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Evaluating the Partial Derivatives of Some Types of Multivariable Functions

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Abstract This paper takes the mathematical software Maple as the auxiliary tool to study the partial differential problem of three types of multivariable functions. We can obtain the infinite series forms of any order partial derivatives of these multivariable functions by using differentiation term by term theorem, and hence greatly reduce the difficulty of calculating their higher order partial derivative values. On the other hand, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple.

Keywords: *partial derivatives, infinite series forms, differentiation term by term theorem, Maple*

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1. Introduction

In calculus and engineering mathematics curricula, evaluating the *r*-th order partial derivative value of a multivariable function at some point, in general, needs to go through two procedures: firstly determining the *r*-th order partial derivative of this function, and then taking the point into this *r*-th order partial derivative. These two procedures will make us face with increasingly complex calculations when calculating higher order partial derivative values (i.e. *r* is large), and hence to obtain the answers by manual calculations is not easy. In this paper, we study the partial differential problem of the following three types of *n*-variables functions

$$f(x_1,x_2,\cdots,x_n)=\prod_{k=1}^{n} x_k^{p_k}\cdot\exp\left(\prod_{k=1}^{n} x_k^{q_k}\right) \tag{1}$$

$$g(x_1,x_2,\cdots,x_n)=\prod_{k=1}^{n} x_k^{p_k}\cdot\sinh\left(\prod_{k=1}^{n} x_k^{q_k}\right) \tag{2}$$

$$h(x_1,x_2,\cdots,x_n)=\prod_{k=1}^{n} x_k^{p_k}\cdot\cosh\left(\prod_{k=1}^{n} x_k^{q_k}\right) \tag{3}$$

where n is a positive integer, p_k, q_k are real numbers for all $k=1,\cdots,n$. We can obtain the infinite series forms of any order partial derivatives of these three types of n-variables functions by using differentiation term by term theorem ; these are the major results of this study (i.e., Theorems 1, 2 and 3), and hence greatly reduce the difficulty of calculating their higher order partial derivative values. The study of related partial differential problems can refer to [1-14]. Additionally, we propose some examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce some notations and formulas used in this paper.

2.1. Notations

2.1.1. $\prod_{k=1}^{n} c_k = c_1\times c_2\times\cdots\times c_n$, where n is a positive integer, c_k are real numbers for all $k=1,\cdots,n$.

2.1.2. Suppose r is any real number, m is any positive integer. Define $(r)_m=r(r-1)\cdots(r-m+1)$, and $(r)_0=1$.

2.1.3. Suppose n is a positive integer, j_k are non-negative integers for all $k=1,\cdots,n$. For the n-variables function $f(x_1,x_2,\cdots,x_n)$, its j_k-times partial derivative with respect to x_k for all $k=1,\cdots,n$, forms a $j_1+j_2+\cdots+j_n$-th order partial derivative, and denoted by $\frac{\partial^{j_1+j_2+\cdots+j_n} f}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n)$.

2.2. Formulas

2.2.1. Taylor series expansion of the exponential function $e^y=\sum_{m=0}^{\infty}\frac{1}{m!}y^m$, where y is any real number.

2.2.2. Taylor series expansion of the hyperbolic sine function $\sinh y=\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}y^{2m+1}$, where y is any real number.

2.2.3. Taylor series expansion of the hyperbolic cosine function $\cosh y=\sum_{m=0}^{\infty}\frac{1}{(2m)!}y^{2m}$, where y is any real number.

Next, we introduce an important theorem used in this study.

2.3. Differentiation Term by Term Theorem ([15])

For all non-negative integers k, if the functions $g_k:(a,b)\to R$ satisfy the following three conditions：(i) there exists a point $x_0\in(a,b)$ such that $\sum_{k=0}^{\infty}g_k(x_0)$ is convergent, (ii) all functions $g_k(x)$ are differentiable on open interval (a, b), (iii) $\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$ is uniformly convergent on (a, b). Then $\sum_{k=0}^{\infty}g_k(x)$ is uniformly convergent and differentiable on (a, b). Moreover, its derivative $\frac{d}{dx}\sum_{k=0}^{\infty}g_k(x)=\sum_{k=0}^{\infty}\frac{d}{dx}g_k(x)$.

The following is the first result in this study, we determine the infinite series forms of any order partial derivatives of the multivariable function (1).

