

A Method for Solving a Class of Boundary Value Problems of Laguerre Equation

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Abstract Based on the analysis of the boundary value problem of Laguerre equation, this paper studies the similar structure of its solution expression. This paper is found that its solution can be obtained by combining similar kernel function with coefficients of left boundary condition. While the similar kernel function is constructed by both the function of guide solution and coefficients of right boundary condition. Hence, we proposed a method for solving this class of boundary value problems: the similar constructing method.

Keywords: Laguerre equation, boundary value problem, similar kernel function, the similar constructing method

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1. Introduction

As is known to all, we often involve solving differential equations in practical problems. Studying the inherent law of the solutions to differential equations plays a crucial role for simplifying the solving process. Then, can the form of the solutions(the solutions belong to the various differential equations in the same boundary value problem) be represented by using a uniform formula? Since 2004, the definite answer has been made in many relevant reports. At the same time, the solutions of some second-order homogeneous linear ordinary differential equations [1-9], the solutions of some second-order homogeneous linear partial differential equations [10,11], and the solutions of the seepage equations in some reservoir engineering [12,13,14] were studied, with gradually forming the similar constructing theory and similar constructing method of the solutions that belong to differential equations.

Laguerre equation has some comprehensive applications in several research areas, such as physics and engineering. Based on the above study, this paper will study a class of boundary value problem (BVP)of the Laguerre equation as follows:

$$\begin{cases} xy'' + (1-x)y' + ny = 0 & x \in (a, b) \\ [Ey + (1+EF)y']_{x=a} = D \\ [Ky + Hy']_{x=b} = 0 \end{cases} \quad (1)$$

where D, E, F, K, H, a, b, n are real constants, $D \neq 0$, $K^2 + H^2 \neq 0$, $0 < a < b$, n is a positive constant.

2. Preliminary Knowledge

Lemma 1 General solution to Laguerre equation can be expressed as[15]:

$$y = D_1 L_n(x) + D_2 G(-n, 1, x) \quad (2)$$

Where D_1, D_2 are arbitrary constants, $L_n(x)$ is the Laguerre polynomial, and $G(-n, 1, x)$ is the second class of Kummer functions.

Lemma 2 Known that $F(\alpha, \gamma, \chi)$ is the first class of Kummer functions, it can get the conclusion as follows [15]:

$$\frac{d}{dx} F(\alpha, \gamma, \chi) = \frac{\alpha}{\gamma} F(\alpha+1, \gamma+1, \chi),$$

where

$$F(\alpha, \gamma, \chi) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!(\gamma)_k} \chi^k,$$

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1).$$

According to the lemma 2, we can be obtained as follows:

$$\frac{d}{dx} L_n(x) = \frac{d}{dx} (n! F(-n, 1, x)) = n!(-n) F(1-n, 2, x),$$

$$\begin{aligned} \frac{d}{dx} G(-n, 1, x) &= \frac{d}{dx} [F(-n, 1, x) \ln x - 1] \\ &= \ln x (-n) F(1-n, 2, x) + \frac{1}{x} F(-n, 1, x). \end{aligned}$$

3. The Main Theorem and Its Proof

The probative thought in this paper is that the similar construction of the solution of the BVP (1) is obtained by extending from the special case of right boundary condition to the general situation. Now, this paper discusses the BVP (1) with the special case of right boundary condition.

Theorem 1 If the boundary value problem

$$\begin{cases} xy'' + (1-x)y' + ny = 0 & x \in (a,b) \\ [Ey + (1+EF)y']_{x=a} = D \\ y(b) = 0, \text{ 或 } y'(b) = 0 \end{cases} \quad (3)$$

has one solution, then the similar construction of the solution can be expressed as:

$$y(x) = \frac{D}{E + \frac{1}{F + \psi(a)}} \frac{1}{F + \psi(a)} \psi(x) \quad (4)$$

And where $\psi(x)$ is kernel function:

$$\psi(x) = \begin{cases} \frac{\phi_1(x)}{\phi_1'(a)}, & y(b) = 0 \\ \frac{\phi_2(x)}{\phi_2'(a)}, & y'(b) = 0 \end{cases} \quad (5)$$

Where

$$\begin{aligned} \phi_1(x) &= G(-n,1,b)L_n(x) - L_n(b)G(-n,1,x), \\ \phi_2(x) &= G'(-n,1,b)L_n(x) - L_n'(b)G(-n,1,x). \end{aligned}$$

Proof. The basic equation in left and right regions of the BVP (1) is Laguerre equation, according to the lemma 1, we can obtain the general solution of (1) as follows:

$$y = D_1L_n(x) + D_2G(-n,1,x) \quad (6)$$

where D_1, D_2 are undetermined constants.

