2)}. \quad (15)$$

**Lemma 4** [15] For a sequence $\{\delta^n\}$ and bounded function $\kappa(x)$, the following equality holds

$$(\kappa\frac{3\delta^{n+1}-4\delta^{n}+\delta^{n-1}}{2\Delta t},\delta^{h})$$
$$=\frac{1}{4\Delta t}[(\|\kappa^{\frac12}\delta^{n+1}\|^2+\|\kappa^{\frac12}(2\delta^{n+1}-\delta^{n})\|^2) \quad (16)$$
$$-(\|\kappa^{\frac12}\delta^{n}\|^2+\|\kappa^{\frac12}(2\delta^{n}-\delta^{n-1})\|^2)$$
$$+\|\kappa^{\frac12}(\delta^{n+1}-2\delta^{n}+\delta^{n-1})\|^2].$$

## 2.3. The Error Estimates of Two-step Mixed Element

**Theorem 1** Assuming that $U^0,U^1\in V_h$ and $Q^0,Q^1\in\mathbf{W}_h$, then, for $1\le J\le M$, $j=0,1$ hold

$$\| u^{J+1}-U^{J+1}\|_j\le C(h^{\min(k+1-j,r)}+2\Delta t^2),\ j=0,1 \quad (17)$$

$$\|\mathbf{q}^{J+1}-Q^{J+1}\|+(\Delta t\sum_{n=0}^{J}\|\mathbf{q}^{n+1}-Q^{n+1}\|^2_{\mathbf{H}(\mathbf{div};\Omega)})^{\frac12} \quad (18)$$
$$\le C(h^{\min(k+1-j,r)}+\Delta t^2).$$

**Proof:** Set $v^h=\varsigma^{n+1}$ in (11) and use lemma 4 to get

$$\| a^{\frac12}\nabla\varsigma^{n+1}\|^2+(a(x)\frac{3\nabla\varsigma^{n+1}-4\nabla\varsigma^{n}+\nabla\varsigma^{n-1}}{2\Delta t},\nabla\varsigma^{n+1})$$
$$=\| a^{\frac12}\nabla\varsigma^{n+1}\|^2+\frac{1}{4\Delta t}[(\| a^{\frac12}\nabla\varsigma^{n+1}\|^2$$
$$+\| a^{\frac12}(2\nabla\varsigma^{n+1}-\nabla\varsigma^{n})\|^2) \quad (19)$$
$$-(\| a^{\frac12}\nabla\varsigma^{n}\|^2+\| a^{\frac12}(2\nabla\varsigma^{n}-\nabla\varsigma^{n-1})\|^2)$$
$$+\| a^{\frac12}(\nabla\varsigma^{n+1}-2\nabla\varsigma^{n}+\nabla\varsigma^{n-1})\|^2]$$
$$=(\rho^{n+1}+\xi^{n+1},\nabla\varsigma^{n+1})-(R_u{}^{n+1},\nabla\varsigma^{n+1})$$

By the application for Cauchy-Schwarz inequality and Young inequality, we arrive at

$$\frac{a_0}{2}\|\nabla\varsigma^{n+1}\|^2+\frac{1}{4\Delta t}[(\| a^{\frac12}\nabla\varsigma^{n+1}\|^2$$
$$+\| a^{\frac12}(2\nabla\varsigma^{n+1}-\nabla\varsigma^{n})\|^2)$$
$$-(\| a^{\frac12}\nabla\varsigma^{n}\|^2+\| a^{\frac12}(2\nabla\varsigma^{n}-\nabla\varsigma^{n-1})\|^2) \quad (20)$$
$$+\| a^{\frac12}(\nabla\varsigma^{n+1}-2\nabla\varsigma^{n}+\nabla\varsigma^{n-1})\|^2]$$
$$\le C(\|\rho^{n+1}\|^2+\|\xi^{n+1}\|^2+\| R_u^{n+1}\|^2).$$

Multiply $4\Delta t$ in (20), sum $n=1$ to $J$ and use lemma 3 to get

$$2\Delta t a_0\sum_{n=1}^{J}\|\nabla\varsigma^{n+1}\|^2+(\| a^{\frac12}\nabla\varsigma^{J+1}\|^2$$
$$+\| a^{\frac12}(2\nabla\varsigma^{J+1}-\nabla\varsigma^{J})\|^2)$$
$$+4\Delta t\sum_{n=1}^{J}\| a^{\frac12}(\nabla\varsigma^{n+1}-2\nabla\varsigma^{n}+\nabla\varsigma^{n-1})\|^2 \quad (21)$$
$$\le(\| a^{\frac12}\nabla\varsigma^{1}\|^2+\| a^{\frac12}(2\nabla\varsigma^{1}-\nabla\varsigma^{0})\|^2)$$
$$+C\Delta t\sum_{n=0}^{J}(\|\rho^{n+1}\|^2+\|\xi^{n+1}\|^2)+\Delta t^4\|\nabla u_{ttt}\|^2_{L^\infty(L^2)}$$

