

Coupled Fixed Point Theorem in Partially Ordered Metric Spaces

Virendra Singh Chouhan^{1,*}, Richa Sharma^{2,*}

¹Department of Mathematics, Lovely Professional University, Jalandhar, India

²Department of Applied Sciences, Rayat Bahra Institute of Engineering & Nano-Technology, Hoshiarpur, India

*Corresponding author: darbarvsingh@yahoo.com; richa.tuknait@yahoo.in

Received March 23, 2014; Revised April 12, 2014; Accepted April 13, 2014

Abstract In this paper, we obtain some unique coupled fixed point theorem in a complete metric space endowed with a partial order. We also give example to validate the main result in this paper.

Keywords: Coupled fixed point, partial ordered, complete metric space

Cite This Article: Virendra Singh Chouhan, and Richa Sharma, "Coupled Fixed Point Theorem in Partially Ordered Metric Spaces." *American Journal of Mathematical Analysis*, vol. 2, no. 2 (2014): 19-22. doi: 10.12691/ajma-2-2-1.

1. Introduction

The famous and probably the well known fixed point theorem is the Banach Contraction Principle. It has been extended and improved by many mathematicians. Its significance lies in its vast applicability in a number of branches of mathematics. Recently, W. Zhong et al [11] give the existence and uniqueness of solutions to the Cauchy problem for the local fractional differential equation with fractal conditions in a generalized Banach space.

In 2006, Bhaskar and Lakshmikantham [3] established some coupled fixed point theorem on ordered metric spaces and give some application in the existence and uniqueness of a solution for periodic boundary value problem. Ćirić and Lakshmikantham [5] later on investigated some more coupled fixed point theorems in partially ordered sets. Also, many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces [1,2,4,6].

In this paper, we prove some unique coupled fixed point theorems in a complete metric space endowed with a partial order. At the end of this paper we give an example to support our main theorem.

The organization of this paper is as follows. In section 2, the preliminary result on partial metric space is discussed. In section 3, we investigated the necessary condition for the uniqueness of coupled fixed point of the given mapping in partially ordered metric space and give an example to illustrate our main theorem.

2. Preliminaries

In this section, we give some definitions, lemma which are useful for main result in this paper.

Definition 2.1. [3,5] An element $(x, y) \in X \times X$ is said to be coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x, F(y, x) = y$.

Definition 2.2. [3] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y , that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.3. [7,8,9] Let X be a non empty set. A partial metric on X is a function $p : X \times X \rightarrow R_n$ such that for all $x, y, z \in X$:

$$(P_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y);$$

$$(P_2) \quad p(x, x) \leq p(x, y);$$

$$(P_3) \quad p(x, y) = p(y, x);$$

$$(P_4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a non empty set and p is a partial metric on X .

If p is a partial metric on X , then the function $p^s : X \times X \rightarrow R^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X .

Definition 2.4. [7,8,9] Let (X, p) be a partial metric space. Then:

(a) a sequence $\{x_n\}$ in partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$,

(b) a sequence $\{x_n\}$ in partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x_n) = \lim_{n \rightarrow \infty} p(x, x_n)$, if and only if $\lim_{n \rightarrow \infty} p^s(x, x_n) = 0$,

(c) a sequence $\{x_n\}$ in partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$,

(d) a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Lemma 2.5. [7,8] Let (X, p) be partial metric space;

a. $\{x_n\}$ is Cauchy sequence in (X, p) if and only if it is Cauchy sequence in the metric space (X, p^s) ,

(b) a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete; furthermore, $\lim_{n \rightarrow \infty} p^s(x_n, x) = 0$ iff $p(x, x) = \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_m)$.

3. Main Theorem

Theorem 3.1. Let (X, \leq) be a partially ordered set and let p be a partial metric on X such that (X, p) is complete. Suppose the mapping $F : X \times X \rightarrow X$ satisfies the following condition for all $x, y, u, v \in X$, we have

- 1) F is continuous or
- 2) X has the following properties,
 - (a) if a non-decreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \leq x, \forall n$,
 - (b) if a non-increasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \leq x, \forall n$.
- 3) $\exists x_0, y_0 \in X$, such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.
- 4) $\exists \psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non decreasing function such that it is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t \rightarrow \infty} \psi(t) = \infty$;

$$p(F(x, y), F(u, v)) \leq p(x, u) + \psi(p(y, v)) \quad (3.1)$$

Then F has a coupled fixed point $(u^*, v^*) \in X \times X$.

