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d-Lucky Labeling of Some Special Graphs

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Abstract Consider $\lambda: V(G) \to \{1, 2, 3, \ldots\}$ as labeling of graph 's vertices. The weight for vertex x is specified as $\omega(x) = \deg(x) + \sum_{y \in N(x)} (\lambda(y))$ where deg(x) shows the vertex x's degree, $N(x)$ shows the u's open neighborhood and $\lambda(y)$ shows the label for vertex y. In [1] M. Miller et al. define d-lucky labeling that is similar to the graph vertex coloring. The labeling λ is said to be $d-$lucky labeling of graph G if $\omega(x) \neq \omega(y)$ for each adjacent pair of vertices x and y in G. The least positive integer n such that G has a d-lucky labeling with $\{1, 2, ..., n\}$ as the set of labels is known as d -lucky number of a graph G represented as $\eta_{dl}(G)$. In this paper we investigated the d-lucky number for jelly fish graph, coconut tree, kite graph, complete binary tree and generalized theta graph.

Keywords: *lucky labeling, d -lucky labeling, jelly fish graph, coconut tree, kite graph, complete binary tree, generalized theta graph*

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1. Introduction

One of the most researched topics in graph theory is graph coloring. It is the process of assigning labels called color to the components of a graph subject to certain constraints. The analysis of proper labeling was started by Karonski, Luczak, and Thomason. The rule about using colors comes from coloring a map's nations, where each face is precisely colored. Vertex coloring, also known as proper labeling, is a method of coloring a graph's vertices such that no two neighboring vertices have the same color. The issue of proper labeling has a wide range of solutions and has recently gained a lot of attention. See [2-5] and [11,12] for more information. Networking, image segmentation, clustering, image capturing, and data mining are only a few of the computer science research fields where graph coloring is used.

There is a wide range of labeling procedures available in the literature that lead to proper graph vertex coloring. For a mapping $f: V(G) \to \{1, 2, 3, \ldots k\}$, a proper vertex coloring is obtained through Lucky labeling [6,7] Vertex-labeling by product [8], Vertex-labeling by gap, Vertex-labeling by degree and Vertex-labeling by maximum degree [8]. For a mapping $f: V(G) \to \{1, 2, 3, \ldots k\}$, a proper vertex coloring is obtained through Edge-labeling by sum [9], Edge-labeling by product [5] and Edge-labeling by gap. In this paper we use the d -lucky labeling and compute the d -lucky number of some special graphs.

2. Jelly Fish Graph *J*(p,q)

Jelly fish graph $J(p,q)$ defined in [10] can be constructed by a 4-cycle c_1, c_2, c_3, c_4 by attaching c_1 and c_3 via an edge and p pendent edges are attached with c_2 and and q pendent edges with c_4. The Jelly fish graph $J(p,q)$ containing $p+q+4$ vertices and $p+q+5$ edges. The vertex and edge collections of $J(p,q)$ are listed below.

$$V(J(p,q)) = \{c_1, c_2, c_3, c_4\} \cup \{a_j : 1 \leq j \leq p\} \cup \{b_i : 1 \leq i \leq q\}$$

and

$$E(J(p,q)) = \{c_1c_2, c_2c_3, c_3c_4, c_4c_1, c_1c_3\} \cup \{c_2a_j : 1 \leq j \leq p\} \cup \{c_4b_i : 1 \leq i \leq q\}.$$

c_1 is known as upper central node, c_2 is known as left central node, c_3 is known as lower central node and c_4 is known as right central node. a_j where $1 \leq j \leq p$ and b_i where $1 \leq i \leq q$ are one-degree pendent nodes.

The Jelly fish graph $J(\mathrm{p},\mathrm{q})$ accepts d-lucky labeling, as shown in the next theorem and $\eta_{dl}\left(J\left(p,q\right)\right)=2.$

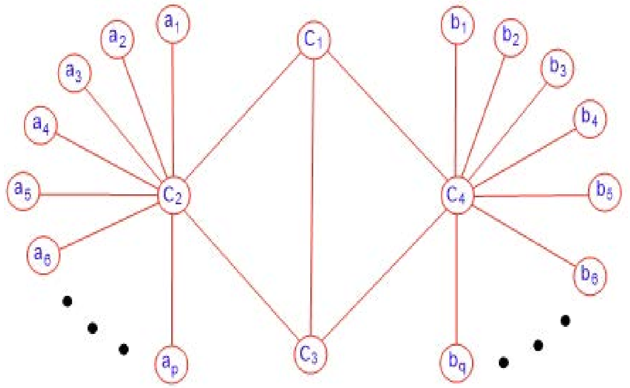

Figure 1. Jelly fish graph $J(p,q)$

Theorem 2.1. The Jelly fish graph $J(\mathrm{p},\mathrm{q})$ where $p,q\geq 1$ accepts d-lucky labeling with $\eta_{dl}\left(J\left(p,q\right)\right)=2.$

Proof: Consider $\lambda:V\left(G\right)\to\{1,\ 2,\ 3,\ldots\}$ as labeling of graph's vertices. The weight for vertex x is specified as

$$\omega(\mathrm{x})=\deg(\mathrm{x})+\sum_{y\in N(\mathrm{x})}(\lambda(\mathrm{y}))$$

where deg(x) shows the vertex x' s degree, $N(\mathrm{x})$ shows the u 's open neighborhood and $\lambda(y)$ shows the label for vertex y. The proof s divided into 4 separate cases for $J(\mathrm{p},\mathrm{q})$ where $p,q\geq 1$.