By an application for Poincare inequality, we have

$$\|\nabla\varsigma^{J+1}\|^2$$
$$\le(\| a^{\frac12}\nabla\varsigma^{1}\|^2+\| a^{\frac12}(2\nabla\varsigma^{1}-\nabla\varsigma^{0})\|^2) \quad (22)$$
$$+C\Delta t\sum_{n=0}^{J}(\|\rho^{n+1}\|^2+\|\xi^{n+1}\|^2)+\Delta t^4\|\nabla u_{ttt}\|^2_{L^\infty(L^2)}.$$

We set $\mathbf{w}^h=\xi^{n+1}$ in (12) to get

$$(\alpha\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2\Delta t},\xi^{n+1})$$
$$+(\nabla\cdot\rho^{n+1}+\nabla\cdot\xi^{n+1},\nabla\cdot\xi^{n+1})$$
$$=-(\alpha\frac{3\rho^{n+1}-4\rho^{n}+\rho^{n-1}}{2\Delta t},\xi^{n+1}) \quad (23)$$
$$-(R_{\mathbf{q}}{}^{n+1},\mathbf{w}^{h})-(R_{f}{}^{n+1},\xi^{n+1})-(2(f^{n}(u)-f^{n}(U))$$
$$-(f^{n-1}(u)-f^{n-1}(U)),\nabla\cdot\xi^{n+1})$$

Apply Cauchy-Schwarz inequality and Young inequality to get

$$(\alpha\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2\Delta t},\xi^{n+1})+\|\nabla\cdot\xi^{n+1}\|^2$$
$$\le C(\|\nabla\cdot\rho^{n+1}\|^2+\|\alpha\frac{3\rho^{n+1}-4\rho^{n}+\rho^{n-1}}{2\Delta t}\|^2$$
$$+\|\xi^{n+1}\|^2+\| R_{\mathbf{q}}{}^{n+1}\|^2+\| R_f{}^{n+1}\|^2+\frac14\|\nabla\cdot\xi^{n+1}\|^2 \quad (24)$$
$$+\Big|(2(f^{n}(u)-f^{n}(U))$$
$$-(f^{n-1}(u)-f^{n-1}(U)),\nabla\cdot\xi^{n+1})\Big|$$

Use lemma 4 to get

$$\begin{aligned}&(\alpha\frac{3\xi^{n+1}-4\xi^{n}+\xi^{n-1}}{2\Delta t},\xi^{n+1})\\&=\frac{1}{4\Delta t}\left[\left(\left\|\sqrt{\alpha}\xi^{n+1}\right\|^{2}+\left\|\left(2\sqrt{\alpha}\xi^{n+1}-\sqrt{\alpha}\xi^{n}\right)\right\|\right)^{2}\right.\\&-(\|2\sqrt{\alpha}\xi^{n}\|^{2}+\|2\sqrt{\alpha}\xi^{n}-\sqrt{\alpha}\xi^{n-1}\|^{2})\\&+\|\sqrt{\alpha}\xi^{n+1}-2\sqrt{\alpha}\xi^{n}+\sqrt{\alpha}\xi^{n-1}\|^{2}]\end{aligned}\tag{25}$$

Using differential mean value theorem, we have for $u_1$ and $u_2$

$$\begin{aligned}&\left(2(f^{n}(u)-f^{n}(U))-(f^{n-1}(u)-f^{n-1}(U)),\nabla\cdot\xi^{n+1}\right)\\&=\Big|(2(f_{u}^{n}(u_{1})(u^{n}-U^{n})\\&-2(f_{u}^{n-1}(u_{2})(u^{n-1}-U^{n-1}),\nabla\cdot\xi^{n+1})\Big|\\&\le C(\|\eta^{n}+\varsigma^{n}\|+\|\eta^{n-1}+\varsigma^{n-1}\|)\|\nabla\cdot\xi^{n+1}\|\\&\le C\begin{pmatrix}\|\eta^{n}\|^{2}+\|\varsigma^{n}\|^{2}\\+\|\eta^{n-1}\|^{2}+\|\varsigma^{n-1}\|^{2}\end{pmatrix}+\frac{1}{4}\|\nabla\cdot\xi^{n+1}\|^{2}\end{aligned}\tag{26}$$

As the result in [15], we have

$$\left\|\alpha\frac{3\rho^{n+1}-4\rho^{n}+\rho^{n}}{2\Delta t}\right\|^{2}\le\frac{C}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\rho_{t}\|^{2}ds.\tag{27}$$

