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The Fractional Sub-Equation Method and Exact Analytical Solutions for Some Nonlinear Fractional PDEs

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Abstract In the present paper, a fractional sub-equation method is proposed to solve fractional differential equations. Being concise and straightforward, this method is applied the space–time fractional Potential Kadomtsev–Petviashvili (PKP) equation and the space–time fractional Symmetric Regularized Long Wave (SRLW) equation. As a result, many exact analytical solutions are obtained including hyperbolic function solutions, trigonometric function solutions, and rational solutions. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear fractional PDEs arising in mathematical physics.

Keywords: *fractional sub-equation method, fractional differential equation, modified Riemann–Liouville derivative, Mittag-Leffler function, analytical solutions*

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order .In recent years, nonlinear fractional differential equations (FDEs) have been attracted great interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering, and biology [1,2,3,4,5,6,7]. For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, the solution of fractional differential equation [8,9,10,11,12,13,14,15] is much involved. In the past, many analytical and numerical methods have been proposed to obtain solutions of nonlinear FDEs, such as finite difference method [16], finite element method [17], differential transform method [18,19], Adomian decomposition method [20,21,22], variational iteration method [23,24,25], homotopy perturbation method [26,27,28] and so on. The fractional differential equations are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification and other areas. Many articles have investigated some aspects of fractional differential equations, such as the existence and uniqueness of solutions to Cauchy type problems, the methods for explicit and numerical solutions, and the stability of solutions [29,30]. Among the investigations for fractional differential equations, research for seeking exact solutions and numerical solutions of fractional differential equations is an important topic, which can also provide valuable reference for other related research.

Recently, Zhang and Zhang [31] introduced a new method called fractional sub-equation method to look for traveling wave solutions of nonlinear FDEs. The method is based on the homogeneous balance principle [32] and Jumarie's modified Riemann-Liouville derivative [33,34]. By using fractional sub-equation method, Zhang et al. successfully obtained traveling wave solutions of nonlinear time fractional biological population model and (4 + 1)-dimensional space-time fractional Fokas equation. More recently, Guo et al. [35] and Lu [36] improved Zhang et al.'s work [31] and obtained exact solutions of some nonlinear FDEs.

In this paper, we will apply the fractional sub-equation method [31] for solving fractional PDEs in the sense of modified Riemann–Liouville derivative by Jumarie [33]. To illustrate the validity and advantages of the method, we will apply it to the space-time fractional PKP equation and the space-time fractional SRLW equation.

The rest of this paper is organized as follows. In Section 2, we will describe the Modified Riemann-Liouville derivative and give the main steps of the method here. In Section 3, we give two applications of the proposed method to nonlinear equations. In Section 4, some conclusions are given.

2. Description of Modified Riemann-Liouville Derivative and the Proposed Method

The Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression [33]

$$D_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\int_0^x (x-\xi)^{-\alpha-1}(f(\xi)-f(0))d\xi, & \alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\int_0^x (x-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1,\\ \left[f^{(\alpha-n)}(x)\right]^{(n)} & n\le\alpha<n+1,\ n\ge1.\end{cases} \tag{1}$$

Some properties for the proposed modified Riemann–Liouville derivative are listed in [33] as follows:

$$D_x^\alpha x^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}x^{\gamma-\alpha},\quad \gamma>0, \tag{2}$$

$$D_x^\alpha(f(x)g(x))=g(x)D_x^\alpha f(x)+f(x)D_x^\alpha g(x), \tag{3}$$

$$D_x^\alpha f\left[g(x)\right]=f_g'\left[g(x)\right]D_x^\alpha g(x)=D_g^\alpha f\left[g(x)\right](g'(x))^\alpha, \tag{4}$$

The above equations play an important role in fractional calculus in the following sections.
We present the main steps of the fractional sub-equation method as follows.