Case. 1: For $p=q=1$

c_1 is upper central node has $deg\left(c_1\right)=3,\ \lambda\left(c_1\right)=2$ and other 3 -adjacent vertices $\mathrm{c}_2,\mathrm{c}_3,\mathrm{c}_4$ are labeled as 1. So

$$\omega(c_1)=\deg(c_1)+\sum_{\mathrm{v}_i\in N(c_1)}(\lambda(\mathrm{v}_i))$$
$$3+1+1+1=6$$

Similarly c_3 is lower central node having $deg\left(c_3\right)=3.$ So

$$\omega(c_3)=\deg(c_3)+\sum_{\mathrm{v}_i\in N(c_3)}(\lambda(\mathrm{v}_i))$$
$$3+1+2+1=7$$

The left central node c_2 and right central node c_4 have $deg\left(c_2\right)=deg\left(c_4\right)=3.$ The vertex c_2 is connected to the vertices c_1 , c_3 and also with pendent vertex a_1 having $\lambda\left(a_1\right)=2.$ Likewise, the vertex c_4 is connected to the vertices $c_1,\ c_3$ and also with pendent vertex b_1 having $\lambda\left(b_1\right)=2.$ So,

$$\omega(c_2)=\deg(c_2)+\sum_{\mathrm{v}_i\in N(c_2)}(\lambda(\mathrm{v}_i))$$
$$3+2+2+1=8$$

and

$$\omega(c_4)=\deg(c_4)+\sum_{\mathrm{v}_i\in N(c_4)}(\lambda(\mathrm{v}_i))$$
$$3+2+2+1=8$$

On the other side, the vertex c_2 is connected to vertex a_1 having $deg\left(a_1\right)\ =\ 1.$ So

$$w\left(a_1\right)=deg\left(a_1\right)+\lambda\left(c_2\right)=1+1=2.$$

Likewise, the vertex c_4 is connected to vertex b_1 having $deg\left(b_1\right)\ =\ 1.$ So

$$w\left(b_1\right)=deg\left(b_1\right)+\lambda\left(c_4\right)=1+1=2.$$

Case. 2: For $p=1$ and $q>1$

The upper central node c_1 having $deg\left(c_1\right)=3,$ $\lambda\left(c_1\right)=2$ and a11 other 3-adjacent vertices $\mathrm{c}_2,\mathrm{c}_3,\mathrm{c}_4$ are labeled as 1. So

$$\omega(c_1)=\deg(c_1)+\sum_{\mathrm{v}_i\in N(c_1)}(\lambda(\mathrm{v}_i))$$
$$3+1+1+1=6$$

Likewise lower central node c_3 also having $deg\left(c_3\right)=3.$ So

$$\omega(c_3)=\deg(c_3)+\sum_{\mathrm{v}_i\in N(c_3)}(\lambda(\mathrm{v}_i))$$
$$3+1+2+1=7$$

The left central node c_2 has $deg\left(c_2\right)=3.$ The vertex c_2 is connected to the vertices $c_1,\ c_3$ and pendent node a_1 having $\lambda\left(a_1\right)=2.$ So

$$\omega(c_2)=\deg(c_2)+\sum_{\mathrm{v}_i\in N(c_2)}(\lambda(\mathrm{v}_i))$$
$$3+2+2+1=8$$

The left central vertex c_2 is connected to a pendent node a_1 having $deg\left(a_1\right)\ =\ 1.$ So

$$w\left(a_1\right)\ =deg\left(a_1\right)\ +\lambda\left(c_2\right)\ =\ 1\ +\ 1\ =\ 2.$$

The right central node c_4 has $deg\left(c_4\right)=q+2.$ The vertex c_4 connected to vertices c_1,c_3 and with q one-degree pendent nodes b_i such that $\lambda\left(b_i\right)=1$ where $1\leq i\leq q.$ So

$$\omega(c_4)=\deg(c_4)+\sum_{\mathrm{v}_i\in N(c_4)}(\lambda(\mathrm{v}_i))$$
$$=2q+5$$

The q pendent nodes b_i having $deg\left(b_i\right)\ =\ 1.$ So

$$w\left(b_i\right)\ =deg\left(b_i\right)\ +\lambda\left(c_4\right)\ =\ 1\ +\ 1\ =\ 2$$

where $1\leq i\leq q.$

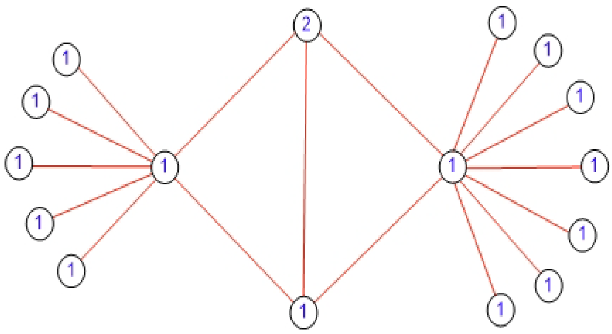

Figure 2. d-lucky labeling of J(5,7)

Case. 3: For $q = 1$, $p > 1$

This case is identical to previous case 2. All central nodes c_1, c_2, c_3, c_4 labeled as in case 2 with $w(c_1) = 6$, $w(c_3) = 7$. c_4 is connected to vertices c_1 and c_3 as well as a single pendent vertex b_1. As a result, $w(c_4) = 8$. Similarly the vertex c_2 connected to vertices c_1 and c_3 as well as the p pendent vertices c_j where $1 \le j \le p$. As a result, $w(c_2) = 2p + 5$. Every pendent vertex has the same weight as mentioned in previous case 2.