Substituting Eq.(6) into the left boundary condition of the BVP(3), yields:

$$\begin{aligned} & [En!F(-n,1,a) + (1+EF)n!(-n)F(1-n,2,a)]D_1 \\ & + \left[\begin{matrix} F(-n,1,a)\ln a - 1 \\ + \ln a(-n)F(1-n,2,a) + \frac{1}{a}F(-n,1,a) \end{matrix} \right] D_2 = D \end{aligned} \quad (7)$$

1. when the right boundary condition of the BVP(3) for $y(b) = 0$, then Substituting Eq.(6) into it :

$$\begin{aligned} D_1n!F(-n,1,b) + D_2G(-n,1,b) \\ = D_1n!F(-n,1,b) + D_2[F(-n,1,b)\ln b - 1] = 0 \end{aligned} \quad (8)$$

Incorporating Eqs.(7)and(8), we can acquire the expressions about coefficients of Eq.(6) as follows:

$$D_1 = \frac{D(-G(-n,1,b))}{M_1G(-n,1,b) - M_2L_n(b)} \quad (9)$$

$$D_2 = \frac{DL_n(b)}{M_1G(-n,1,b) - M_2L_n(b)} \quad (10)$$

Substituting Eqs.(9) and (10) into Eq.(6), we can gain the solution:

$$y(x) = \frac{D[G(-n,1,b)L_n(x) - L_n(b)G(-n,1,x)]}{M_1G(-n,1,b) - M_2L_n(b)} \quad (11)$$

where

$$\begin{aligned} M_1 &= EL_n(a) + (1+EF)L_n'(a), \\ M_2 &= EG(-n,1,a) + (1+EF)G'(-n,1,a). \end{aligned}$$

Let

$$\psi(x) = \frac{\phi_1(x)}{\phi_1'(a)} = \frac{G(-n,1,b)L_n(x) - L_n(b)G(-n,1,x)}{G(-n,1,b)L_n'(a) - L_n(b)G'(-n,1,a)}.$$

After simplifying, then

$$y(x) = \frac{D}{E + \frac{1}{F + \psi(a)}} \frac{1}{F + \psi(a)} \psi(x) \quad (12)$$

2. when the right boundary condition of the BVP (3) for $y'(b) = 0$, then Substituting Eq.(6) into it:

$$\begin{aligned} D_1n!(-n)F(1-n,2,b) \\ + D_2 \left[\ln b(-n)F(1-n,2,b) + \frac{1}{b}F(-n,1,b) \right] = 0 \end{aligned} \quad (13)$$

Incorporating Eqs.(7) and (13), we can acquire the expressions about coefficients of Eq.(6) as follows:

$$D_1 = \frac{D(-G'(-n,1,b))}{M_1G'(-n,1,b) - M_2L_n'(b)} \quad (14)$$

$$D_2 = \frac{DL_n'(b)}{M_1G'(-n,1,b) - M_2L_n'(b)} \quad (15)$$

Substituting Eqs.(14) and(15)into Eq.(6), we can gain the solution:

$$y(x) = \frac{D[G'(-n,1,b)L_n(x) - L_n'(b)G(-n,1,x)]}{M_1G'(-n,1,b) - M_2L_n'(b)}$$

where

$$\begin{aligned} M_1 &= EL_n(a) + (1+EF)L_n'(a), \\ M_2 &= EG(-n,1,a) + (1+EF)G'(-n,1,a). \end{aligned}$$

Let

$$\begin{aligned} \psi(x) &= \frac{\phi_2(x)}{\phi_2'(a)} \\ &= \frac{G'(-n,1,b)L_n(x) - L_n'(b)G(-n,1,x)}{G'(-n,1,b)L_n'(a) - L_n'(b)G'(-n,1,a)}. \end{aligned}$$

After simplification, we can gain the solution as same as Eq.(12).

After discussing the structure of solution of the BVP(4), that is the special situation of the BVP(1). Now, we return to the theme in this paper--the BVP(1)that includes the general situation of the right boundary condition, and study the BVP(1) by the above way.