Setp1. Suppose that a nonlinear FDEs, say in two independent variables x and t, is given by

$$P(u,u_x,u_t,D_x^\alpha u,D_y^\alpha u,D_t^\alpha u,\ldots)=0,\quad 0<\alpha\le1, \tag{5}$$

where $D_x^\alpha u, D_y^\alpha u$ and $D_t^\alpha u$ are Jumarie's modified Riemann–Liouville derivatives of u, $u=(x,r,t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

Setp2. By using the traveling wave transformation:

$$u(x,y,t)=u(\xi),\quad \xi=kx+ly+ct, \tag{6}$$

where k, l and c are constants to be determined later, the FDE (5) is reduced to the following nonlinear fractional ordinary differential equation (ODE) for $u=u(\xi)$:

$$P(u,ku',cu',k^\alpha D_\xi^\alpha u,l^\alpha D_\xi^\alpha u,c^\alpha D_\xi^\alpha u,\ldots)=0. \tag{7}$$

Setp3. We suppose that Eq. (7) has the following solution:

$$u(\xi)=\sum_{i=0}^{n}a_i\varphi^i, \tag{8}$$

where $a_i\,(i=0,1,2,\ldots,n)$ are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (5) or Eq. (7) (see Ref.[37] for details), and $\varphi=\varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha\varphi=\sigma+\varphi^2, \tag{9}$$

where σ is a constant. By using the generalized Exp-function method via Mittag-Leffler functions [38], Zhang et al. first obtained the following solutions of fractional Riccati equation (9)

$$\varphi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh_\alpha(\sqrt{-\sigma}\,\xi), & \sigma<0,\\ -\sqrt{-\sigma}\coth_\alpha(\sqrt{-\sigma}\,\xi), & \sigma<0,\\ \sqrt{\sigma}\tan_\alpha(\sqrt{\sigma}\,\xi), & \sigma>0,\\ -\sqrt{\sigma}\cot_\alpha(\sqrt{\sigma}\,\xi), & \sigma>0,\\ -\dfrac{\Gamma(1+\alpha)}{\xi^\alpha+\omega},\quad \omega=\text{const.}, & \sigma=0,\end{cases} \tag{10}$$

where the generalized hyperbolic and trigonometric functions are defined as

$$\sinh_\alpha(x)=\frac{E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)}{2},\quad \cosh_\alpha(x)=\frac{E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)}{2},\quad \tanh_\alpha(x)=\frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},$$

$$\coth_\alpha(x)=\frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},\quad \sin_\alpha(x)=\frac{E_\alpha(ix^\alpha)-E_\alpha(-ix^\alpha)}{2i},\quad \cos_\alpha(x)=\frac{E_\alpha(ix^\alpha)+E_\alpha(-ix^\alpha)}{2},$$

$$\tan_\alpha(x)=\frac{\sin_\alpha(x)}{\cos_\alpha(x)},\quad \cot_\alpha(x)=\frac{\cos_\alpha(x)}{\sin_\alpha(x)},$$

where $E_\alpha(z)$ denotes the Mittag-Leffler function, given as

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}.$$

Setp4. Substituting Eq. (8) along with Eq. (9) into Eq. (7) and using the properties of Jumarie's modified Riemann–Liouville derivative (2)–(4), we can get a polynomial in $\varphi(\xi)$. Setting all the coefficients of $\varphi^m(m=0,1,2,...)$ to zero, yields a set of overdetermined nonlinear algebraic equations for $a_i\,(i=0,1,2,...,n),k,l$ and c.

Setp5. Assuming that the constants $a_i\,(i=0,1,2,...,n),k,l$ and c can be obtained by solving the algebraic equations in Step 4, substituting these constants and the solutions of Eq.(9) into Eq.(8), we can obtain the explicit solutions of Eq.(5) immediately.

Remark: If $\alpha\to1$, the Riccati equation become $\varphi'(\xi)=\sigma+\varphi^2(\xi)$ used in [39]. So the method in this example can be used to solve integer-order differential equations. In this sense, we would like to conclude that our method includes the existing tanh-function method as special case.

