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Effect of Elevated Carbon Dioxide Concentration on Plant Growth: A Mathematical Model

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Abstract The enhanced emission of carbon dioxide (CO_2) due to increased population density has significant effect on the growth of plant biomass. It is noted here that increased atmospheric carbon dioxide is absorbed by plant biomass during photosynthesis. In this paper, therefore, a nonlinear mathematical model is proposed to study the dynamics of population density dependent emission of carbon dioxide in the atmosphere. The phenomenon is assumed to be governed by three nonlinearly dependent variables namely; plant biomass density, population density and the concentration of carbon dioxide. The model is analyzed using stability theory of ordinary differential equations and numerical simulations. It is shown that the density of plant biomass increases as the concentration of carbon dioxide increases. It is, further, shown that the equilibrium density of plant biomass decreases as the density of human population increases but the concentration of carbon dioxide increases in the atmosphere. The numerical simulation confirms these analytical results.

Keywords: *mathematical model, population density, plant biomass density, carbon dioxide (CO_2), stability*

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1. Introduction

The increased concentration of atmospheric CO_2 has direct effect on the growth of plant biomass such as vegetation, forests, etc. This effect is known as 'CO_2 fertilization'. According to 'The Intergovernmental Panel on Climate Change' CO_2 fertilization is defined as "the enhancement in the net primary productivity of terrestrial vegetation that occurs as a result of elevated atmospheric CO_2 concentration" [12]. It has also been stated in [12] that in standard carbon cycle model calculations, CO_2 fertilization acts as a sink in the carbon cycle. In ecological modeling studies, it has been observed that the terrestrial biomass grows due to uptake of atmospheric carbon dioxide introduced into the atmosphere naturally as well as by human activities. During interaction of plant biomass with CO_2, carbon is exchanged naturally through photosynthesis, respiration, decomposition and combustion, etc. [12]. The transformation of CO_2 in to glucose by plant leaves during photosynthesis process is given by the following simple chemical equation [11],

$$Carbon\ dioxide + water + light\ energy$$
$$= glucose + oxygen$$
$$6CO_2 + 6H_2O + light\,energy \rightarrow C_6H_{12}O_6 + 6O_2$$

Some investigations have been conducted to study the effect of elevated level of atmospheric CO_2 on growth of plant species [2,3,6,8,14,15,16,17,20,22,23,26,27,28]. In particular, Albertine et al. [2] have conducted an experiment to study the growth of pollen from flowers at elevated levels of ozone and carbon dioxide concentrations and found considerable increase in pollens produced at elevated carbon dioxide level. Ambavaram et al. [3] have investigated the effect of elevated atmospheric carbon dioxide on cereal crops such as rice (Oryza sativa) and found considerable increase in rice yield. Madhu and Hatfield [14] have shown that the growth and yield of most of the agricultural crops are significantly enhanced by above and belowground environmental conditions and elevated level of atmospheric carbon dioxide. It has been observed that if the rate of photosynthesis is increased, plants can grow faster attaining their equilibrium with increased biomass [15]. Miri et al. [17] studied the effect of elevated CO_2 on vegetative growth of plants such as soybean and lamb's-quarter and found significant increase in their growth. Poorter and Perez-Soba [20] presented an analysis to study the effect of elevated carbon dioxide on growth of plant species. They have shown that increased level of atmospheric carbon dioxide stimulates the rate of photosynthesis enhancing the biomass of plant species. Prior et al. [22] presented a review of elevated atmospheric CO_2 effects on plant growth and water relations. They have shown that plant growth can be enhanced by increasing the concentration of atmospheric CO_2. Wolfe-Bellin et al. [28] examined the effect of rising concentration of atmospheric CO_2 on C_3 forb phytolacca americana L. (Phytolaccacea) and found considerable increase in its growth.

Further, the depletion of plant biomass is chiefly caused by human activities such as cutting trees for agriculture, food and energy, establishment of industries, household appliances, creating new houses, farmlands, etc. Several investigations have been made to study the effect of population density on degradation of plant biomass [1,5,7,10,13,19,24,25]. In this regard, Dubey et al. [7] have presented a mathematical model to study the effect of population and population pressure augmented industrialization on forestry resources. They have shown that the equilibrium density of biomass decreases as the equilibrium densities of population and population pressure augmented industrialization increases. Shukla et al. [24] have studied the effect of population on forestry biomass using a nonlinear mathematical model. They have shown that the forestry biomass may become extinct if population increases without control.

It is pointed out here that increase in the carbon dioxide concentration in the atmosphere due to human activities is a key factor to stimulate the growth of biomass [4,5,18,21]. Further, increase in the population density causes depletion of plant biomass but it increases the concentration of CO_2 in the atmosphere which is used by plant leaves during photosynthesis thus regulating the concentration of CO_2 in the atmosphere. In view of the above, in this paper, we have proposed and analyzed a nonlinear mathematical model to study the effect of increased carbon dioxide concentration (due to population density) on plant growth.