Case 4: For p, $q > 1$

The upper central node c_1 having $deg(c_1) = 3$, $\lambda(c_1) = 2$ and a11 other 3-adjacent vertices c_2, c_3, c_4 are labeled as 1. So.

$$\omega(c_1) = \deg(c_1) + \sum_{v_i \in N(c_1)} (\lambda(v_i))$$

$$3 + 1 + 1 + 1 = 6$$

Similarly c_3 is lower central node also having $deg(c_3) = 3$. So

$$\omega(c_3) = \deg(c_3) + \sum_{v_i \in N(c_3)} (\lambda(v_i))$$

$$3 + 1 + 2 + 1 = 7$$

The left central node c_2 has $deg(c_2) = p + 2$. The left central node c_2 is connected to vertices c_1 and c_3 as well as p pendent vertices a_j such that $\lambda(a_j) = 1$ where $1 \le j \le p$. So

$$\omega(c_2) = \deg(c_2) + \sum_{v_i \in N(c_2)} (\lambda(v_i))$$

$$= 2p + 5$$

The p pendent nodes a_j having $deg(a_j) = 1$. As a result

$$w(a_j) = deg(a_j) + \lambda(c_4) = 1 + 1 = 2$$

where $1 \le j \le p$.

The right central node c_4 has $deg(c_4) = q + 2$. The vertex c_4 connected to vertices c_1 and c_3 as well as q pendent vertices b_i such that $\lambda(b_i) = 1$ where $1 \le i \le q$. So

$$\omega(c_4) = \deg(c_4) + \sum_{v_i \in N(c_4)} (\lambda(v_i))$$

$$= 2q + 5$$

The q pendent nodes b_i having $deg(b_i) = 1$. So

$$w(b_i) = deg(b_i) + \lambda(c_4) = 1 + 1 = 2$$

where $1 \le i \le q$.

It's simple to verify that the weights of all neighboring nodes are different and brings the evidence to a close.

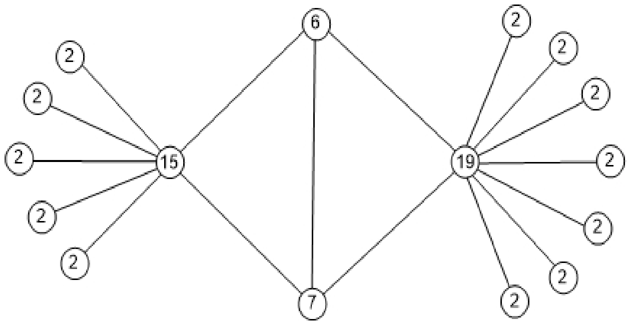

Figure 3. Weights of J(5,7)

3. Coconut Tree *CT*(p,q)

Coconut tree is formed by appending q new pendent edges to the last node of path P_p as definein [10].

The Coconut tree CT (p, q) containing $p + q + 4$ vertices and $p + q + 5$ edges. The vertex and edge collections of $CT(p, q)$ are listed below.

$$V(CT(p,q)) = \{x_i : 1 \le i \le p\} \cup \{y_j : 1 \le j \le q\}$$

and

$$E(CT(p,q)) = \{x_i x_{i+1} : 1 \le i \le p-1\} \cup \{v_p y_j : 1 \le j \le q\}.$$

The vertices of path P_p are represented by x_i and the pendent nodes of degree one are represented by y_j.

Coconut tree CT (p, q) accepts d-lucky labeling, as shown in the next theorem and $\eta_{dl}(CT(p,q)) = 2$.

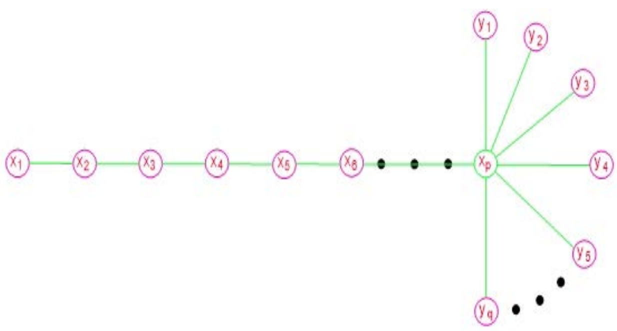

Figure 4. Coconut Tree *CT(p,q)*

Theorem 3.1. Coconut tree CT (p, q) accepts d-lucky labeling with $\eta_{dl}(CT(p,q)) = 2$.

Proof: Consider $\lambda : V(G) \rightarrow \{1, 2, 3, \ldots\}$ as labeling of graph's vertices. The weight for vertex x is specified as

$$\omega(\mathrm{x}) = \deg(\mathrm{x}) + \sum_{y \in N(\mathrm{x})} (\lambda(\mathrm{y}))$$

where deg(x) shows the vertex x's degree, $N(\mathrm{x})$ shows the u's open neighborhood and $\lambda(y)$ shows the label for vertex y. The proof s divided into 2 separate cases for $CT(p,q)$ where $p, q \geq 1$.

Case. 1: For $p = q = 1$

The path P_p has single node x_1 with $deg(x_1) = 1$ and $\lambda(x_1) = 2$. The pendent node is y_1 with $deg(y_1) = 1$, $\lambda(y_1) =$ □ As a result,

$$w(x_1) = deg(x_1) + \lambda(y_1) = 1 + 1 = 2$$

and

$$w(y_1) = deg(y_1) + \lambda(x_1) = 1 + 2 = 3.$$

Case 2**: For** $p > 1, q \geq 1$ **Or** $p \geq 1, q > 1$

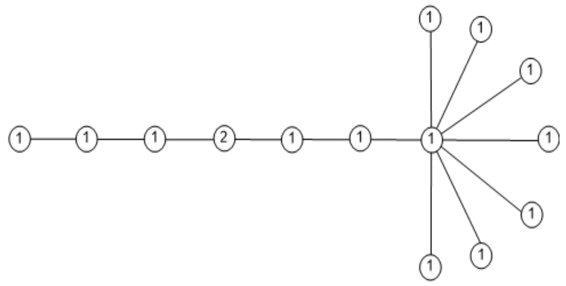

Figure 5. d-lucky labeling of CT(7,7)