2.4. Theorem 1

Suppose n is a positive integer, p_k, q_k are real numbers, and j_k are non-negative integers for all $k=1,\cdots,n$. If the n-variables function

$$f(x_1,x_2,\cdots,x_n)=\prod_{k=1}^{n}x_k^{p_k}\cdot\exp\left(\prod_{k=1}^{n}x_k^{q_k}\right)$$

satisfies $x_k^{p_k},x_k^{q_k}$ exist, and $x_k\neq 0$ for all $k=1,\cdots,n$. Then the $j_1+j_2+\cdots+j_n$-th order partial derivative of $f(x_1,x_2,\cdots,x_n)$,

$$\frac{\partial^{j_1+j_2+\cdots+j_n}f}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n)=\sum_{m=0}^{\infty}\frac{1}{m!}\prod_{k=1}^{n}(mq_k+p_k)_{j_k}\cdot\prod_{k=1}^{n}x_k^{mq_k+p_k-j_k} \tag{4}$$

Proof Because

$$\begin{aligned}&f(x_1,x_2,\cdots,x_n)\\&=\prod_{k=1}^{n}x_k^{p_k}\cdot\exp\left(\prod_{k=1}^{n}x_k^{q_k}\right)\\&=\prod_{k=1}^{n}x_k^{p_k}\cdot\sum_{m=0}^{\infty}\frac{1}{m!}\left(\prod_{k=1}^{n}x_k^{q_k}\right)^m\end{aligned}$$

(By Formula 2.2.1.)

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\prod_{k=1}^{n}x_k^{mq_k+p_k} \tag{5}$$

By differentiation term by term theorem, differentiating j_k-times with respect to $x_k(k=1,\cdots,n)$ on both sides of (5), we obtain the $j_1+j_2+\cdots+j_n$-th order partial derivative of $f(x_1,x_2,\cdots,x_n)$,

$$\begin{aligned}&\frac{\partial^{j_1+j_2+\cdots+j_n}f}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n)\\&=\sum_{m=0}^{\infty}\frac{1}{m!}\prod_{k=1}^{n}(mq_k+p_k)_{j_k}\cdot\prod_{k=1}^{n}x_k^{mq_k+p_k-j_k}\end{aligned}$$

Next, we determine the infinite series forms of any order partial derivatives of the multivariable function (2).

2.5. Theorem 2

If the assumptions are the same as Theorem 1. Suppose the n-variables function

$$g(x_1,x_2,\cdots,x_n)=\prod_{k=1}^{n}x_k^{p_k}\cdot\sinh\left(\prod_{k=1}^{n}x_k^{q_k}\right)$$

satisfies $x_k^{p_k},x_k^{q_k}$ exist, and $x_k\neq 0$ for all $k=1,\cdots,n$. Then the $j_1+j_2+\cdots+j_n$-th order partial derivative of $g(x_1,x_2,\cdots,x_n)$,

$$\begin{aligned}&\frac{\partial^{j_1+j_2+\cdots+j_n}g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n)\\&=\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\prod_{k=1}^{n}((2m+1)_{q_k+p_k})_{j_k}\cdot\prod_{k=1}^{n}x_k^{(2m+1)q_k+p_k-j_k}\end{aligned} \tag{6}$$

Proof Because

$$\begin{aligned}&g(x_1,x_2,\cdots,x_n)\\&=\prod_{k=1}^{n}x_k^{p_k}\cdot\sinh\left(\prod_{k=1}^{n}x_k^{q_k}\right)\\&=\prod_{k=1}^{n}x_k^{p_k}\cdot\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\prod_{k=1}^{n}x_k^{q_k}\right)^{2m+1}\end{aligned}$$

(By Formula 2.2.2.)

$$=\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\prod_{k=1}^{n}x_k^{(2m+1)q_k+p_k} \tag{7}$$

Using differentiation term by term theorem, differentiating j_k-times with respect to $x_k(k=1,\cdots,n)$ on both sides of (7), we obtain

$$\begin{aligned}&\frac{\partial^{j_1+j_2+\cdots+j_n}g}{\partial x_n^{j_n}\cdots\partial x_2^{j_2}\partial x_1^{j_1}}(x_1,x_2,\cdots,x_n)\\&=\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\prod_{k=1}^{n}((2m+1)_{q_k+p_k})_{j_k}\cdot\prod_{k=1}^{n}x_k^{(2m+1)q_k+p_k-j_k}\end{aligned}$$

Finally, we obtain the infinite series forms of any order partial derivatives of the multivariable function (3).