2. Mathematical Model

To model the dynamics of density dependent emission of carbon dioxide, we have made the following assumptions,

1. The equilibrium level of carbon dioxide is enhanced significantly due to increase in population density and is assumed to be in the direct proportion of population density.
2. The growth rates of plant biomass and population density are assumed to be following logistic equation.
3. The depletion of plant biomass is in the direct proportion of its density as well as the human population density.
4. The growth of plant biomass is in the direct proportion of its density as well as the concentration of carbon dioxide.
5. The growth of population density is in the direct proportion of its density as well as the density of plant biomass.

In the atmosphere, under consideration, let B and N denote the densities of plant biomass and human population at any time t, respectively. Let C be the concentration of carbon dioxide (CO_2), emitted into the atmosphere naturally as well as by human activities, at any time t. It is pointed out here that the plant biomass density decreases due to human activities such as cutting of trees for household appliances, paper industries, agriculture, creating new houses and other purposes. Thus, we assume that the plant biomass density decreases due to human population while human population density increases due to plant biomass and therefore the decrease in plant biomass density is assumed to be proportional to the human population as well as plant biomass density. Further, the increase in human population density is assumed to be proportional to the human population as well as plant biomass density. As pointed out in previous section that the growth and yield of plant biomass depend on carbon dioxide concentration and hence it is obvious to assume that the growth rate of plant biomass increases due to carbon dioxide whereas the concentration of carbon dioxide decreases due to plant biomass. Thus, the growth rate of biomass is assumed to be proportional to the density of plant biomass as well as the concentration of carbon dioxide. In the modeling process, the dynamics of the biomass density is assumed to be governed by logistic equation as follows,

$$\frac{dB}{dt} = sB\left(1-\frac{B}{L}\right) - s_1 NB + s_2 BC$$

The constants s and L are intrinsic growth rate and carrying capacity of plant biomass respectively. The constant s_1 represents the depletion rate coefficient of plant biomass due human population, s_2 $(0 \le s_2 \le \delta_1)$ is the growth rate coefficient of plant biomass due to uptake of CO_2. Further, let r and K be intrinsic growth rate and carrying capacity of human population, respectively. It is assumed that the growth of human population due to biomass is directly proportional to the densities of plant biomass as well as human population (i.e. $r_1 NB$). The constant r_1 $(0 \le r_1 \le s_1)$ denotes the growth rate coefficient of human population due to plant biomass. In view of these assumptions, the differential equation governing the dynamics of the population is given by,

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) + r_1 NB$$

Increase in carbon dioxide concentration into the atmosphere due to natural sources (e.g. ocean release, the combustion of organic matter, wildfires, the respiration processes of living organisms, etc.) is assumed to be constant (let Q). It is noted here that the anthropogenic emissions of carbon dioxide into the atmosphere is due to human population and therefore the growth rate of atmospheric carbon dioxide is assumed to be proportional to the human population density (i.e. δN). Further, as discussed above, the depletion of carbon dioxide is assumed to be proportional to the biomass density as well as the concentration of carbon dioxide. The constant δ denotes the growth rate coefficient of CO_2 due to human population, δ_0 be its natural depletion rate coefficient and δ_1 is its depletion rate coefficient due to plant biomass. Thus, the equation governing the concentration of CO_2 in the atmosphere is given as follows,

$$\frac{dC}{dt} = Q + \delta N - \delta_0 C - \delta_1 BC$$

Thus, the dynamics of the system is governed by the following set of nonlinear ordinary differential equations,

$$\frac{dB}{dt} = sB\left(1-\frac{B}{L}\right) - s_1 NB + s_2 BC \quad (1)$$

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) + r_1 NB \quad (2)$$

$$\frac{dC}{dt} = Q + \delta N - \delta_0 C - \delta_1 BC \quad (3)$$

$$B(0) \ge 0, N(0) \ge 0, C(0) \ge 0$$

All the constants taken here are positive.

Now, we analyze the model (1) – (3) under the following two cases:

(I) $Q > 0,\ \delta > 0$

(II) $Q > 0,\ \delta = 0$

2.1. Case I. $Q > 0,\ \delta > 0$

To analyze the model system (1) – (3), we need the bounds of dependent variables. For this, we establish the region of attraction in the following lemma [9].

Lemma 2.1.1. The set

$$\Omega = \{(B,N,C) \in R_+^3 : \\ 0 \le B \le B_m; 0 \le N \le N_m; 0 \le C \le C_m\}$$

is the region of attraction for all solutions of the model system (1) – (3) initiating in the interior of positive octant, where,

$$B_m = \frac{\frac{L}{s}\left\{s + \frac{s_2}{\delta_0}(Q+\delta K)\right\}}{\left\{1 - \frac{L}{s}\frac{s_2 r_1 \delta K}{\delta_0 r}\right\}}, N_m = \frac{K}{r}(r + r_1 B_m)$$

$$C_m = \frac{Q + \delta N_m}{\delta_0}$$

provided,

$$\frac{L}{s}\frac{s_2 r_1 \delta K}{\delta_0 r} < 1$$

2.1.1. Equilibrium Analysis

The model system (1) – (3) has four non-negative equilibria;

(i) $E_0\left(0,\ 0,\ \frac{Q}{\delta_0}\right)$

(ii) $E_1\left(0,\ K,\ \frac{Q+\delta K}{\delta_0}\right)$

(iii) $E_2(\bar{B},\ 0,\ \bar{C})$

(iv) $E^*(B^*,\ N^*,\ C^*)$

The existence of E_0 or E_1 is obvious and hence omitted.