The vertices of path P_p at locations $4k$ where $k = 1, 2, \ldots, h$: $4h \leq p$, are labeled with 2 and all other vertices x_i $(i = 1, 2, \ldots, p)$ are labeled with 1. All q pendent nodes y_j where $1 \leq j \leq q$ are labeled with 1. The first node x_1 of path P_p has

$$w(x_1) = deg(x_1) + \lambda(x_2) = 1 + 1 = 2$$

With the exception of end node, all other vertices that located at even locations of path P_p have weights

$$w(x_i) = deg(x_i) + \Sigma\lambda(N(x_i)) = 2 + 1 + 1 = 4$$

Similarly except the end node, all other vertices that located at odd locations of path P_p have weights

$$w(x_i) = deg(x_i) + \Sigma\lambda(N(x_i)) = 2 + 1 + 2 = 5$$

If pendent nodes are attached to a vertex which located at location $4k$ where $k = 1, 2, \ldots, h : 4h \leq p$ of path P_p then

$$w(\mathrm{y}_j) = deg(\mathrm{y}_j) + \Sigma\lambda(N(\mathrm{y}_j)) = 1 + 2 = 3$$

Otherwise

$$w(\mathrm{y}_j) = deg(\mathrm{y}_j) + \Sigma\lambda(N(\mathrm{y}_j)) = 1 + 1 = 2$$

where $j = 1, 2, \ldots, q$. If the end node x_p of path P_p is located at position $4k + 1$ then $w(x_m) = 2q + 3$ otherwise $w(x_m) = 2q + 2$.

It's simple to check that the weights of all neighboring nodes are different and brings the evidence to a close.

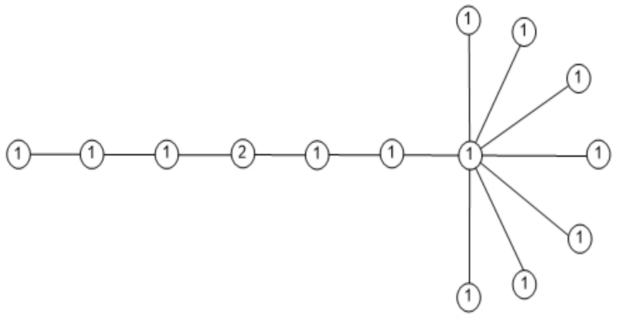

Figure 6. d-lucky labeling of CT(7,7)

4. (p, q)−Kite Graph

As specified in [10], a (p, q)−kite is a cycle of length p with q−edges path (the tail) connects to one node of cycle. In particular, the $(p, 1)$−kite is a cycle of length p with an edge fixed to one vertex of cycle. The flag graph denoted by Fl_p is also known as (p, q)−kite. A (p, q)−kite containing $p + q$ vertices and $p + q$ edges. The vertex and edge collections of (p, q)−kite are listed below.

$$V((p,q)-kite) = \{y_i : 1 \leq i \leq p\} \cup \{x_j : 1 \leq j \leq q\}$$

and

$$E((p,q)-kite) = \{y_i y_{i+1} : 1 \leq i \leq p-1\} \cup \{y_1 x_1, y_1 y_p\} \cup \{x_j y_{j+1} : 1 \leq j \leq q-1\}.$$

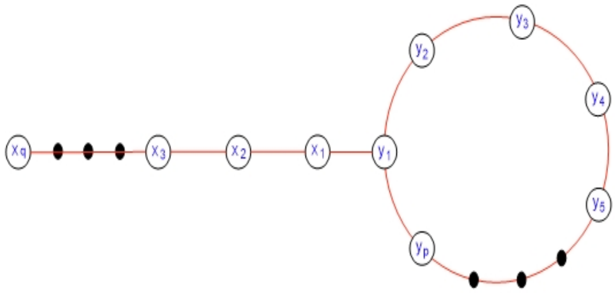

Figure 7. (p, q)-Kite

The vertices of cycle C_p are represented by y_i where $1 \leq i \leq p$ and the vertices of path P_q where $1 \leq j \leq q$ are represented by x_j.

The (p, q) kite graph accepts d-lucky labeling, as shown in the next theorem and $\eta_{dl}((p,q)-kite) = 2$.

Theorem. *A* (p, q)*−kite accepts* d-*lucky labeling with* $\eta_{dl}((p,q)-kite) = 2$.

Proof: Consider $\lambda : V(G) \rightarrow \{1, 2, 3, \ldots\}$ as labeling of graph 's vertices. The weight for vertex x is specified as

$$\omega(\mathrm{x}) = \deg(\mathrm{x}) + \sum_{y \in N(\mathrm{x})} (\lambda(\mathrm{y}))$$

where deg(x) shows the vertex x' s degree, N(x) shows the u 's open neighborhood and $\lambda(y)$ shows the label for vertex y. The proof s divided into 3 separate cases for $(p, q)-$kite graph.

Case. 1**:** For $p = 3$, $(3, q)$−kite $\forall\ q \geq 1$.

