

# Using Parseval's Theorem to Solve Some Definite Integrals

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**Abstract** This paper uses the mathematical software Maple for the auxiliary tool to study three types of definite integrals. The closed forms of these definite integrals can be obtained using Parseval's theorem. In addition, we propose some definite integrals to do calculation practically. The research methods adopted in this study is to find solutions through manual calculations and verify these solutions using Maple.

**Keywords:** *definite integrals, closed forms, Parseval's theorem, Maple*

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## 1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics, we learnt many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, etc. In this paper, we study the following three types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx \quad (1)$$

$$\int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx \quad (2)$$

$$\int_0^{2\pi} \frac{1}{1+2r \cos x+r^2} dx \quad (3)$$

where  $r$  is a real number, and  $|r| \neq 1$ . We can obtain the closed forms of these definite integrals using Parseval's theorem; this is the major result of this paper (i.e., Theorem 2.5). The study of integral problems can refer to [1,2,3]. In addition, the findings of these papers [4-30] show that the closed forms or infinite series forms of some types of integrals can be obtained using integration term by term theorem, differentiation with respect to a parameter, Parseval's theorem, etc. In this article, we provide some definite integrals to do calculation practically, and the research method adopted in this study is to find solutions through manual calculations and verifying these solutions using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations.

## 2. Preliminaries and Main Results

Firstly, we introduce a definition and some formulas used in this paper.

### 2.1. Definition

Suppose that  $f(x)$  is a continuous function defined on  $[0, 2\pi]$ . If the Fourier series expansion of  $f(x)$  is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \text{ where } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx,$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx, \text{ and } b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

for all positive integers  $k$ .

## 2.2 Formulas

### 2.2.1. Euler's Formula

$$e^{ix} = \cos x + i \sin x \quad (4)$$

where  $x$  is any real number.

### 2.2.2. DeMoivre's Formula

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx \quad (5)$$

where  $n$  is any integer, and  $x$  is any real number.

### 2.2.3.

Suppose that  $z$  is a complex number, and  $|z| \neq 1$ , then

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k \quad \text{if } |z| < 1 \quad (6)$$

$$\frac{1}{1+z} = \sum_{k=1}^{\infty} (-1)^{k-1} z^{-k} \quad \text{if } |z| > 1 \quad (7)$$

The following is an important theorem used in this paper which can be found in [31].

## 2.3. Parseval's Theorem

If  $f(x)$  is a continuous function defined on  $[0, 2\pi]$ , and  $f(0) = f(2\pi)$ . Suppose that the Fourier series expansion of  $f(x)$  is

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \text{ then}$$

$$\frac{1}{\pi} \int_0^{2\pi} f^2(x) dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2). \quad (8)$$

Before deriving the major results of this study, we need a lemma.

**Lemma 2.4** Suppose that  $r$  is a real number, and  $|r| \neq 1$ . The Fourier series expansions of the following two trigonometric functions are

$$\frac{1+r \cos x}{1+2r \cos x+r^2} = 1 + \sum_{k=1}^{\infty} (-1)^k r^k \cos kx \quad \text{if } |r| < 1 \quad (9)$$

$$\frac{1+r \cos x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^{k-1} r^{-k} \cos kx \quad \text{if } |r| > 1 \quad (10)$$

$$\frac{r \sin x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^{k+1} r^k \sin kx \quad \text{if } |r| < 1 \quad (11)$$

$$\frac{r \sin x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^k r^{-k} \sin kx \quad \text{if } |r| > 1 \quad (12)$$

**Proof** Let  $z = re^{ix}$ .

Case 1. If  $|z| < 1$  (i.e.,  $|r| < 1$ ). By Eq. (6), we obtain

$$\frac{1}{1+re^{ix}} = \sum_{k=0}^{\infty} (-1)^k (re^{ix})^k.$$

Using Euler's formula and DeMoivre's formula yields

$$\frac{(1+r \cos x) - ir \sin x}{(1+r \cos x)^2 + (r \sin x)^2} = \sum_{k=0}^{\infty} (-1)^k r^k e^{ikx} \quad (13)$$

Using the real parts of both sides of Eq. (13) are equal yields

$$\frac{r \cos x}{1+2r \cos x+r^2} = 1 + \sum_{k=1}^{\infty} (-1)^k r^k \cos kx.$$

By the equality of the imaginary parts of both sides of Eq. (13) are equal, we obtain

$$\frac{r \sin x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^{k+1} r^k \sin kx.$$

Case 2. If  $|z| > 1$ , (i.e.,  $|r| > 1$ ). Using Eq. (7) yields

$$\frac{1}{1+re^{ix}} = \sum_{k=1}^{\infty} (-1)^{k-1} (re^{ix})^{-k}$$

Therefore,

$$\frac{1+r \cos x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^{k-1} r^{-k} \cos kx$$

$$\frac{r \sin x}{1+2r \cos x+r^2} = \sum_{k=1}^{\infty} (-1)^k r^{-k} \sin kx$$

Next, we determine the closed forms of the definite integrals (1), (2), and (3).