2.1.1.1. Existence of $E_2(\bar{B},\ 0,\ \bar{C})$.

The positive solution of $E_2(\bar{B},\ 0,\ \bar{C})$ can be obtained by solving the following algebraic equations,

$$s - \frac{sB}{L} + s_2 C = 0 \quad (4)$$

$$Q - \delta_0 C - \delta_1 BC = 0 \quad (5)$$

Eliminating B from (4) and (5), we get,

$$C = \frac{-(\delta_0 + \delta_1 L) + \sqrt{(\delta_0 + \delta_1 L)^2 + 4Q s_2 \delta_1 (L/s)}}{2 s_2 \delta_1 (L/s)} = \bar{C}(\text{say})$$

Using the value of $\bar{C}$ we can find the value of $\bar{B}$ from equation (4) or (5).

2.1.1.2. Existence of $E^*(B^*,\ N^*,\ C^*)$

The positive solution of $E^*(B^*,\ N^*,\ C^*)$ can be obtained by solving the following algebraic equations,

$$s\left(1-\frac{B}{L}\right) - s_1 N + s_2 C = 0 \quad (6)$$

$$r\left(1-\frac{N}{K}\right) + r_1 B = 0 \quad (7)$$

$$Q + \delta N - \delta_0 C - \delta_1 BC = 0 \quad (8)$$

From (7) and (8), we get respectively,

$$N = K\left(1 + \frac{r_1 B}{r}\right) \quad (9)$$

$$C = \frac{Q + \delta K\left(1 + \frac{r_1 B}{r}\right)}{\delta_0 + \delta_1 B} \quad (10)$$

Using (9) and (10) in (6), we get

$$aB^2 + bB - c = 0 \quad (11)$$

Where

$$a = \delta_1\left(\frac{s}{L} + \frac{K}{r} r_1 s_1\right)$$

$$b = \frac{s\delta_0}{L} + s_1 \delta_1 K + \frac{K}{r} r_1 s_1 \delta_0 - \delta_1 s - \frac{K}{r} r_1 s_2 \delta$$

$$c = \delta_0 (s - s_1 K) + s_2 (Q + \delta K)$$

From (11), we get, $B = \frac{-b + \sqrt{b^2 + 4ac}}{2a} = B^*$ (say).

Thus, we have a unique positive root (say B^*), provided,

$$s - s_1 K \ge 0 \quad (12)$$

which makes $c > 0$.

Using this value of B^*, we will get the values of N^* and C^* from equations (9) and (10) respectively.

Remark: From equation (11), it can be checked that $\frac{dB}{dQ} > 0$ and $\frac{dB}{d\delta} > 0$. This implies that the equilibrium density of plant biomass increases due to increase in the rates of introduction of carbon dioxide by natural factors as well as by population density.

Similarly, it can also be proved that, $\frac{dB}{dr} < 0$ and $\frac{dB}{ds_1} < 0$. This implies that equilibrium density of plant biomass decreases as the growth rate of population density increases.

2.1.2. Stability Analysis

To establish the local stability behaviour of E_0, E_1 and E_2 , we compute the following general variational matrix M for model system (1) – (3),

$$M = \begin{bmatrix} s\left(1-\frac{2B}{L}\right)-s_1N+s_2C & -s_1B & s_2B \\ r_1N & r\left(1-\frac{2N}{K}\right)+r_1B & 0 \\ -\delta_1 C & \delta & -(\delta_0+\delta_1 B) \end{bmatrix}$$

From the matrix, corresponding to $E_0\left(0,\ 0,\ \frac{Q}{\delta_0}\right)$, it can be easily seen that two eigenvalues of the variational matrix will always be positive and hence it is unstable. Also corresponding to $E_1\left(0,\ K,\ \frac{Q+\delta K}{\delta_0}\right)$, we may easily note that one eigenvalue of the variational matrix is $(s-s_1K)+s_2\frac{Q+\delta K}{\delta_0}$, which is positive for the existence of E^* and hence $E_1\left(0,\ K,\ \frac{Q+\delta K}{\delta_0}\right)$ is unstable whenever E^* exist. Further, corresponding to $E_2(\bar{B},\ 0,\ \bar{C})$, we may note that one eigenvalue of the variational matrix is always positive and hence it is also unstable.

The local stability behaviour of interior equilibrium $E^*(B^*,\ N^*,\ C^*)$ is given in the following theorem.

Theorem 2.1.1. Let the following inequality holds

$$S_1 = \frac{r}{K}C^*(\delta_0+\delta_1 B^*) - \frac{r_1 s_2}{\delta_1 s_1}\delta^2 > 0 \tag{13}$$

Then the equilibrium $E^*(B^*,\ N^*,\ C^*)$ is locally asymptotically stable.