Consider y_1 as the first node in cycle C_3 with a tail, so $deg(y_1) = 3$ and $\lambda(y_1) = 1$. y_2 is a node with $deg(y_2) = 2$, $\lambda(y_2) = 2$ and y_3 is a node with $deg(y_3) = 2$, $\lambda(y_3) = 1$. If the path's first vertex is labeled with 1 than

$$w(y_1) = deg(y_1) + \sum \lambda(N(y_1)) = 3 + 1 + 2 + 1 = 7$$

and

$$w(y_2) = deg(y_2) + \sum \lambda(N(y_2)) = 2 + 1 + 1 = 4$$

$$w(y_3) = deg(y_3) + \sum \lambda(N(y_3)) = 2 + 1 + 2 = 5$$

Consider vertex x_1 as the first vertex of path P_q that is attached to vertex y_1 of cycle C_3. So x_1 has $deg(x_1) = 2,$ $\lambda(x_1) = 1$ and each $N(x_1)$ abeled as 1. So

$$w(x_1) = deg(x_1) + \sum \lambda(N(x_1)) = 2 + 1 + 1 = 4$$

The node x_2 of P_q has $deg(x_2) = 2$ and each $N(x_2)$ labeled as 1 and 2. So

$$w(x_2) = deg(x_2) + \sum \lambda(N(x_2)) = 2 + 2 + 1 = 5$$

After assigning the labels to x_1, x_2 *and* x_3 of P_q, each node x_j where $1 \leq j \leq q$ is labeled as 1 i.e. $\lambda(x_j) = 1$ with exception of nodes that located at locations $4k + 3$ where $k = 1, 2, ..., h : 4h + 3 \leq q$ are labeled with 2. The nodes x_j where $1 \leq j \leq q$ which are located at locations at even locations of path P_q have

$$w(x_j) = deg(x_j) + \sum \lambda(N(x_j)) = 2 + 1 + 2 = 5$$

Likewise, the nodes x_j where $1 \leq j \leq q$ that lie at odd locations of path P_q have

$$w(x_j) = deg(x_j) + \sum \lambda(N(x_j)) = 2 + 1 + 1 = 4$$

If end vertex x_q path P_q lies at position $4a$ where $a \geq 1$ then weight

$$w(x_q) = deg(x_q) + \sum \lambda(N(x_q)) = 1 + 2 = 3$$

otherwise

$$w(x_q) = deg(x_q) + \sum \lambda(N(x_q)) = 1 + 1 = 2.$$

Case 2: For $p > 3$, $p \neq 4k - 1$ where $k \in N$ and $\forall\ q \geq 1$.

Each node y_i where $1 \leq i \leq p$ of C_p which located at location $4a$ where $a = 1, 2, ..., l \mid 4l \leq p$ is labeled with 2 and remaining nodes are labeled with 1. As a result, nodes which located at even locations of cycle C_p have

$$w(y_i) = deg(y_i) + \sum \lambda(N(y_i)) = 2 + 1 + 1 = 4$$

Likewise the vertices that lies at odd positions of cycle C_p have weights

$$w(y_i) = deg(y_i) + \sum \lambda(N(y_i)) = 2 + 1 + 2 = 5.$$

If p is a multiple of 4 then the first vertex y_1 has weight

$$w(y_1) = deg(y_1) + \sum \lambda(N(y_1)) = 3 + 1 + 1 + 2 = 7$$

otherwise weight

$$w(y_1) = deg(y_1) + \sum \lambda(N(y_1)) = 3 + 1 + 1 + 1 = 6$$

Now consider the vertex x_1 as the first vertex of path P_q that is adjacent to vertex y_1 of cycle C_p. The vertex x_1 has $deg(x_1) = 2$, $\lambda(x_1) = 1$ and each $N(x_1)$ is labeled with 1. As a result, weight

$$w(x_1) = \text{deg}(x_1) + \sum \lambda(N(x_1)) = 2 + 1 + 1 = 4.$$

The node x_2 of P_q has $deg(x_2) = 2$ and $N(x_2)$ are labeled as 1 and 2. So,

$$w(x_2) = \text{deg}(x_2) + \sum \lambda(N(x_2)) = 2 + 1 + 2 = 5.$$

After assigning the labels to x_1, x_2 and x_3 of path P_q every node x_j where $1 \leq j \leq q$ has $\lambda(x_j) = 1$ except nodes that located at positions $4k + 3$ where $k = 1, 2, ..., h : 4h + 3 \leq q$ are labeled with 2. Vertices x_j where $1 \leq j \leq q$ that lies at even positions of path P_q have weights

$$w(x_j) = deg(x_j) + \sum \lambda(N(x_j)) = 2 + 1 + 2 = 5.$$

Likewise, the vertices x_j where $1 \leq j \leq q$ that lies at even positions of path P_q have weights

$$w(x_j) = deg(x_j) + \sum \lambda(N(x_j)) = 2 + 1 + 1 = 4.$$

If the end vertex x_q of path P_q located at location $4a$ where $a \geq 1$ then

$$w(x_q) = deg(x_q) + \sum \lambda(N(x_q)) = 1 + 2 = 3$$

Otherwise weight

$$w(x_q) = deg(x_q) + \sum \lambda(N(x_q)) = 1 + 1 = 2$$

Case 3**:** when $p > 3$ and $p = 4k - 1$ where $k \geq 1$ and $\forall\ q \geq 1$.

Consider y_1 as the first node of C_p with a tail. Each node y_i where $1 \leq i \leq p$ which located at location $4a$ where $a = 1, 2, ..., l : 4l \leq p$ and node which located at location $C_p - 1$ are labeled with 2 i.e. $\lambda(y_i) = 2$ and remaining nodes are labeled with 1. The vertices which located at even locations of cycle C_p have

$$w(y_i) = deg(y_i) + \sum \lambda(N(y_i)) = 2 + 1 + 1 = 4.$$

The vertex y_{p-2} has weight

$$w(y_{p-2}) = deg(y_{p-2}) + \sum \lambda(N(y_{p-2})) = 2 + 2 + 2 = 6.$$

Vertex y_p has

$$w(y_p) = deg(y_p) + \sum \lambda(N(y_p)) = 2 + 1 + 2 = 5.$$

The vertices which located at odd locations of cycle C_p have

$$w(y_i) = deg(y_i) + \sum \lambda(N(y_i))$$
$$= 2+1+2 = 5.$$

The first vertex y_1 has weight

$$w(y_1) = deg(y_1) + \sum \lambda(N(y_1)) = 3+1+1+1 = 6.$$

The path P_q is labeled as described in Case 2.