**Theorem 2.5** Assume that  $r$  is a real number, and  $|r| \neq 1$ , then we have

$$\int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx = \frac{\pi(2-r^2)}{1-r^2} \quad \text{if } |r| < 1 \quad (14)$$

$$\int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx = \frac{\pi}{r^2-1} \quad \text{if } |r| > 1 \quad (15)$$

$$\int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx = \frac{\pi r^2}{1-r^2} \quad \text{if } |r| < 1 \quad (16)$$

$$\int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx = \frac{\pi}{r^2-1} \quad \text{if } |r| > 1 \quad (17)$$

$$\int_0^{2\pi} \frac{1}{1+2r \cos x+r^2} dx = \frac{2\pi}{1-r^2} \quad \text{if } |r| > 1 \quad (18)$$

$$\int_0^{2\pi} \frac{1}{1+2r \cos x+r^2} dx = \frac{2\pi}{r^2-1} \quad \text{if } |r| < 1 \quad (19)$$

**Proof** Case 1. If  $|r| < 1$ , then

$$\begin{aligned} & \frac{1}{\pi} \int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx \\ &= 2 + \sum_{k=1}^{\infty} r^{2k} \quad (\text{by Eqs. (8) and (9)}) \quad (20) \\ &= 2 + \frac{r^2}{1-r^2} = \frac{2-r^2}{1-r^2} \end{aligned}$$

Therefore,

$$\int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx = \frac{\pi(2-r^2)}{1-r^2}.$$

Also, using Eqs. (8) and (9) yields

$$\frac{1}{\pi} \cdot \int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx = \sum_{k=1}^{\infty} r^{2k} = \frac{r^2}{1-r^2}$$

Hence,

$$\int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx = \frac{\pi r^2}{1-r^2}.$$

In addition, from the summation of Eqs. (14) and (16), we obtain Eq. (18).

Case 2. If  $|r| > 1$ , then

$$\begin{aligned} \frac{1}{\pi} \cdot \int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx &= \sum_{k=1}^{\infty} r^{-2k} \\ (\text{by Eqs. (8) and (10)}) & \\ &= \frac{r^{-2}}{1-r^{-2}} = \frac{1}{r^2-1} \end{aligned} \quad (21)$$

Thus,

$$\int_0^{2\pi} \frac{(1+r \cos x)^2}{(1+2r \cos x+r^2)^2} dx = \frac{\pi}{r^2-1}.$$

Also, by Eqs. (8) and (10) we have

$$\int_0^{2\pi} \frac{r^2 \sin^2 x}{(1+2r \cos x+r^2)^2} dx = \frac{\pi}{r^2-1}$$

From the summation of Eqs. (15) and (17), we obtain Eq. (19).

### 3. Examples

In the following, for the three types of definite integrals in this study, we provide some definite integrals and use Theorem 2.5 to find their solutions. In addition, we employ Maple to calculate the approximations of these definite integrals and their solutions for verifying our answers.

#### 3.1. If $r = 1/4$ in Eq. (14), then the definite integral

$$\int_0^{2\pi} \frac{\left(1 + \frac{1}{4} \cos x\right)^2}{\left(\frac{17}{16} + \frac{1}{2} \cos x\right)^2} dx = \frac{31\pi}{15}. \quad (22)$$

Next, we use Maple to verify the correctness of Eq. (22).

```
>evalf(int((1+1/4*cos(x))^2/(17/16+1/2*cos(x))^2,x=0..2*Pi),18);
6.49262481741890604
>evalf(31*Pi/15,18);
6.49262481741890604
```

#### 3.2. If $r = 1/\sqrt{2}$ in Eq. (16), then

$$\int_0^{2\pi} \frac{\frac{1}{2} \sin^2 x}{\left(\frac{3}{2} + \sqrt{2} \cos x\right)^2} dx = \pi \quad (23)$$

```
>evalf(int(1/2*(sin(x))^2/(3/2+sqrt(2)*cos(x))^2,x=0..2*Pi),18);
3.14159265358979324
>evalf(Pi,18);
3.14159265358979324
```

#### 3.3. Let $r = 2/3$ in Eq. (18), we have

$$\int_0^{2\pi} \frac{1}{\frac{13}{9} + \frac{4}{3} \cos x} dx = \frac{18\pi}{5} \quad (24)$$

```
>evalf(int(1/(13/9+4/3*cos(x)),x=0..2*Pi),18);
11.3097335529232557
>evalf(18*Pi/5,18);
11.3097335529232557
```

#### 3.4. Taking $r = 2$ into Eq. (15) yields

$$\int_0^{2\pi} \frac{(1+2 \cos x)^2}{(5+4 \cos x)^2} dx = \frac{\pi}{3} \quad (25)$$

```
>evalf(int((1+2*cos(x))^2/(5+4*cos(x))^2,x=0..2*Pi),18);
1.04719755119659775
>evalf(Pi/3,18);
1.04719755119659775
```

#### 3.5. Let $r = 5$ in Eq. (17), then we obtain

$$\int_0^{2\pi} \frac{25 \sin^2 x}{(26+10 \cos x)^2} dx = \frac{\pi}{24} \quad (26)$$

```
>evalf(int(25*(sin(x))^2/(26+10*cos(x))^2,x=0..2*Pi),18);
0.130899693899574718
>evalf(Pi/24,18);
0.130899693899574718
```

#### 3.6. If $r = 6$ in Eq. (19), then

$$\int_0^{2\pi} \frac{1}{37+12 \cos x} dx = \frac{2\pi}{35} \quad (27)$$

```
>evalf(int(1/(37+12*cos(x)),x=0..2*Pi),18);
0.179519580205131042
>evalf(2*Pi/35,18);
0.179519580205131042
```

## 4. Conclusion

In this paper, we use Parseval's theorem to solve some definite integrals. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and

engineering mathematics problems and solve these problems using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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