Proof: Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, set $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then by (3.1), we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq p(x_{n-1}, x_n) + \psi(p(y_{n-1}, y_n)) \end{aligned} \quad (3.2)$$

and similarly,

$$\begin{aligned} p(y_n, y_{n+1}) &= p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq p(y_{n-1}, y_n) + \psi(p(x_{n-1}, x_n)). \end{aligned} \quad (3.3)$$

By adding, we have

$$p_n \leq p_{n-1} + \psi(p_{n-1}). \quad (3.4)$$

Let

$$p_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1}).$$

If $\exists n_1 \in N^*$ such that $p(x_{n_1}, x_{n_1-1}) = 0$, $p(y_{n_1}, y_{n_1-1}) = 0$, then $x_{n_1-1} = x_{n_1} = F(x_{n_1-1}, y_{n_1-1})$, $y_{n_1-1} = y_{n_1} = F(y_{n_1-1}, x_{n_1-1})$ and $x_{n_1-1}; y_{n_1-1}$ is fixed point of F and the proof is finished. In other case $p(x_{n+1}, x_n) \neq 0; p(y_{n+1}, y_n) \neq 0$ for all $n \in N$. Then by using assumption on ψ , we have,

$$p_n \leq p_{n-1} + \psi(p_{n-1}) \leq p_{n-1} \quad (3.5)$$

p_n is a non - negative sequence and hence posses a limit p^* . Taking limit when $n \rightarrow \infty$, we get,

$$p^* \leq p^* + \psi(p^*)$$

and consequently $\psi(p^*) = 0$. By our assumption on ψ , we conclude $p^* = 0$, ie. $\lim_{n \rightarrow \infty} (p_n) = 0$

$$\begin{aligned} \lim_{n \rightarrow \infty} p(x_{n+1}, x_n) + p(y_{n+1}, y_n) &= 0 \\ \Rightarrow \lim_{n \rightarrow \infty} p(x_{n+1}, x_n) &= \lim_{n \rightarrow \infty} p(y_{n+1}, y_n) = 0 \end{aligned} \quad (3.6)$$

Next, we prove that $\{x_n\}, \{y_n\}$ are Cauchy sequences. Suppose that at least one $\{x_n\}$ or $\{y_n\}$ be not a Cauchy sequence. Then $\exists \varepsilon > 0$ and two subsequence of integers n_k, m_k with $n_k > m_k \geq k$, such that

$$r_k = p(x_{m_k}, x_{n_k}) + p(y_{m_k}, y_{n_k}) \geq \varepsilon, \forall k = 1, 2, 3, \dots \quad (3.7)$$

Further, corresponding to m_k , we can choose n_k in such a way that it is smallest integer with $n_k > m_k \geq k$ satisfying equation (3.7), we have

$$p(x_{m_k}, x_{n_k-1}) + p(y_{m_k}, y_{n_k-1}) < \varepsilon. \quad (3.8)$$

Using (3.7) and (3.8) and triangle inequality, we get

$$\begin{aligned} \varepsilon \leq r_k &= p(x_{m_k}, x_{n_k}) + p(y_{m_k}, y_{n_k}) \\ &\leq p(x_{m_k}, x_{n_k-1}) + p(x_{n_k-1}, x_{n_k}) \\ &\quad + p(y_{m_k}, y_{n_k-1}) + p(y_{n_k-1}, y_{n_k}) \\ &= p(x_{m_k}, x_{n_k-1}) + p(y_{m_k}, y_{n_k-1}) \\ &\quad + p(x_{n_k-1}, x_{n_k}) + p(y_{n_k-1}, y_{n_k}) \\ &< \varepsilon + p_{n_k-1}. \end{aligned} \quad (3.9)$$

Letting $k \rightarrow \infty$ and using (3.6), we have

$$\lim_{n, m \rightarrow \infty} r_k = \varepsilon > 0.$$

Now, we get

$$\begin{aligned} p(x_{m_k+1}, x_{n_k+1}) &= p(F(x_{m_k}, y_{m_k}), F(x_{n_k}, y_{n_k})) \\ &= p(F(x_{n_k}, y_{n_k}), F(x_{m_k}, y_{m_k})) \quad (3.10) \\ &\leq p(F(x_{n_k}, x_{m_k}), \psi(p(y_{n_k}, y_{m_k}))) \end{aligned}$$

similarly,

$$\begin{aligned} p(y_{m_k+1}, y_{n_k+1}) &= p(F(y_{m_k}, x_{m_k}), F(y_{n_k}, x_{n_k})) \\ &= p(F(y_{n_k}, x_{n_k}), F(y_{m_k}, x_{m_k})) \quad (3.11) \\ &\leq p(y_{n_k}, y_{m_k}), \psi(p(x_{n_k}, x_{m_k})) \end{aligned}$$