2.6. Theorem 3

If the assumptions are the same as Theorem 1. Suppose the *n*-variables function

$$h(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} x_k^{p_k} \cdot \cosh\left(\prod_{k=1}^{n} x_k^{q_k}\right)$$

satisfies $x_k^{p_k}, x_k^{q_k}$ exist, and $x_k \neq 0$ for all $k = 1, \cdots, n$. Then the $j_1 + j_2 + \cdots + j_n$-th order partial derivative of $h(x_1, x_2, \cdots, x_n)$,

$$\frac{\partial^{j_1+j_2+\cdots+j_n} h}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \prod_{k=1}^{n} (2m_{q_k+p_k})_{j_k} \cdot \prod_{k=1}^{n} x_k^{2mq_k+p_k-j_k} \tag{8}$$

Proof Because

$$\begin{aligned} & h(x_1, x_2, \cdots, x_n) \\ & = \prod_{k=1}^{n} x_k^{p_k} \cdot \cosh\left(\prod_{k=1}^{n} x_k^{q_k}\right) \\ & = \prod_{k=1}^{n} x_k^{p_k} \cdot \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\prod_{k=1}^{n} x_k^{q_k}\right)^{2m} \end{aligned}$$

(By Formula 2.2.3.)

$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!} \prod_{k=1}^{n} x_k^{2mq_k+p_k} \tag{9}$$

Also, by differentiation term by term theorem, differentiating j_k-times with respect to $x_k \, (k = 1, \cdots, n)$ on both sides of (9), then

$$\frac{\partial^{j_1+j_2+\cdots+j_n} h}{\partial x_n^{j_n} \cdots \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, \cdots, x_n) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \prod_{k=1}^{n} (2m_{q_k+p_k})_{j_k} \cdot \prod_{k=1}^{n} x_k^{2mq_k+p_k-j_k}$$

3. Examples

In the following, for the partial differential problem of the three types of multivariable functions in this study, we provide three examples and use Theorems 1-3 to determine the infinite series forms of any order partial derivatives of these functions and evaluate some of their higher order partial derivative values. On the other hand, we employ Maple to calculate the approximations of these higher order partial derivative values and their solutions for verifying our answers.

3.1. Example 1

Suppose the domain of the two-variables function

$$f(x_1, x_2) = x_1^{7/2} x_2^{-9/4} \exp(x_1^{-5/3} x_2^{6/11}) \tag{10}$$

is $\left\{(x_1, x_2) \in R^2 \,\middle|\, x_1 > 0, x_2 > 0\right\}$. By Theorem 1, we obtain any $j_1 + j_2$-th order partial derivative of $f(x_1, x_2)$,

$$\frac{\partial^{j_1+j_2} f}{\partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-5m}{3}+\frac{7}{2}\right)_{j_1}\left(\frac{6m}{11}-\frac{9}{4}\right)_{j_2} \times x_1^{\frac{-5m}{3}+\frac{7}{2}-j_1} \cdot x_2^{\frac{6m}{11}-\frac{9}{4}-j_2} \tag{11}$$

for all $x_1 > 0, x_2 > 0$. Thus, we can evaluate the 13-th order partial derivative value of $f(x_1, x_2)$ at $(4, 6)$,

$$\frac{\partial^{13} f}{\partial x_2^{8} \partial x_1^{5}}(4, 6) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{-5m}{3}+\frac{7}{2}\right)_{5}\left(\frac{6m}{11}-\frac{9}{4}\right)_{8} \times 4^{\frac{-5m}{3}+\frac{3}{2}} \cdot 6^{\frac{6m}{11}-\frac{41}{4}} \tag{12}$$

We use Maple to verify the correctness of (12) as follows:

```
>f:=(x1,x2)->x1^(7/2)*x2^(-9/4)*exp(x1^(-5/3)*x2^(6/11)
);
>evalf(D[1$5,2$8](f)(4,6),14);
```

$$-0.0024999213914188$$

```
>evalf(sum(1/m!*product(-5*m/3+7/2-j,j=0..4)*product(6
*m/11-9/4-p,p=0..7)*4^(-5*m/3-3/2)*6^(6*m/11-41/4),m
=0..infinity),14);
```

$$-0.0024999213914188$$

3.2. Example 2

Let the domain of the three-variables function

$$g(x_1, x_2, x_3) = x_1^{6/5} x_2^{-9/4} x_3^{-11/3} \sinh(x_1^{8/7} x_2^{-12/5} x_3^{20/3}) \tag{13}$$

be $\left\{(x_1, x_2, x_3) \in R^3 \,\middle|\, x_1 \neq 0, x_2 > 0, x_3 \neq 0\right\}$. Using Theorem 2, we can determine any $j_1 + j_2 + j_3$-th order partial derivative of $g(x_1, x_2, x_3)$,

$$\frac{\partial^{j_1+j_2+j_3} g}{\partial x_3^{j_3} \partial x_2^{j_2} \partial x_1^{j_1}}(x_1, x_2, x_3) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}\left(\frac{16m}{7}+\frac{82}{35}\right)_{j_1}\left(\frac{-24m}{5}-\frac{93}{20}\right)_{j_2}\left(\frac{40m}{3}+3\right)_{j_3} \times x_1^{\frac{16m}{7}+\frac{822}{35}-j_1} \cdot x_2^{\frac{-24m}{5}-\frac{93}{20}-j_2} \cdot x_3^{\frac{40m}{3}+3-j_3} \tag{14}$$