It's simple to check that the weights of all neighboring nodes are different and brings the evidence to a close.

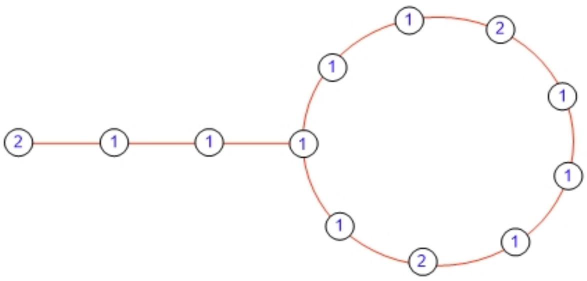

Figure 8. *d* -lucky labeling of (9,3)-Kite

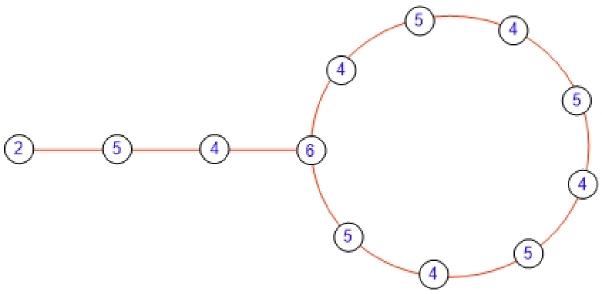

Figure 9. Weights of (9,3)-Kite

5. The Complete Binary Tree *CBT* (*l*)

According to [10], a binary tree has precisely a single vertex (the root node) with degree 2 and each of remaining nodes has degree 1 or 3. A l level complete binary tree *CBT* (l) having depth l is a binary tree where every internal node (except root node) has a degree of three and all the pendent nodes are at level l.

A l level complete binary tree *CBT* (l), $\forall l \geq 1$ containing $2^{l+1}-1$ vertices and $2^{l+1}-2$ edges. The vertex and edge collections of l level complete binary tree *CBT* (l) are listed as below.

$$V\left(CBT\left(l\right)\right) = \{x_0\} \cup \{x_{p,q} : 1 \leq p \leq l, 1 \leq q \leq 2^p\}$$

and

$$E\left(CBT\left(l\right)\right) = \{x_0 x_{1,1}, x_0 x_{1,2}\} \cup$$
$$\{x_{p,q}x_{p+1,2q-1}, x_{p,q}x_{p+1,2q} : 1 \leq p \leq l-1, 1 \leq q \leq 2^p\}.$$

The root vertex at level 0 is represented by x_0. The vertices at level p are represented by $x_{p,q}$ where $1 \leq p \leq l, 1 \leq q \leq 2^p$.

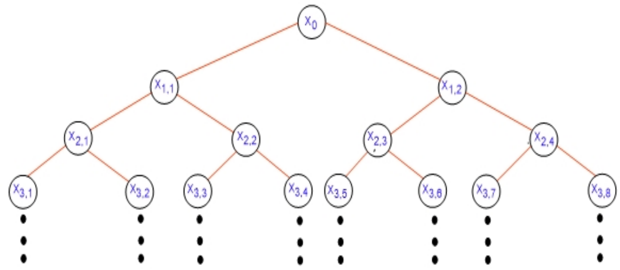

Figure 10. Complete Binary Tree *CBT* (l)

The l level complete binary tree *CBT* (l) $\forall$ $l \geq 1$ accepts d-lucky labeling, as shown in the next theorem and $\eta_{dl}(CBT(l)) = 2$.

Theorem 5.1. A complete binary tree of level l accepts d−lucky labeling with $\eta_{dl}(CBT(l)) = 2$.

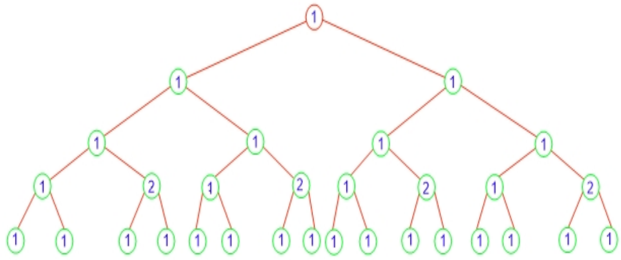

Figure 11. *d* -lucky labeling of *CBT* (4)

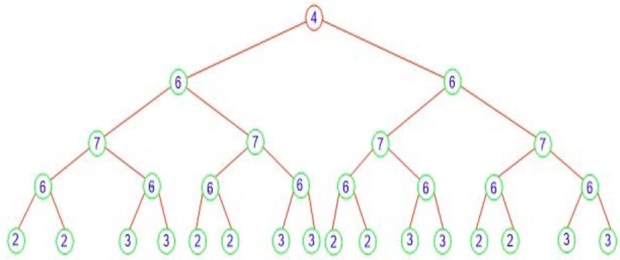

Figure 12. Weights of *CBT* (4)

Proof: Consider $\lambda : V(G) \rightarrow \{1, 2, 3, \ldots\}$ as labeling of graph 's vertices. The weight for vertex x is specified as

$$\omega(\mathrm{x}) = \deg(\mathrm{x}) + \sum_{y \in N(x)} (\lambda(\mathrm{y}))$$

where $\deg(\mathrm{x})$ shows the vertex x' s degree, $N(\mathrm{x})$ shows the u 's open neighborhood and $\lambda(y)$ shows the label for vertex y. Assume vertex x_0 as root vertex at level 0 with two adjacent nodes at level 1. As a result, $deg(x_0) = 2$ with every $N(x_0)$ is labeled with 1. So weight

$$w(x_0) \ = deg(x_0) + \sum \lambda(N(x_0)) = 2+1+1 = 4.$$

Except root vertex (vertex at level 0) all other vertices have degree 1 or 3.