Using(3.10) and (3.11), we get

$$r_{k+1} \leq r_k + \psi(r_k) \quad (3.12)$$

$\forall k \in 1, 2, 3, \dots$ taking $k \rightarrow \infty$ of both sides of equation (3.12) from $\lim_{t \rightarrow \infty} \psi(t) = \infty$ it follows that

$$\varepsilon = \lim_{k \rightarrow \infty} r_{k+1} \leq \lim_{k \rightarrow \infty} r_k + \psi(r_k) < \varepsilon$$

which is a contraction. Therefore $\{x_n\}$ and $\{y_n\}$ are cauchy sequences. By lemma (2.5), $\{x_n\}$ and $\{y_n\}$ are cauchy sequence in (X, p^s) . Since (X, p) is complete, hence (X, p^s) is also complete, so $\exists u^*, v^* \in X$ such that

$$\lim_{n \rightarrow \infty} p^s(x_n, u^*) = \lim_{n \rightarrow \infty} p^s(y_n, v^*) = 0.$$

By lemma, we have

$$\begin{aligned} p(u^*, u^*) &= \lim_{n \rightarrow \infty} p(x_n, u^*) = \lim_{n \rightarrow \infty} p(x_n, x_n) \\ p(v^*, v^*) &= \lim_{n \rightarrow \infty} p(y_n, v^*) = \lim_{n \rightarrow \infty} p(y_n, y_n). \end{aligned}$$

By condition and equation, we get

$$\lim_{n \rightarrow \infty} p(x_n, x_n) = 0.$$

It follows that

$$p(u^*, u^*) = \lim_{n \rightarrow \infty} p(x_n, u^*) = \lim_{n \rightarrow \infty} p(x_n, x_n) = 0$$

Similarly,

$$p(v^*, v^*) = \lim_{n \rightarrow \infty} p(y_n, v^*) = \lim_{n \rightarrow \infty} p(y_n, y_n) = 0.$$

We now prove that $F(u^*, v^*) = u^*, F(v^*, u^*) = v^*$. We shall distinguish the cases (1), 2(a) and 2(b) of the Theorem 3.1.

Since X is a complete metric space, $\exists u^*, v^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = u^*, \lim_{n \rightarrow \infty} y_n = v^*$. We now show that if the assumption (1) holds, then (u^*, v^*) is coupled fixed point of F .

As, we have

$$\begin{aligned} x &= \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} F(x_n, y_n) \\ &= F\left(\lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n\right) = F(u^*, v^*) \end{aligned}$$

and

$$\begin{aligned} y &= \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} F(y_n, x_n) \\ &= F\left(\lim_{n \rightarrow \infty} y_n, \lim_{n \rightarrow \infty} x_n\right) = F(v^*, u^*). \end{aligned}$$

Therefore, (u^*, v^*) is coupled fixed point of F .

Suppose now that the condition 2(a) and 2(b) of the theorem holds.

The sequence $\{x_n\} \rightarrow u^*, \{y_n\} \rightarrow v^*$

$$\begin{aligned} p(F(u^*, v^*), u^*) &\leq p(F(u^*, v^*), x_{n+1}) + p(x_{n+1}, u^*) \\ &= p(F(u^*, v^*), F(x_n, y_n)) + p(x_{n+1}, u^*) \\ &\leq p(u^*, x_n) + \psi(p(y_n, v^*)) + p(x_{n+1}, u^*). \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$p(F(x, y), x) \leq 0 + \psi(0) = 0.$$

This implies that $F(u^*, v^*) = u^*$, similarly, we can show that $F(v^*, u^*) = v^*$. This completes the theorem.

Theorem 3.2. Let the hypotheses of Theorem 3.1 hold. In addition, suppose that there exists $z \in X$ which is comparable to u and v for all $u, v \in X$. Then F has a unique coupled fixed point.

Proof: Suppose that there exists $(u', v'), (u^*, v^*) \in X \times X$ are coupled fixed points of F . Consider the following two cases:

Case1: (u', v') and (u^*, v^*) are comparable. We have

$$\begin{aligned} p(u', u^*) &= p(F(u', v'), F(u^*, v^*)) \\ &\leq p(u', u^*) + \psi(p(v', v^*)) \end{aligned}$$

similarly,

$$\begin{aligned} p(v', v^*) &= p(F(v', u'), F(v^*, u^*)) \\ &\leq p(v', v^*) + \psi(p(u', u^*)). \end{aligned}$$

It follows that

$$\begin{aligned} &p(u', u^*) + p(v', v^*) \\ &\leq p(u', u^*) + p(v', v^*) + \psi[p(v', v^*) + p(u', u^*)] \\ &\Rightarrow p(u', u^*) + p(v', v^*) = 0. \end{aligned}$$

So, $u^* = u', v^* = v'$. The proof is complete.