(See appendix A for proof)

Theorem 2.1.2. Let the following inequality holds in Ω

$$S_2 = \frac{r}{K}\delta_0 C^* - \frac{r_1 s_2}{\delta_1 s_1}\delta^2 > 0 \tag{14}$$

Then the interior equilibrium $E^*(B^*,\ N^*,\ C^*)$ is nonlinearly asymptotically stable.

(See appendix B for proof)

The above theorems imply that under appropriate conditions the dynamical system would remain in equilibrium. The density of plant biomass decreases as the density of population increases and it increases as the rate of introduction of CO_2 into the atmosphere increases.

Remark: If the rate of introduction of CO_2 due to population is zero (i.e. $\delta = 0$), the stability conditions (13) and (14) are satisfied automatically. This implies that δ has destabilizing effect on the system.

2.2 Case II. $Q > 0,\ \delta = 0$

In this case also, we need the bounds of dependent variables, to analyze the model system (1) – (3). For this purpose, we establish the region of attraction in the following lemma [9].

Lemma 2.2.1. The set

$$\Pi = \left\{ \begin{array}{l} (B,N,C) \in R_+^3 : 0 \le B \le B_n; \\ 0 \le N \le \frac{K}{r}(r + r_1 B_n);\ 0 \le C \le \frac{Q}{\delta_0} \end{array} \right\}$$

is region of attraction for all solutions of the model system (1) – (3) initiating in the interior of positive octant, where, $B_n = \frac{L}{s}\left(s + s_2\frac{Q}{\delta_0}\right)$.

2.2.1. Equilibrium Analysis

In this case also, the model system (1) – (3) has four non-negative equilibria;

(i) $\tilde{E}_0\left(0,\ 0,\ \frac{Q}{\delta_0}\right)$

(ii) $\tilde{E}_1\left(0,\ K,\ \frac{Q}{\delta_0}\right)$

(iii) $\tilde{E}_2(\tilde{B}_1,\ 0,\ \tilde{C}_1)$

(iv) $\tilde{E}(\tilde{B},\ \tilde{N},\ \tilde{C})$

The existence of $\tilde{E}_0$ and $\tilde{E}_1$ is obvious hence omitted. The proof of existing $\tilde{E}_2(\tilde{B}_1,\ 0,\ \tilde{C}_1)$ in this case is the same as in the case (i) hence it is omitted.

2.2.1.1. Existence of $\tilde{E}(\tilde{B},\ \tilde{N},\ \tilde{C})$

The positive solution of $\tilde{E}(\tilde{B},\ \tilde{N},\ \tilde{C})$ can be obtained by solving the following algebraic equations,

$$s\left(1-\frac{B}{L}\right) - s_1N + s_2C = 0 \tag{15}$$

$$r\left(1-\frac{N}{K}\right) + r_1B = 0 \tag{16}$$

$$Q - \delta_0 C - \delta_1 BC = 0 \tag{17}$$

From (16) and (17), we get respectively,

$$N = K\left(1+\frac{r_1 B}{r}\right) \tag{18}$$

$$C = \frac{Q}{\delta_0 + \delta_1 B} \tag{19}$$

Using (18) and (19) in (15), we get

$$\alpha B^2 + \beta B - \gamma = 0 \tag{20}$$

where

$$\alpha = \delta_1\left(\frac{s}{L} + \frac{K}{r}r_1 s_1\right)$$

$$\beta = \frac{s\delta_0}{L} + s_1\delta_1 K + \frac{K}{r}r_1 s_1\delta_0 - \delta_1 s$$

$$\gamma = \delta_0(s - s_1 K) + s_2 Q$$

From (20), we get,

$$B = \frac{-\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha} = \tilde{B}(\text{say}) \quad (21)$$

Thus, we have a unique positive root (say $\tilde{B}$), provided (12), Using this value of $\tilde{B}$, we will get the values of $\tilde{N}$ and $\tilde{C}$ from equations (18) and (19) respectively.

Remark: From equation (20), as before, it can be checked that $\frac{dB}{dQ} > 0$. This implies that the equilibrium density of plant biomass increases as the rate of introduction of carbon dioxide increases due to natural factors. Similarly, it can also be proved that, $\frac{dB}{dr} < 0$ and $\frac{dB}{ds_1} < 0$. It means that equilibrium density of plant biomass decreases as the growth rate of population increases.

2.2.2. Stability analysis

As in case (I), the local stability behaviour of $\tilde{E}_0$, $\tilde{E}_1$ and $\tilde{E}_2$ can be easily checked by computing the variational matrix for model system (1) – (3) and it is found that $\tilde{E}_0$, $\tilde{E}_1$ and $\tilde{E}_2$ are unstable.

The stability behaviour of interior equilibrium $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$ is given in the following theorems.

Theorem 2.2.1. The interior equilibrium $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$ is locally asymptotically stable without any condition.

(See appendix C for proof)

Theorem 2.2.2. The interior equilibrium $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$ is nonlinearly asymptotically stable inside the region of attraction Π without any condition.