3. Applications of the Method

In this section, we apply the fractional sub-equation method to construct the exact analytical solutions for some nonlinear fractional PDEs, namely the space–time fractional PKP equation and the space–time fractional SRLW equation which are very important nonlinear fractional PDEs in mathematical physics and have been paid attention by many researchers.

3.1. Example 1. The Space–time Fractional PKP Equation

We first consider the space-time fractional PKP equation [40] in the form:

$$\frac{1}{4}D_x^{4\alpha}u+\frac{3}{2}D_x^{\alpha}u\,D_x^{2\alpha}u+\frac{3}{4}D_y^{2\alpha}u+D_t^{\alpha}(D_x^{\alpha}u)=0, \tag{11}$$

By considering the traveling wave transformation $u=u(\xi),\ \xi=kx+ly+ct$, Eq.(11) can be reduced to the following nonlinear fractional ODE:

$$\frac{1}{4}k^{4\alpha}D_\xi^{4\alpha}u+\frac{3}{2}k^{3\alpha}D_\xi^{\alpha}u\,D_\xi^{2\alpha}u+\frac{3}{4}l^{2\alpha}D_\xi^{2\alpha}u+k^\alpha c^\alpha D_\xi^{2\alpha}u=0. \tag{12}$$

By balancing the highest order derivative terms and nonlinear terms in Eq. (12), we suppose that Eq. (12) has the following formal solution:

$$u(\xi)=a_0+a_1\varphi(\xi), \tag{13}$$

where $\varphi(\xi)$ satisfies Eq. (9).

Substituting Eq.(13) along with Eq.(9) into Eq.(12) and then setting the coefficients of $\varphi^i\,(i=1,3,5)$ to zero, we can obtain a set of algebraic equations for k,l,c,a_o,a_1 as follows:

$$\begin{aligned}
&\varphi^1: 4k^{4\alpha}a_1\sigma^2+\frac{3}{2}a_1 l^{2\alpha}\sigma+3k^{3\alpha}a_1^2\sigma^2+2k^{\alpha}c^{\alpha}a_1\sigma=0,\\
&\varphi^3: 10k^{4\alpha}a_1\sigma+2k^{\alpha}c^{\alpha}a_1+\frac{3}{2}l^{2\alpha}a_1+6k^{3\alpha}a_1^2\sigma=0,\\
&\varphi^5: 6k^{4\alpha}a_1+3k^{3\alpha}a_1^2=0.
\end{aligned}\tag{14}$$

Solving the algebraic equations(14) by Maple or Mathematica, we have:

$$a_0=a_0,\qquad a_1=-2k^{\alpha},\qquad \sigma=\frac{3l^{2\alpha}+4k^{\alpha}c^{\alpha}}{4k^{4\alpha}}.\tag{15}$$

where a_0 is an arbitrary constant.

We, therefore, obtain from Eqs. (10), (13) and (15) three types of exact solutions of Eq.(11), namely, two generalized hyperbolic function solutions, two generalized trigonometric function solutions and one rational solution as follows:

$$u=a_0+\frac{\sqrt{-(3l^{2\alpha}+4k^{\alpha}c^{\alpha})}}{k^{\alpha}}\tanh_{\alpha}\left[\frac{\sqrt{-(3l^{2\alpha}+4k^{\alpha}c^{\alpha})}}{2k^{2\alpha}}(kx+ly+ct)\right],\quad 3l^{2\alpha}+4k^{\alpha}c^{\alpha}<0,\quad k^{4\alpha}>0,\tag{16}$$

$$u=a_0+\frac{\sqrt{-(3l^{2\alpha}+4k^{\alpha}c^{\alpha})}}{k^{\alpha}}\coth_{\alpha}\left[\frac{\sqrt{-(3l^{2\alpha}+4k^{\alpha}c^{\alpha})}}{2k^{2\alpha}}(kx+ly+ct)\right],\quad 3l^{2\alpha}+4k^{\alpha}c^{\alpha}<0,\quad k^{4\alpha}>0,\tag{17}$$