Case 1: Vertices with a degree of 3.

The odd-level vertices of degree 3.

Let $x_{p,q}$ lies at level p with $deg(x_{p,q})$=3 and every $N(x_{p,q})$ at levels $p-1$ and $p+1$ are labeled with 1. So weight

$$w\left(x_{p,q}\right) = deg\left(x_{p,q}\right)+\sum\lambda\left(N\left(x_{p,q}\right)\right)$$
$$=3+1+1+1=6.$$

The even-level vertices of degree 3.

Let $x_{p,q}$ lies at level p with $deg(x_{p,q})=3$, there is also a single $N(x_{p,q})$ with the label 2 and the other two vertices with the label 1. So

$$w\left(x_{p,q}\right) = deg\left(x_{p,q}\right)+\sum\lambda\left(N\left(x_{p,q}\right)\right)$$
$$=3+2+1+1=7.$$

Case 2: Vertices with a degree of 1**.**

The node $x_{l,q}$ lies at level l with $deg\left(x_{l,q}\right)$ = 1. If $N\left(x_{l,q}\right)$ is labeled with 1 then

$$w\left(x_{l,q}\right) = deg\left(x_{l,q}\right)+\lambda\left(N\left(x_{l,q}\right)\right)=1+1=2.$$

Otherwise $N\left(x_{l,q}\right)$ is labeled with 2 having

$$w(x_{l,q})=deg\left(x_{l,q}\right)+\lambda\left(N\left(x_{l,q}\right)\right)=1+2=3.$$

It's simple to check that the weights of all neighboring nodes are different and brings the evidence to a close.

6. Generalized Theta Graph $\theta_g(L^{(t)})$

Each theta graph in $\theta_g\left(L^{(t)}\right)$ consists of t internally disjoint paths $l_{a,c}$ and each one has t + 2 vertices. Each path in each theta has a specific common node c_n where $1 \le n \le g$ and these internally disjoint paths have a common end vertex c_0.

A generalized theta graph $\theta_g\left(L^{(t)}\right)$ containing $p(t(l-1)+1)+1$ vertices and gtl edges. The vertex and edge collections for $\theta_g\left(L^{(t)}\right)$ are listed below.

$$V\left(\theta_g\left(L^{(t)}\right)\right)=\{x_{a,b,c} : 1\le a\le t : 1\le c\le g : 1\le b\le h$$
$$\text{where } h=1,2,...,l-1\}\cup\{c_0, c_n : 1\le n\le g\}$$

and

$$E\left(\theta_g\left(L^{(t)}\right)\right)$$
$$=\{x_{a,b,c}x_{a,b+1,c} : 1\le b\le l-1, 1\le a\le t, 1\le c\le g\}$$
$$\{c_0x_{a,1,c} : 1\le a\le t, 1\le c\le g\}$$
$$\cup\{c_nx_{a,l-1,k} : 1\le a\le t, 1\le c\le g\}.$$

The generalized theta graph $\theta_g\left(L^{(t)}\right)$ accepts d-lucky labeling, as shown in the next theorem with $\eta_{dl}\left(\theta_g\left(L^{(t)}\right)\right)=2.$

Theorem 6.1. A generalized theta graph accepts d-lucky labeling with $\eta_{dl}\left(\theta_g\left(L^{(t)}\right)\right)=2.$

Proof: Consider $\lambda : V(G)\rightarrow\{1, 2, 3,\dots\}$ as labeling of graph's vertices. The weight for vertex x is specified as

$$\omega(\mathrm{x})=\deg(\mathrm{x})+\sum_{y\in N(\mathrm{x})}(\lambda(\mathrm{y}))$$

where deg(x) shows the vertex x' s degree, $N(\mathrm{x})$ shows the u 's open neighborhood and $\lambda(y)$ shows the label for vertex y.