Case 2: Suppose now that (u', v') and (u^*, v^*) are not comparable. Choose an element $(w, z) \in X$ comparable with both of them.

$$\text{Monotomcity} \Rightarrow (F^n(w, z), F^n(z, w))$$

$$\begin{aligned} p\left(\begin{matrix} (u^*, v^*) \\ (u', v') \end{matrix}\right) &= p\left(\begin{matrix} (F^n(u^*, v^*)) \\ (F^n(v^*, u^*)) \end{matrix}, \begin{matrix} (F^n(u', v')) \\ (F^n(v', u')) \end{matrix}\right) \\ &\leq p\left(\begin{matrix} (F^n(u^*, v^*)) \\ (F^n(v^*, u^*)) \end{matrix}, \begin{matrix} (F^n(w, z)) \\ (F^n(z, w)) \end{matrix}\right) \\ &\quad + p\left(\begin{matrix} (F^n(w, z)) \\ (F^n(z, w)) \end{matrix}, \begin{matrix} (F^n(u', v')) \\ (F^n(v', u')) \end{matrix}\right) \end{aligned}$$

$$\begin{aligned} &\leq p(u^*, w) + \psi(p(v^*, z)) + (p(v^*, z) + \psi(p(u^*, w))) \\ &\quad + (p(w, u') + \psi(p(z, v'))) + (p(z, v') + \psi(p(w, u'))) \\ &= 0 \end{aligned}$$

So, $u^* = u', v^* = v'$. The proof is complete.

Example 3.3. Let $X = [0, 1]$ endowed with the usual partial metric p defined by $p: X \times X \rightarrow [0, 1]$ with $p(x, y) = \max(x, y)$. The partial metric space (X, p) is complete because (X, p^s) is complete for any $x, y \in X$,

$$\begin{aligned} p^s(x, y) &= 2p(x, y) - p(x, x) - p(y, y) \\ &= 2\max(x, y) - (x + y) = |x - y| \end{aligned}$$

Thus (X, p^s) is Euclidean metric space which is complete.

Consider the mapping $F: X \times X \rightarrow X$ defined by

$$F(x, y) = x + \frac{y}{5}; x \geq y.$$

Let us take $\psi: [0, \infty) \rightarrow [0, \infty)$ such that $\psi(t) = \frac{t}{5}$.

Clearly F is continuous and has the mixed monotone property. Also there are $x_0 = 0; y_0 = 0$ in X such that

$$\begin{aligned} x_0 = 0 &\leq F(0, 0) = F(x_0, y_0) \\ \text{and } y_0 = 0 &\geq F(0, 0) = F(y_0, x_0). \end{aligned}$$

Then it is obvious that $(0, 0)$ is the coupled fixed point of F .

Now, we have following possibilities for values of (x, y) and (u, v) such that $x \leq u, y \geq v$

$$\begin{aligned} p(F(x, y), F(u, v)) &= p\left(x + \frac{y}{5}, u + \frac{v}{5}\right) \\ &= \max\left(x + \frac{y}{5}, u + \frac{v}{5}\right) \\ &\leq \max(x, u) + \frac{1}{5}\max(y, v) \\ &= p(x, u) + \psi(p(u, v)). \end{aligned}$$

Thus all the conditions of theorem 3.1 are satisfied. Therefore F has a coupled fixed point in X .

References

- [1] H. Alaeidizaji, V. Parvaneh, *Coupled fixed point results in complete partial metric space*, International Journal of mathematical sciences, Artical ID 670410 (2012).
- [2] H. Aydi, *Some coupled fixed point results on partial metric spaces*, International Journal of Mathematical Sciences, Artical ID 647091 (2011).
- [3] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis: Theory, Methods and Applications, 65(7) (2006), 1379-1393.
- [4] L.Ciric, V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stochastic and Applications 27, (2009), 1246-1259.
- [5] L.Ciric, V. Lakshmikantham, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Analysis: Theory, Methods and Applications 70 (12), (2009), 4341-4349.
- [6] L.Ciric, M.O. Olatinwo, D. Gopal, G. Akinbo, *Coupled fixed point theorems for mappings satisfying a contractions of rational type on a partially ordered metric space*, Advances in Fixed Point Theory, 2 No. 1, (2012), 1-8.
- [7] S.G. Matthews, *Partial metric topology*, in: *Proc 8th summer conference on general topology and application*, in Annals of the New York Academy of Sciences, Vol.728, pp. (1994), 183-197.
- [8] S. J. O'Neill, *Two topologies are better than one*, Tech. Rep., University of warwick, Coventry, UK (1995).
- [9] S. J. O'Neil, *Partial metrics, valuations and domain theory*, in: *Proc 11th summer conference on general topology and application*, in Annals of the New York Academy of Sciences, Vol.806, (1996), 304-315.
- [10] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory and Applications, Article ID 125426 (2009).
- [11] W.Zhong, X. Yang, F. Gao, *A cauchy problem for some local fractional abstract differential equation with fractal conditions*, J. Applied Functional Analysis, Vol.8, No. 1(2013), 92-99.