$$u=a_0-\frac{\sqrt{3l^{2\alpha}+4k^{\alpha}c^{\alpha}}}{k^{\alpha}}\tan_{\alpha}\left[\frac{\sqrt{3l^{2\alpha}+4k^{\alpha}c^{\alpha}}}{2k^{2\alpha}}(kx+ly+ct)\right],\quad 3l^{2\alpha}+4k^{\alpha}c^{\alpha}>0,\quad k^{4\alpha}>0,\tag{18}$$

$$u=a_0+\frac{\sqrt{3l^{2\alpha}+4k^{\alpha}c^{\alpha}}}{k^{\alpha}}\cot_{\alpha}\left[\frac{\sqrt{3l^{2\alpha}+4k^{\alpha}c^{\alpha}}}{2k^{2\alpha}}(kx+ly+ct)\right],\quad 3l^{2\alpha}+4k^{\alpha}c^{\alpha}>0,\quad k^{4\alpha}>0,\tag{19}$$

$$u=a_0+\frac{2k^{\alpha}\Gamma(1+\alpha)}{(kx+ly+ct)^{\alpha}+\omega},\quad 3l^{2\alpha}+4k^{\alpha}c^{\alpha}=0,\quad \omega=\text{const.}\tag{20}$$

As $\alpha\to 1$, these obtained exact solutions give the ones of the standard form equation of the space-time fractional PKP equation (11).

3.2. Example 2. The Space-time Fractional SRLW Equation

We next consider the following space-time fractional SRLW equation [41]

$$D_t^{2\alpha}u+D_x^{2\alpha}u+u\,D_t^{\alpha}(D_x^{\alpha}u)+D_x^{\alpha}u\,D_t^{\alpha}u+D_t^{2\alpha}(D_x^{2\alpha}u)=0,\tag{21}$$

which arises in severed physical applications including ion sound waves in plasma. Using the traveling wave transformation $u=u(\xi),\ \xi=kx+ct$, Eq.(21) can be reduced to the following nonlinear fractional ODE:

$$(c^{2\alpha}+k^{2\alpha})D_{\xi}^{2\alpha}u+k^{\alpha}c^{\alpha}u\,D_{\xi}^{2\alpha}u+k^{\alpha}c^{\alpha}(D_{\xi}^{\alpha}u)^2+k^{2\alpha}c^{2\alpha}D_{\xi}^{4\alpha}u=0.\tag{22}$$

According to the method described in Section 2, we suppose that Eq.(22) has the following formal solution:

$$u(\xi)=a_0+a_1\varphi(\xi)+a_2\varphi^2(\xi),\tag{23}$$

where $\varphi(\xi)$ satisfies Eq. (9).

Substituting Eq.(23) along with Eq.(9) into Eq.(22) and collect the coefficients of $\varphi^i\ (i=0,1,2,3,4,5,6)$ and set them to be zero, a set of algebraic equations are obtained as follows:

$$\begin{aligned}
&\varphi^0 : 2k^{2\alpha} a_2 \sigma^2 + k^{\alpha} c^{\alpha} a_1^2 \sigma^2 + 2c^{2\alpha} a_2 \sigma^2 + 2c^{\alpha} k^{\alpha} a_0 a_2 \sigma^2 + 16k^{2\alpha} c^{2\alpha} a_2 \sigma^3 = 0,\\
&\varphi^1 : 2a_0 a_1 k^{\alpha} c^{\alpha} \sigma + 2a_1 k^{2\alpha} \sigma + 2a_1 c^{2\alpha} \sigma + 6a_1 a_2 k^{\alpha} c^{\alpha} \sigma^2 + 16a_1 c^{2\alpha} k^{2\alpha} \sigma^2 = 0,\\
&\varphi^2 : 6a_2^2 k^{\alpha} c^{\alpha} \sigma^2 + 8a_2 k^{2\alpha} \sigma + 8a_2 c^{2\alpha} \sigma + 136k^{2\alpha} c^{2\alpha} a_2 \sigma^2 + 4k^{\alpha} c^{\alpha} a_1^2 \sigma + 8a_0 a_2 k^{\alpha} c^{\alpha} \sigma = 0,\\
&\varphi^3 : 2k^{2\alpha} a_1 + 18k^{\alpha} c^{\alpha} a_1 a_2 \sigma + 2a_0 a_1 k^{\alpha} c^{\alpha} + 40k^{2\alpha} c^{2\alpha} a_1 \sigma + 2c^{2\alpha} a_1 = 0,\\
&\varphi^4 : 6a_0 a_2 k^{\alpha} c^{\alpha} + 3a_1^2 k^{\alpha} c^{\alpha} + 6a_2 k^{2\alpha} + 16a_2^2 k^{\alpha} c^{\alpha} \sigma + 6a_2 c^{2\alpha} + 240k^{2\alpha} c^{2\alpha} a_2 \sigma = 0,\\
&\varphi^5 : 12a_1 a_2 k^{\alpha} c^{\alpha} + 24k^{2\alpha} c^{2\alpha} a_1 = 0,\\
&\varphi^6 : 120k^{2\alpha} c^{2\alpha} a_2 + 10a_2^2 k^{\alpha} c^{\alpha} = 0.
\end{aligned} \tag{24}$$