for all $x_1 \neq 0, x_2 > 0, x_3 \neq 0$. Therefore, the 16-th order partial derivative value of $g(x_1, x_2, x_3)$ at $\left(\frac{1}{3}, \frac{2}{5}, \frac{5}{8}\right)$,

$$\frac{\partial^{16} g}{\partial x_3{}^7 \partial x_2{}^3 \partial x_1{}^6}\left(\frac{1}{3},\frac{2}{5},\frac{5}{8}\right)$$
$$= \sum_{m=0}^{\infty} \frac{1}{(2m+1)!}\left(\frac{16m}{7}+\frac{82}{35}\right)_6 \left(\frac{-24m}{5}-\frac{93}{20}\right)_3 \left(\frac{40m}{3}+3\right)_7 \times$$
$$\left(\frac{1}{3}\right)^{\frac{16m}{7}-\frac{128}{35}} \cdot \left(\frac{2}{5}\right)^{\frac{-24m}{5}-\frac{153}{20}} \cdot \left(\frac{5}{8}\right)^{\frac{40m}{3}-4} \tag{15}$$

>g:=(x1,x2,x3)->x1^(6/5)*x2^(-9/4)*x3^(-11/3)*sinh(x1^(8/7)*x2^(-12/5)*x3^(20/3));
>evalf(D[1$6,2$3,3$7](g)(1/3,2/5,5/8),18);

$$-9.11268042306315752 \cdot 10^{16}$$

>evalf(sum(1/(2*m+1)!*product(16*m/7+82/35-j,j=0..5)*product(-24*m/5-93/20-p,p=0..2)*product(40*m/3+3-r,r=0..6)*(1/3)^(16*m/7-128/35)*(2/5)^(-24*m/5-153/20)*(5/8)^(40*m/3-4),m=0..infinity),18);

$$-9.11268042306315753 \cdot 10^{16}$$

3.3. Example 3

If the domain of the two-variables function

$$h(x_1,x_2) = x_1^{-5/7} x_2^{13/9} \cosh(x_1^{-6/5} x_2^{9/7}) \tag{16}$$

is $\left\{(x_1,x_2)\in R^2 \,\middle|\, x_1 \neq 0, x_2 \neq 0\right\}$. By Theorem 3, we obtain any $j_1 + j_2$ -th order partial derivative of $h(x_1,x_2)$,

$$\frac{\partial^{j_1+j_2} h}{\partial x_2{}^{j_2} \partial x_1{}^{j_1}}(x_1,x_2)$$
$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{-12m}{5}-\frac{5}{7}\right)_{j_1} \left(\frac{18m}{7}+\frac{13}{9}\right)_{j_2} \times \tag{17}$$
$$x_1^{\frac{-12m}{5}-\frac{5}{7}-j_1} \cdot x_2^{\frac{18m}{7}+\frac{13}{9}-j_2}$$

for all $x_1 \neq 0, x_2 \neq 0$. Hence, the 11-th order partial derivative value of $h(x_1,x_2)$ at $\left(\frac{4}{9},\frac{7}{11}\right)$,

$$\frac{\partial^{11} h}{\partial x_2{}^6 \partial x_1{}^5}\left(\frac{4}{9},\frac{7}{11}\right)$$
$$= \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{-12m}{5}-\frac{5}{7}\right)_5 \left(\frac{18m}{7}+\frac{13}{9}\right)_6 \times \tag{18}$$
$$\left(\frac{4}{9}\right)^{\frac{-12m}{5}-\frac{40}{7}} \cdot \left(\frac{7}{11}\right)^{\frac{18m}{7}-\frac{41}{9}}$$

>h:=(x1,x2)->x1^(-5/7)*x2^(13/9)*cosh(x1^(-6/5)*x2^(9/7));
>evalf(D[1$5,2$6](h)(4/9,7/11),18);

$$-1.76120755683623639 \cdot 10^{11}$$

>evalf(sum(1/(2*m)!*product(-12*m/5-5/7-j,j=0..4)*product(18*m/7+13/9-p,p=0..5)*(4/9)^(-12*m/5-40/7)*(7/11)^(18*m/7-41/9),m=0..infinity),18);

$$-1.76120755683623639 \cdot 10^{11}$$

4. Conclusion

In this article, we provide a new technique to determine any order partial derivatives of some multivariable functions. We hope this technique can be applied to solve another partial differential problems. On the other hand, the differentiation term by term theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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