(See appendix D for proof)

The above theorems imply that the dynamical system would remain in equilibrium without any condition.

3. Numerical Simulation and Discussion

In this section, we conduct some numerical simulations using MAPLE 7 to check the feasibility of the dynamical system regarding stability conditions by choosing the following set of parameter values in model system (1) – (3). We have performed the numerical simulation only for case (I).

$$s = 0.2, L = 2000, s_1 = 0.00001,$$
$$s_2 = 0.000005, r = 0.1, K = 1500,$$
$$r_1 = 0.0000001, Q = 5, \delta = 0.1,$$
$$\delta_0 = 0.001, \delta_1 = 0.0005$$

The equilibrium values of $E^*(B^*, N^*, C^*)$ corresponding to above data are given as,

$$B^* = 1858.0692, N^* = 1502.7871, C^* = 166.9601$$

The eigenvalues of variational matrix corresponding to equilibrium $E^*(B^*, N^*, C^*)$ for the model system (1) – (3) are -0.1868, -0.9289 and -0.1002. It is noted here that all eigenvalues of the variational matrix are negative. Hence the interior equilibrium $E^*(B^*, N^*, C^*)$ is locally asymptotically stable. For the above set of parameter values, the condition of existence of interior equilibrium $E^*(B^*, N^*, C^*)$ i.e. (12), local stability condition i.e. (13) and nonlinear stability condition i.e. (14) are satisfied.

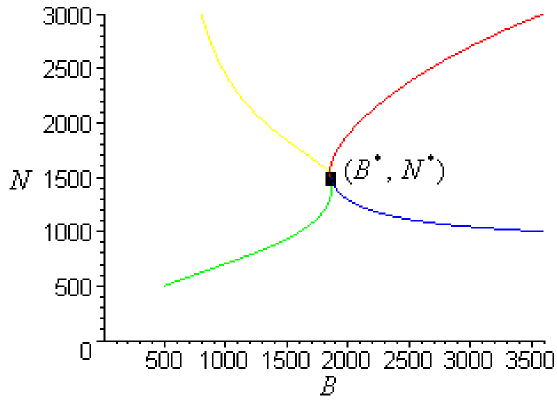

Figure 1. Nonlinear stability in B-N plane

The nonlinear stability behaviour of $E^*(B^*, N^*, C^*)$ in $B-N$ and $B-C$ plane has been shown in figure 1 and figure 2 respectively, with different initial starts. From these figures, it can be seen that all the trajectories initiating inside the region of attraction Ω are approaching towards the equilibrium values (B^*, N^*) and (B^*, C^*) respectively. The variation of concentration of carbon dioxide and density of biomass with time 't' for different values of δ (i.e. at $\delta = 0.1, 0.3, 0.5$) is shown in figure 3 and figure 4, respectively. From these figures, it is shown as the rate of introduction of CO_2 due to population density increases, the equilibrium concentration of carbon dioxide into the atmosphere increases and hence the density of biomass. The variation of concentration of carbon dioxide and biomass density with time 't' for different values of s_2 (i.e. at $s_2 = 0.000005, 0.000025, 0.00005$) is shown in figure 5 and figure 6, respectively. From these figures it is observed that as the rate of uptake of carbon dioxide with biomass increases, the equilibrium concentration of carbon dioxide in the atmosphere decreases while the density of biomass increases [3,14,17,20,28]. The variation of densities of biomass and population with time 't' for different values of r_1 is shown in figures 7 and 8 respectively. From figure 7, we note that the equilibrium density of plant biomass decreases as the growth rate of population density increases [1,10,13,19]. From figure 8, we depict that the density of population increases as r_1 increases.

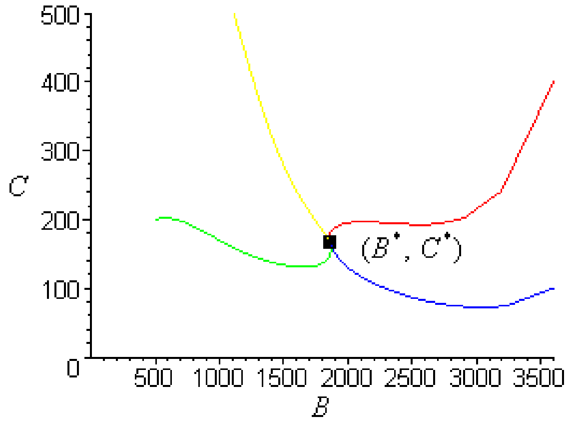

Figure 2. Nonlinear stability in B-C plane

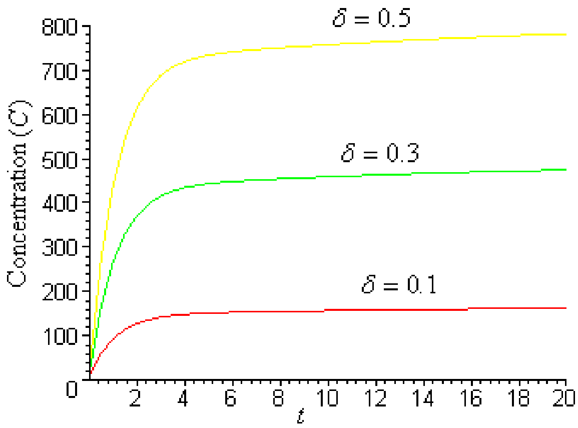

Figure 3. Variation of concentration of CO_2 with time 't' for different values of δ