Substitute (25)-(27) into (24) and note that lemma 3 to get

$$\begin{aligned}&\frac{1}{4\Delta t}[(\|\sqrt{\alpha}\xi^{n+1}\|^{2}+\|2\sqrt{\alpha}\xi^{n+1}-\sqrt{\alpha}\xi^{n}\|^{2})\\&-(\|\sqrt{\alpha}\xi^{n}\|^{2}+\|2\sqrt{\alpha}\xi^{n}-\sqrt{\alpha}\xi^{n-1}\|^{2})\\&+\|\sqrt{\alpha}\xi^{n+1}-2\sqrt{\alpha}\xi^{n}+\sqrt{\alpha}\xi^{n-1}\|^{2}]+2\|\nabla\cdot\xi^{n+1}\|^{2}\\&\le C\begin{pmatrix}\|\nabla\cdot\rho^{n+1}\|^{2}+\|\eta^{n}\|^{2}\\+\|\eta^{n-1}\|^{2}+\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\|\rho_{t}\|^{2}ds\end{pmatrix}\\&+C\Delta t^{4}(\|\nabla\mathbf{q}_{ttt}\|_{L^{\infty}(L^{2})}^{2}+\|f_{tt}(u)\|_{L^{\infty}(L^{2})}^{2}\\&+C(\|\varsigma^{n}\|^{2}+\|\varsigma^{n-1}\|^{2}+\|\xi^{n+1}\|^{2})\end{aligned}\tag{28}$$

Multiply $4\Delta t$ in (28), sum $n=1$ to $J$ and use (22) to get

$$\begin{aligned}&(\|\sqrt{\alpha}\xi^{J+1}\|^{2}+\|2\sqrt{\alpha}\xi^{J+1}-\sqrt{\alpha}\xi^{J}\|^{2})\\&+\|\sqrt{\alpha}\xi^{J+1}-2\sqrt{\alpha}\xi^{J}+\sqrt{\alpha}\xi^{J-1}\|^{2}+2\Delta t\sum_{n=1}^{J}\|\nabla\cdot\xi^{J+1}\|^{2}\\&\le(\|\sqrt{\alpha}\xi^{1}\|^{2}+\|2\sqrt{\alpha}\xi^{1}+\sqrt{\alpha}\xi^{0}\|^{2}\\&+C\Delta t\sum_{n=1}^{J}(\|\nabla\cdot\rho^{n+1}\|^{2}+\|\eta^{n}\|^{2}+\|\eta^{n-1}\|^{2})\\&+C\int_{0}^{t_{J+1}}\|\rho_{t}\|^{2}ds)+C\Delta t^{4}(\|\nabla\mathbf{q}_{ttt}\|_{L^{\infty}(L^{2})}^{2}\\&+\|f_{tt}(u)\|_{L^{\infty}(L^{2})}^{2}+\|\nabla u_{ttt}\|_{L^{\infty}(L^{2})}^{2})\\&+C\Delta t\sum_{n=1}^{J}[\Delta t\sum_{k=0}^{n-1}(\|\rho^{k+1}\|^{2}+\|\xi^{n+1}\|^{2})+\|\xi^{n+1}\|^{2}].\end{aligned}\tag{29}$$

By a combination of (9) and (10) with Gronwall lemma, we have

$$\begin{aligned}&(\|\sqrt{\alpha}\xi^{J+1}\|^{2}+\|2\sqrt{\alpha}\xi^{J+1}-\sqrt{\alpha}\xi^{J}\|^{2})\\&+\|\sqrt{\alpha}\xi^{J+1}-2\sqrt{\alpha}\xi^{J}+\sqrt{\alpha}\xi^{J-1}\|^{2}\\&+2\Delta t\sum_{n=1}^{J}\|\nabla\cdot\xi^{J+1}\|^{2}\\&\le C(h^{2\min(k+1,r)}+\Delta t^{4}).\end{aligned}\tag{30}$$

Substitute (30) into (21) and (22) to get

$$\begin{aligned}&\|\varsigma^{J+1}\|^{2}+2\Delta t a_{0}\sum_{n=1}^{J}\|\nabla\varsigma^{n+1}\|^{2}+(\|a^{\frac{1}{2}}\nabla\varsigma^{J+1}\|^{2}\\&+\|a^{\frac{1}{2}}(2\nabla\varsigma^{J+1}-\nabla\varsigma^{J})\|^{2})\\&+4\Delta t\sum_{n=1}^{J}\|a^{\frac{1}{2}}(\nabla\varsigma^{n+1}-2\nabla\varsigma^{n}+\nabla\varsigma^{n-1})\|^{2}\\&\le C(h^{2\min(k+1,r)}+\Delta t^{4}).\end{aligned}\tag{31}$$

By a combination (28), (31), (9) and (10) with triangle inequality, we get the conclusion of theorem 1.