Theorem 2 If the BVP(1) has one solution, then the similar structure of the solution can be expressed as:

$$y(x) = \frac{D}{E + \frac{1}{F + \psi(a)}} \frac{1}{F + \psi(a)} \psi(x) \quad (16)$$

And where $\psi(x)$ is kernel function:

$$\psi(x) = \frac{K\phi_1(x) + H\phi_2(x)}{K\phi_1'(a) + H\phi_2'(a)} \quad (17)$$

Proof. The basic equation in left and right regions of the BVP (1) is Laguerre equation, according to the lemma 1, we can obtain the general solution of (1) as follows:

$$y = D_1 L_n(x) + D_2 G(-n, 1, x) \quad (18)$$

where D_1, D_2 are undetermined constants. when the right boundary condition of the BVP(1) for $[Ky + Hy']_{x=b} = 0$, then substituting Eq.(18) into it:

$$K[D_1 n! F(-n, 1, b) + D_2 G(-n, 1, b)] + H[D_1 L_n'(b) + D_2 G'(-n, 1, b)] = 0 \quad (19)$$

Substituting Eq.(18) into the left boundary condition of the BVP(1), yields:

$$\left[\begin{matrix} En! F(-n, 1, a) \\ + (1 + EF)n!(-n)F(1-n, 2, a) \end{matrix} \right] D_1 + \left[\begin{matrix} F(-n, 1, a) \ln a - 1 \\ + \ln a(-n)F(1-n, 2, a) + \frac{1}{a} F(-n, 1, a) \end{matrix} \right] D_2 = D \quad (20)$$

Incorporating Eqs.(19)and(20), we can acquire the expressions about coefficients of Eq.(18) as follows:

$$D_1 = \frac{-DY_1}{M_1 Y_1 - M_2 Y_2} \quad (21)$$

$$D_2 = \frac{DY_2}{M_1 Y_1 - M_2 Y_2} \quad (22)$$

Substituting Eqs.(21) and(22)into Eq.(18), we can gain the solution of the BVP(1):

$$y(x) = \frac{D[Y_2 G(-n, 1, X) - Y_1 L_n(x)]}{M_1 Y_1 - M_2 Y_2}$$

Where

$$\begin{aligned} Y_1 &= KG(-n, 1, b) + HG'(-n, 1, b), \\ Y_2 &= KL_n(b) + HL_n'(b), \\ M_1 &= EL_n(a) + (1 + EF)L_n'(a), \\ M_2 &= EG(-n, 1, a) + (1 + EF)G'(-n, 1, a). \end{aligned}$$

Let

$$\psi(x) = \frac{K\phi_1(x) + H\phi_2(x)}{K\phi_1'(a) + H\phi_2'(a)}$$

After simplification, we can gain the solution as same as Eq.(16).

4. The Ideology and Steps of the Similar Constructing Method

To observe and analyze theorems 1 and 2, it is not hard to find :

1) Through observing Eqs.(5) and (6), we can obtain the similar kernel function $\psi(x)$ which consists of $\phi(x_1, x_2) = G(-n, 1, x_2)L_n(x_1) - L_n(x_2)G(-n, 1, x_1)$ and coefficients K, H of right boundary condition $[Ky + Hy']_{x=b} = 0$, as follows:

$$\psi(x) = \frac{K\phi_1(x) + H\phi_2(x)}{K\phi_1'(a) + H\phi_2'(a)} \quad (23)$$

where

$$\begin{aligned} \phi(x_1 = x, x_2 = b) &= \phi_1(x) \\ &= G(-n, 1, b)L_n(x) - L_n(b)G(-n, 1, x), \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial}{\partial x_2} \phi(x_1, x_2) \Big|_{x_2=b} &= \phi_2(x) \\ &= G'(-n, 1, b)L_n(x) - L_n'(b)G(-n, 1, x). \end{aligned} \quad (25)$$

2) According to theorem 2, we can know the solution of BVP(1) by combining the coefficients E, F, D of left boundary condition $[Ey + (1 + EF)y']_{x=a} = D$.

From the above, it is not difficult to gain a method, which is called the similar constructing method, for solving a class of boundary value problems of Laguerre equation. And it is easy to induce steps of the similar constructing method. The concrete steps are as follows:

Step 1 constructing binary function $\phi(x_1, x_2)$

We structure binary function $\phi(x_1, x_2)$ by combining the second class of Kummer function $G(-n, 1, x)$ and the Laguerre polynomial $L_n(x)$, and that $G(-n, 1, x)$ and $L_n(x)$ are linearly independent. Then, we can base $\phi_1(x)$ and $\phi_2(x)$ on lemma 2.