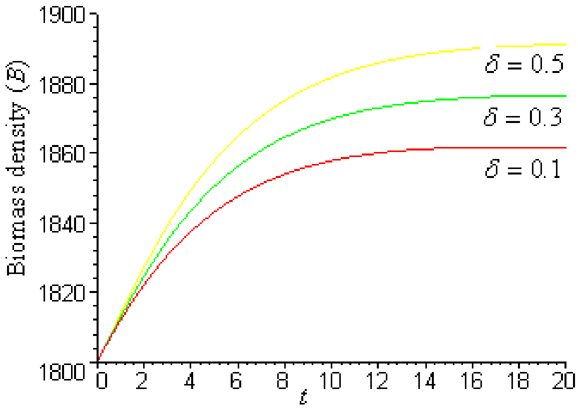

Figure 4. Variation of biomass density with time 't' for different values of δ

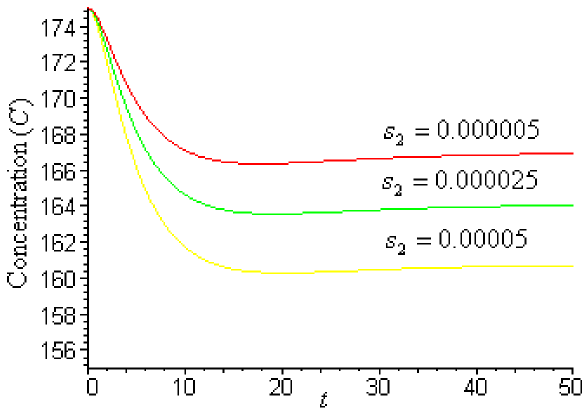

Figure 5. Variation of concentration of $CO_2$with time 't' for different values of S_2

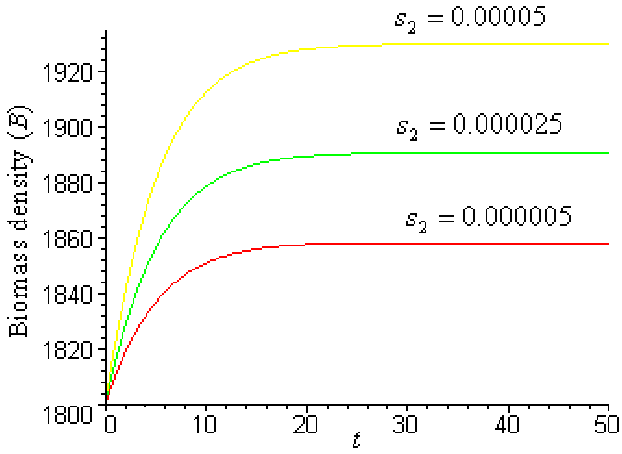

Figure 6. Variation of biomass density with time 't' for different values of S_2

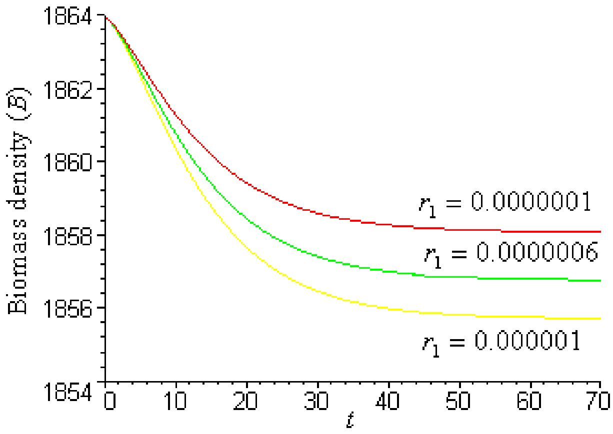

Figure 7. Variation of biomass with time 't' for different values of r_1

The variation of plant biomass density and concentration of CO_2 for different values of Q and δ with time 't' is shown in figures 9 and 10 respectively. From these figures, it is observed that, when $Q = 5$ and $\delta = 0$, the equilibrium density of biomass and concentration of CO_2 are much less than their values when $Q = 5$ and $\delta = 0.1$; $Q = 0$ and $\delta = 0.1$. This implies that the population density has significant effect on carbon dioxide concentration in the atmosphere.