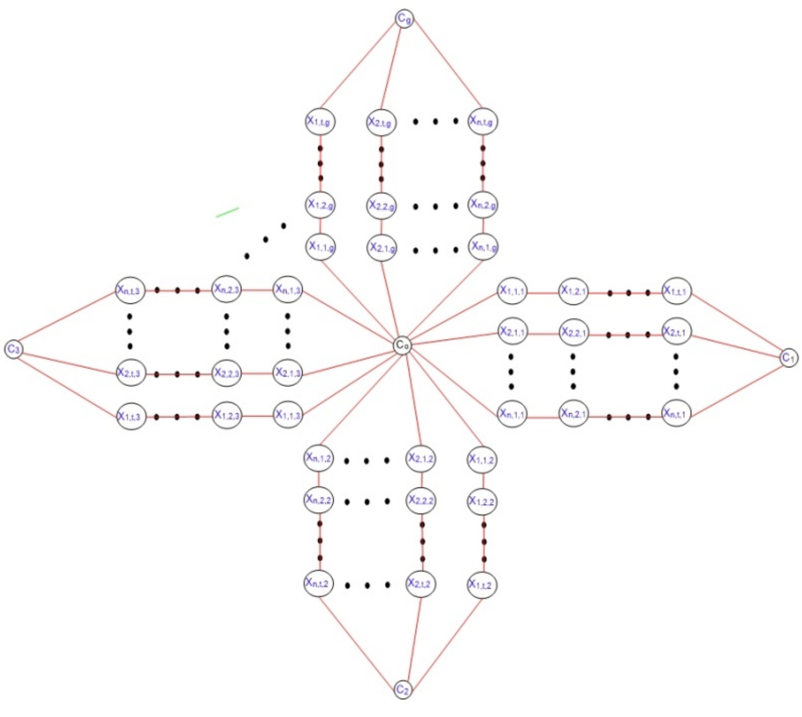

Figure 13. Generalized theta graph $\theta_g(L^{(t)})$

Each path $l_{a,c}$ which starting at node c_0 and ending at node c_g has $\lambda(c_o)=1$ and $\lambda(x_{a,b,c})=1$ where a = 1, 2, ..., t, , b = 1, 2, ..., t , c = 1, 2, ..., g , with exception of nodes that located at locations $4k$ where $4k \leq t$, are labeled with 2. If last node c_g is located at location 4 k than $\lambda(c_g)=2$ otherwise $\lambda(c_g)=1$. Common starting node c_0 of every path $l_{a,c}$ has

$$w(c_0)=deg(c_o)+\sum\lambda(N(c_0))=2gt.$$

The nodes $x_{a,b,c}$ of path $l_{a,c}$ which placed at even locations of path (except ending node c_g) have

$$w(x_{a,b,c})=deg(x_{a,b,c})+\sum\lambda(N(x_{a,b,c}))$$
$$=2+1+1=4$$

The nodes $x_{a,b,c}$ of path $l_{a,c}$ which placed at odd locations of path (except ending node c_g) have weights

$$w(x_{a,b,c})=deg(x_{a,b,c})+\sum\lambda(N(x_{a,b,c}))$$
$$=2+1+2=5.$$

If last node c_g of path $l_{a,c}$ placed at location $4k$ + 1 than

$$w(c_g)=deg(c_g)+\sum\lambda(N(c_g))=3t.$$

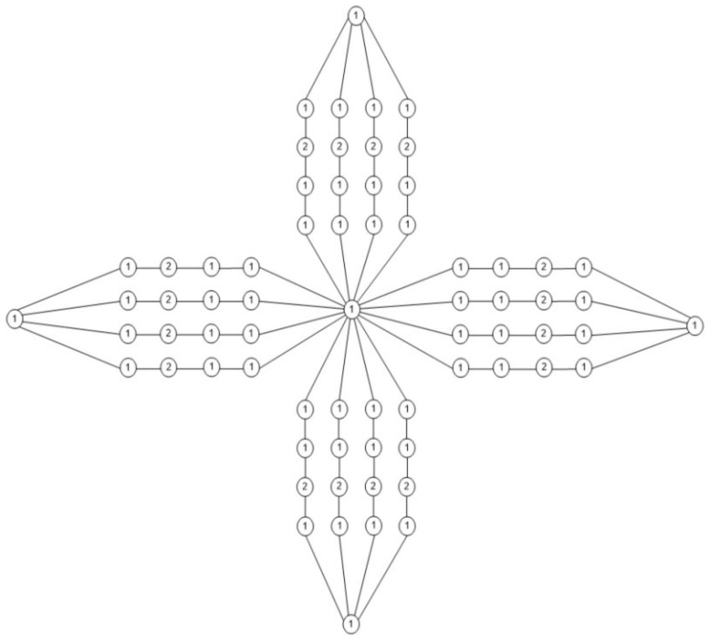

Figure 14. d -lucky labeling of $\theta_4(5^4)$

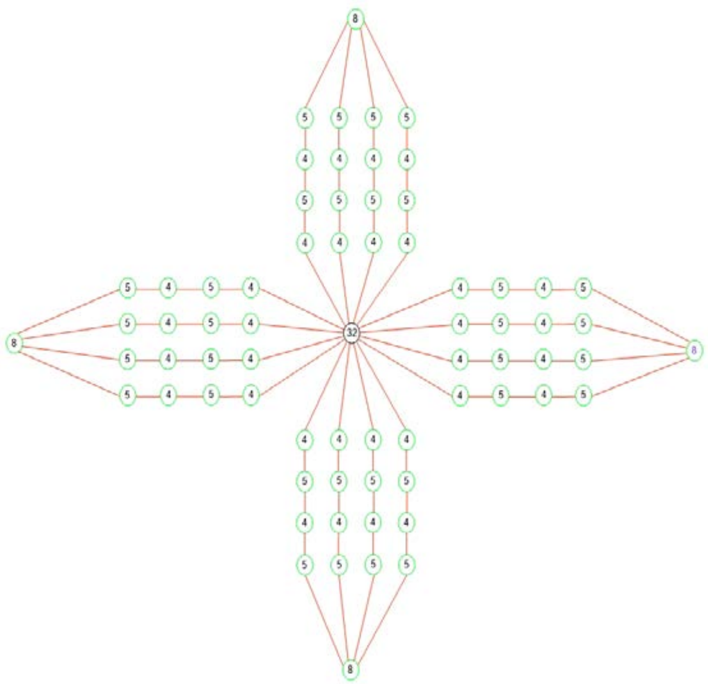

Figure 15. Weights of $\theta_4(5^4)$

Otherwise weight,

$$w\left(c_g\right) = deg\left(c_g\right) + \sum \lambda\left(N\left(c_g\right)\right) = 2t.$$

It's simple to check that the weights of all neighboring nodes are different and brings the evidence to a close.

7. Conclusion

In this paper we have used a newly defined labeling called *d*-lucky labeling and a graph that satisfies *d*-lucky labeling called *d*-lucky graph. *d*-lucky labeling of some special classes of graphs like jelly fish graph, coconut tree graph, kite graph, complete binary tree graph, and generalized theta graph are investigated.

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