Step 2 constructing the similar kernel function $\psi(x)$

The similar kernel function $\psi(x)$ can be structured by using binary function $\phi(x_1, x_2)$ and coefficients K, H of right boundary condition $[Ky + Hy']_{x=b} = 0$, that is to say, the equation(17). If the right boundary condition is $y(b) = 0$ (i.e. $K = 1, H = 0$) or $y'(b) = 0$ (i.e. $K = 0, H = 1$), it can respectively combine binary function $\phi(x_1, x_2)$ to obtain the corresponding similar kernel function, in other words, the equation(5).

Step 3 obtaining the solution to the BVP $y(x)$

To the BVP(1), the solution can be obtained by assembling coefficients D, E, F of left boundary condition $[Ey + (1 + EF)y']_{x=a} = D$, that is, the equation (16).

5. The Application of the Similar Constructing Method

Solving the boundary value problem as follows:

$$\begin{cases} xy'' + (1-x)y' + y = 0 & x \in (a, b) \\ y'(x)_{x=1} = 1 \\ 2y'(x)_{x=2} = 0 \end{cases} \quad (26)$$

Comparing with the boundary value problem (1) and (26), we know that $n = 1$,

$E = 0, a = 1, D = 1, K = 0, H = 2, b = 2$. The basic equation in left and right regions of the BVP (26) is Laguerre equation, then the two linear independent solutions of the main equation is $L_1(x)$ and $G(-1, 1, x)$.

According to steps of the similar constructing method, we solve the boundary value problem(26).

Step 1 constructing binary function $\phi(x_1, x_2)$

We structure binary function $\phi(x_1, x_2) = G(-1, 1, x_2)L_1(x_1) - L_1(x_2)G(-1, 1, x_1)$. Then, according to lemma 2 and Eqs(24)~(25), we can obtain $\phi_1(x)$ and $\phi_2(x)$, as follows:

$$\begin{aligned} \phi_1(x) &= G(-1, 1, 2)L_1(x) - L_1(2)G(-1, 1, x), \\ \phi_2(x) &= G'(-1, 1, 2)L_1(x) - L'(2)G(-1, 1, x). \end{aligned}$$

Step 2 constructing the similar kernel function $\psi(x)$

According to Eqs.(17), we can structure the similar kernel function $\psi(x)$, as the follow:

$$\psi(x) = \frac{0 \times \phi_1(x) + 2 \cdot \phi_2(x)}{0 \times \phi_1'(1) + 2 \cdot \phi_2'(1)} = \frac{\phi_2(x)}{\phi_2'(1)}$$

Step 3 obtaining the solution to the BVP $y(x)$

According to Eqs.(16), the boundary value problem(26) can be obtained respectively as the follow:

$$\begin{aligned} y(x) &= \psi(x) = \frac{\phi_2(x)}{\phi_2'(1)} \\ &= \frac{G'(-1, 1, 2)L_1(x) - L_1'(2)G(-1, 1, x)}{G'(-1, 1, 2)L_1'(1) - L_1'(2)G'(-1, 1, 1)} \end{aligned}$$

Where

$$\begin{aligned} G'(-1, 1, 2) &= -\ln 2F(0, 2, 2) + \frac{1}{2}F(-1, 1, 2), \\ L_1(x) &= F(-1, 1, x) \\ G'(-1, 1, 1) &= F(-1, 1, 1), \\ G(-1, 1, x) &= F(-1, 1, x) \ln x - 1 \\ F(0, 2, 2) &= \sum_{k=0}^{\infty} \frac{(0)_k}{k!(2)_k} 2^k = 0, \\ F(-1, 1, x) &= \sum_{k=0}^{\infty} \frac{(-1)_k}{k!(1)_k} x^k. \end{aligned}$$

6. Conclusions

The Eq.(16), that is the solution of the boundary value of Laguerre equation, can be obtained by assembling coefficients of left boundary condition and similar kernel function. It can be expressed by one uniform formula.

In the research process, we know that we only need to change the corresponding similar kernel functions to obtain the solution to the BVP (1) when the right boundary condition of the BVP (1) changes. The method is simple and effective for solving the boundary value of differential equation.

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