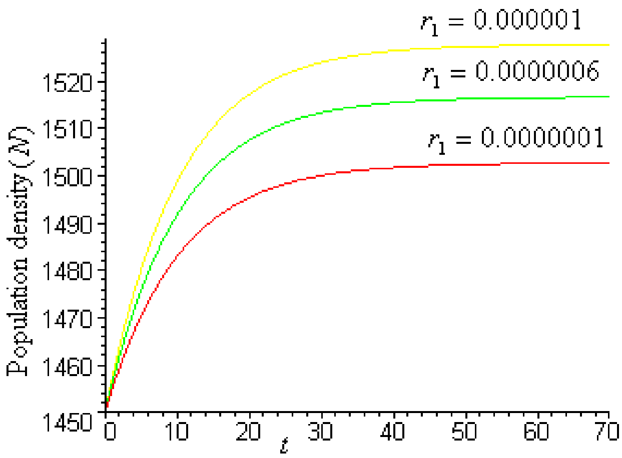

Figure 8. Variation of population density with time 't' for different values of r_1

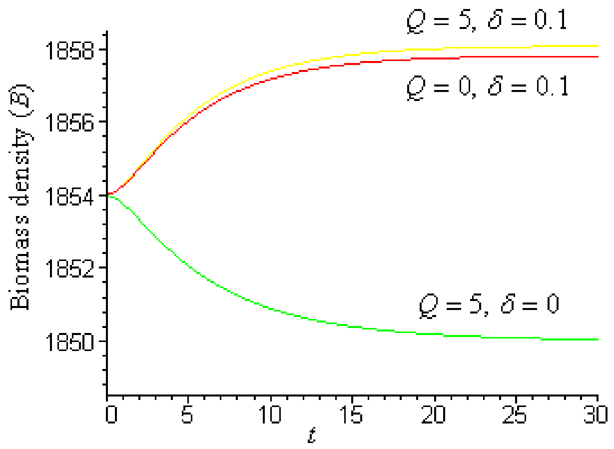

Figure 9. Variation of biomass (B)with time 't' for different values of Q and δ

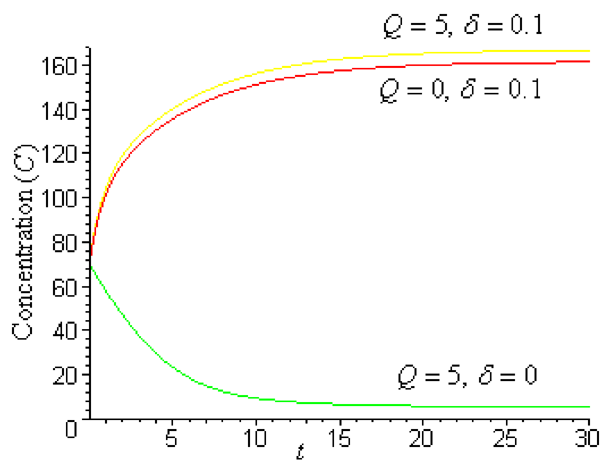

Figure 10. Variation of concentration of CO_2 with time 't' for different values of Q and δ

4. Conclusion

In this paper, a nonlinear mathematical model is proposed and analyzed to study the effect of human population density dependent emission of carbon dioxide on the growth of plant biomass. The dynamics of the model system consists of three nonlinearly interacting dependent variables, namely; plant biomass density, population density and concentration of carbon dioxide. The model analysis is conducted using stability theory of nonlinear ordinary differential equations. It is shown, analytically and numerically, that the density of plant biomass decreases due to increase in population density [1,5,19] and increases due to the presence of carbon dioxide in the atmosphere [4,18,21]. Thus, the plant biomass density increases as the rate of introduction of CO_2, due to natural as well as human activity related factors, increases and it decreases as the growth rate of population increases. It is also shown that the concentration of carbon dioxide increases as the density of human population increases.

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Appendix A

Proof of the theorem 2.1.1.

To establish the local stability behaviour of $E^*(B^*, N^*, C^*)$, we consider the following positive definite function

$$V = \frac{1}{2}[B_1{}^2 + k_1N_1{}^2 + k_2C_1{}^2] \qquad \text{(A1)}$$

Where B_1, N_1, C_1 are small perturbations about $E^*(B^*, N^*, C^*)$. The constants $k_i \ (i = 1,2)$ are all positive to be chosen appropriately.

Differentiating (A1) with respect to $'t'$, we get

$$\frac{dV}{dt} = B_1\frac{dB}{dt} + k_1N_1\frac{dN}{dt} + k_2C_1\frac{dC}{dt}$$

Now using the linearized system of (1) – (3), we get,

$$\frac{dV}{dt} = -\frac{sB^*}{L}B_1{}^2 - k_1\frac{rN^*}{K}N_1{}^2 - k_2(\delta_0 + \delta_1B^*)C_1{}^2 +$$
$$(-s_1B^* + k_1r_1N^*)B_1N_1 + (s_2B^* - k_2\delta_1C^*)B_1C_1 + (k_2\delta)N_1C_1$$

Choosing $k_1 = \frac{s_1B^*}{r_1N^*}$ and $k_2 = \frac{s_2B^*}{\delta_1C^*}$, $\frac{dV}{dt}$ will be negative definite provided the condition (13) is satisfied and hence the theorem.

Appendix B

Proof of the theorem 2.1.2.

To establish the nonlinear stability behaviour of $E^*(B^*, N^*, C^*)$, we consider the following positive definite function

$$U = \left(B - B^* - B^*\log\frac{B}{B^*}\right) + m_1\left(N - N^* - N^*\log\frac{N}{N^*}\right) + \frac{1}{2}m_2(C - C^*)^2 \qquad \text{(B1)}$$

where $m_i \ (i = 1,2)$ are positive constants to be chosen appropriately.