## 3. Numerical Test

In this section, for verifying the theoretical analysis, we take 1-D nonlinear pseudo-hyperbolic equation with space-time domain $[0,1]\times[0,1]$

$$\begin{cases}u_{tt}+u_{t}-(u_{xt}+u_{x})_{x}=u^{2}+f(x,t),\\u(x,0)=\sin(\pi x),u_{t}(x,0)=\sin(\pi x),\\u(0,t)=u(1,t)=0,\end{cases}\tag{32}$$

where the exact solution is $e^{t}\sin(\pi x)$ and $f(x,t)$ is determined by the exact solution.

We choose the piecewise linear space in the spatial direction and two-step difference scheme in time direction and give some numerical results by Matlab procedure. In Table 1, we list some a priori error results for $u$ and $q$ in $L^2$-norm with space-time step length $h=2\Delta t$. From Table 1, we can clearly see that the rate of convergence for our method is close to 2, which confirms our theoretical error results.

In Figure 1, the comparison figures are shown for the exact solution $u$ and the numerical solution $u^h$. The similar results for $q$ and $q^h$ are also described in Figure 2. From Figure 1 and Figure 2, we can intuitively see that the proposed numerical method is feasible and effective.

**Table 1. Optimal errors in $L^{\infty}(L^{2})$ norm**

| $h=2\Delta t$ | $\|u-u_h\|_{L^\infty(L^2)}$ | Order | $\|q-q_h\|_{L^\infty(L^2)}$ | Order |
|---|---|---|---|---|
| 1/10 | 1.0396E-02 | | 2.2453E-02 | |
| 1/20 | 2.6083E-03 | 1.9948 | 5.5425E-03 | 2.0183 |
| 1/40 | 6.5341E-04 | 1.9970 | 1.3753E-03 | 2.0108 |
| 1/80 | 1.6353E-04 | 1.9984 | 3.4243E-04 | 2.0059 |
| 1/160 | 4.0907E-05 | 1.9992 | 8.5426E-05 | 2.0030 |

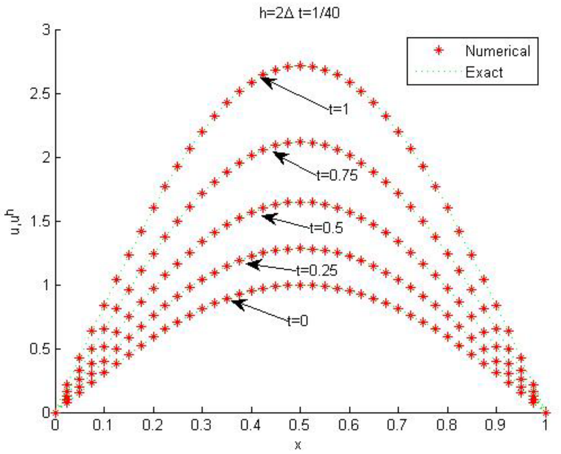


**Figure 1**. Exact solution $u$ and numerical solution $u^h$

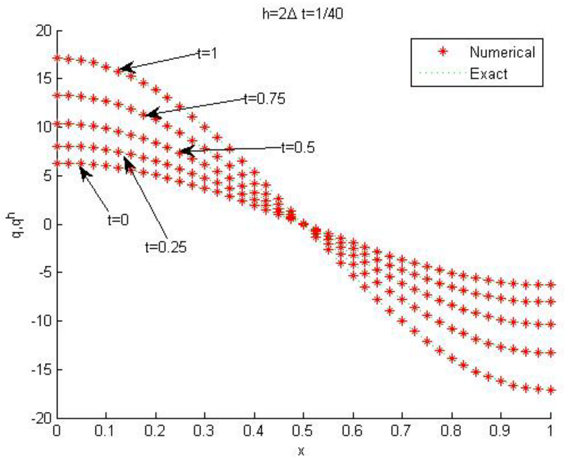


**Figure 2**. Exact solution $q$ and numerical solution $q^h$

# 4. Some Conclusions and Remarks

So far, the $H^1$-Galerkin mixed element method [10] has been considered to seek for the numerical solutions by more and more people. In the literatures on the $H^1$-Galerkin mixed method, the backward Euler method and Crank-Nicolson scheme are widely presented. However, the two-step difference method based on the $H^1$-Galerkin mixed method is sparingly studied and analyzed. In this article, we consider a two-step difference method in time direction combining $H^1$-Galerkin mixed method and obtain some optimal time second-order rates of convergence.

From the results of numerical calculations, we can clearly find that fully discrete two-step $H^1$-Galerkin mixed method is efficient. In the near future, the fully discrete two-step difference method combined with $H^1$-Galerkin mixed method for nonlinear pseudo-hyperbolic integro-differential equations is analyzed.

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