Solving the set of algebraic equations(24) by Maple or Mathematica yields

$$a_0 = -c^{-\alpha} k^{\alpha} - c^{\alpha}(k^{-\alpha} + 8k^{\alpha}\sigma), \qquad a_1 = 0, \qquad a_2 = -12c^{\alpha} k^{\alpha}. \tag{25}$$

Finally, from Eqs.(10), (23) and (25) we obtain the following generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solution of Eq.(21)

$$u = -c^{-\alpha} k^{\alpha} - c^{\alpha}(k^{-\alpha} + 8k^{\alpha}\sigma) + 12c^{\alpha} k^{\alpha}\sigma \tanh_{\alpha}^2(\sqrt{-\sigma}\xi), \quad \sigma < 0, \tag{26}$$

$$u = -c^{-\alpha} k^{\alpha} - c^{\alpha}(k^{-\alpha} + 8k^{\alpha}\sigma) + 12c^{\alpha} k^{\alpha}\sigma \coth_{\alpha}^2(\sqrt{-\sigma}\xi), \quad \sigma < 0, \tag{27}$$

$$u = -c^{-\alpha} k^{\alpha} - c^{\alpha}(k^{-\alpha} + 8k^{\alpha}\sigma) - 12c^{\alpha} k^{\alpha}\sigma \tan_{\alpha}^2(\sqrt{\sigma}\xi), \quad \sigma > 0, \tag{28}$$

$$u = -c^{-\alpha} k^{\alpha} - c^{\alpha}(k^{-\alpha} + 8k^{\alpha}\sigma) - 12c^{\alpha} k^{\alpha}\sigma \cot_{\alpha}^2(\sqrt{\sigma}\xi), \quad \sigma > 0, \tag{29}$$

$$u = -c^{-\alpha} k^{\alpha} - c^{\alpha} k^{-\alpha} - \frac{12c^{\alpha} k^{\alpha}\Gamma^2(1+\alpha)}{(\xi^{\alpha} + \omega)^2}, \quad \sigma = 0, \quad \omega = \text{conts}, \tag{30}$$

where $\xi = kx + ct$.

4. Conclusion

In this paper, we have seen that three types of exact analytical solutions including the generalized hyperbolic function solutions, generalized trigonometric function solutions and rational solutions for the space-time fractional KPK equation and the space-time fractional SRLW equation are successfully found out by using the fractional sub-equation method.

From our results obtained in this paper, we conclude that the fractional sub-equation method is powerful, effective and convenient for nonlinear fractional PDEs . Also, the solutions of the proposed nonlinear fractional PDEs in this paper have many potential applications in physics and engineering. Finally, this method provides a powerful mathematical tool to obtain more general exact analytical solutions of a great many nonlinear fractional PDEs in mathematical physics.

To the best of our knowledge, the solutions obtained in this paper have not been reported in literature.

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