Differentiating (B1) with respect to $'t'$ along the model system (1) – (3), we get,

$$\frac{dU}{dt} = (B - B^*)\frac{1}{B}\frac{dB}{dt} + m_1(N - N^*)\frac{1}{N}\frac{dN}{dt}$$
$$+m_2(C - C^*)\frac{dC}{dt}$$

Now substituting the values of $\frac{dB}{dt}$, $\frac{dN}{dt}$ and $\frac{dC}{dt}$ from the model system (1) – (3) about $E^*(B^*, N^*, C^*)$, we get

$$\frac{dV}{dt} = -\frac{s}{L}(B - B^*)^2 - m_1\frac{r}{K}(N - N^*)^2$$
$$-m_2\delta_0(C - C^*)^2 - m_2\delta_1B(C - C^*)^2$$
$$+(-s_1 + m_1r_1)(B - B^*)(N - N^*)$$
$$+(s_2 - m_2\delta_1C^*)(B - B^*)(C - C^*)$$
$$+(m_2\delta)(N - N^*)(C - C^*)$$

Choosing $m_1 = \frac{s_1}{r_1}$ and $m_2 = \frac{s_2}{\delta_1C^*}$, $\frac{dU}{dt}$ will be negative definite inside the region of attraction Ω_s provided the condition (14) is satisfied and hence the theorem.

Appendix C

Proof of the theorem 2.2.1.

To establish the local stability behaviour of $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$, we consider the following positive definite function

$$W_1 = \frac{1}{2}[B_{11}{}^2 + k_{11}N_{11}{}^2 + k_{21}C_{11}{}^2] \qquad \text{(C1)}$$

Where B_{11}, N_{11}, C_{11} are small perturbations about $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$. The constants k_{11} and k_{21} are positive to be chosen appropriately.

Differentiating (C1) with respect to $'t'$, we get

$$\frac{dW_1}{dt} = B_{11}\frac{dB_{11}}{dt} + k_{11}N_{11}\frac{dN_{11}}{dt} + k_{21}C_{11}\frac{dC_{11}}{dt}$$

Now using the linearized system of (1) – (3), we get,

$$\frac{dW_1}{dt} = -\frac{s\tilde{B}}{L}B_{11}{}^2 - k_{11}\frac{r\tilde{N}}{K}N_{11}{}^2 - k_{21}(\delta_0 + \delta_1\tilde{B})C_{11}{}^2$$
$$+(-s_1\tilde{B} + k_{11}r_1\tilde{N})B_{11}N_{11} + (s_2\tilde{B} - k_{21}\delta_1\tilde{C})B_{11}C_{11}$$

Choosing $k_{11} = \frac{s_1\tilde{B}}{r_1\tilde{N}}$ and $k_{21} = \frac{s_2\tilde{B}}{\delta_1\tilde{C}}$, $\frac{dW_1}{dt}$ will be negative definite without any condition and hence the theorem.

Appendix D

Proof of the theorem 2.2.2.

To establish the nonlinear stability behaviour of $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$, we consider the following positive definite function

$$W_2 = \left(B - \tilde{B} - \tilde{B}\log\frac{B}{\tilde{B}} \right) + m_{11}\left(N - \tilde{N} - \tilde{N}\log\frac{N}{\tilde{N}} \right) + \frac{1}{2}m_{21}(C - \tilde{C})^2 \quad \text{(D1)}$$

where m_{11} and m_{21} are positive constants to be chosen appropriately.

Differentiating (D1) with respect to $'t'$ along the model system (1) – (3), we get,

$$\frac{dW_2}{dt} = (B - \tilde{B})\frac{1}{B}\frac{dB}{dt} + m_{11}(N - \tilde{N})\frac{1}{N}\frac{dN}{dt} + m_{21}(C - \tilde{C})\frac{dC}{dt}$$

Now substituting the values of $\frac{dB}{dt}$, $\frac{dN}{dt}$ and $\frac{dC}{dt}$ from the model system (1) – (3) about $\tilde{E}(\tilde{B}, \tilde{N}, \tilde{C})$, we get

$$\begin{aligned}\frac{dW_2}{dt} &= -\frac{s}{L}(B - \tilde{B})^2 - m_{11}\frac{r}{K}(N - \tilde{N})^2 \\ &-m_{21}\delta_0(C - \tilde{C})^2 - m_{21}\delta_1 B(C - \tilde{C})^2 \\ &+(-s_1 + m_{11}r_1)(B - \tilde{B})(N - \tilde{N}) \\ &+(s_2 - m_{21}\delta_1\tilde{C})(B - \tilde{B})(C - \tilde{C})\end{aligned}$$

Choosing $m_{11} = \frac{s_1}{r_1}$ and $m_{21} = \frac{s_2}{\delta_1\tilde{C}}$, $\frac{dW_2}{dt}$ will be negative definite inside the region of attraction Π without any condition and hence the theorem.