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Recurrence Relations for Moments of Generalized Order Statistics from Marshall – Olkin extended Kumaraswamy distribution and its Characterization

Ibrahim B. Abdul-Moniem*

Department of Quantitative Methods, College of Commerce – Aswan University, Aswan, Egypt
*Corresponding author: ibtaib@hotmail.com

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Abstract In this paper, Marshall – Olkin extended Kumaraswamy distribution will be obtained. We give some properties for this distribution. Recurrence relations for single and product moments of generalized order statistics from Marshall – Olkin extended Kumaraswamy distribution have been obtained. Specializations to order statistics and records have been made. Further, using a recurrence relation for single moments we obtain characterization of Marshall – Olkin extended Kumaraswamy distribution.

Keywords: *Kumaraswamy distribution, marshall, Olkin extended distributions, generalized order statistics, order statistics, records, single and product moments, recurrence relations, characterization*

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1. Introduction

A random variable X is said to have a Kumaraswamy distribution (KD) if its probability density function is (*pdf*) in the form:

$$f(x;\lambda,\beta)=\lambda\beta x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}; \quad 0\le x\le 1,\quad \lambda,\beta>0 \tag{1.1}$$

The cumulative distribution func-tion (*CDF*) and survival function (*SF*) are:

$$F(x;\lambda,\beta)=1-\left(1-x^{\lambda}\right)^{\beta}; \quad 0\le x\le 1,\quad \lambda,\beta>0 \tag{1.2}$$

and

$$\overline{F}(x;\lambda,\beta)=\left(1-x^{\lambda}\right)^{\beta}; \quad 0\le x\le 1,\quad \lambda,\beta>0 \tag{1.3}$$

More details on this distribu-tion and its applications can be found in Kumaraswamy [19], Sundar and Subbiah [24], Fletcher and Ponnambalam [10], Seifi et al. [23], Ganji et al. [11], Sanchez et al. [22] and Courard-Hauri [9].

Marshall and Olkin [19] intr-oduced a new method of adding a parameter into a family of distrib-utions. According to them if $\overline{F}(x)$ denote the survival or reliability function of a continuous random variable X then the timely honored device of adding a new parameter results in another survival function $\overline{G}(x)$ defined by

$$\overline{G}(x)=\frac{\alpha\overline{F}(x)}{1-\overline{\alpha}\,\overline{F}(x)}; \quad -\infty<x<\infty,\alpha>0 \tag{1.4}$$

and $\overline{\alpha}=1-\alpha$.

The *pdf* and hazard rate function (*HRF*) corresponding to $\overline{G}(x)$ are:

$$g(x)=\frac{\alpha f(x)}{\left[1-\overline{\alpha}\,\overline{F}(x)\right]^{2}}; \quad -\infty<x<\infty, \quad \alpha>0 \text{ and } \overline{\alpha}=1-\alpha, \tag{1.5}$$

and

$$r(x)=\frac{\alpha h(x)}{1-\overline{\alpha}\,\overline{F}(x)}; \quad -\infty<x<\infty, \quad \alpha>0 \text{ and } \overline{\alpha}=1-\alpha, \tag{1.6}$$

where, $h(x)$ is the HRF correspo-nding $f(x)$.

Here, we substituting from (1.1) and (1.3) in (1.5) to obtain a new distribution called Marshall-Olkin extended Kumaraswamy distribu-tion (*M-OEKD*) as follows

$$g(x)=\frac{\alpha\lambda\beta x^{\lambda-1}\left(1-x^{\lambda}\right)^{\beta-1}}{\left[1-\overline{\alpha}\left(1-x^{\lambda}\right)^{\beta}\right]^{2}}; 0\le x\le 1, \quad \lambda,\beta>0 \text{ and } \overline{\alpha}=1-\alpha \tag{1.7}$$

The *CDF*, *SF* and *HRF* corresponding to $g(x)$ are:

$$G(x)=\frac{\alpha\left(1-x^{\lambda}\right)^{\beta}}{1-\bar{\alpha}\left(1-x^{\lambda}\right)^{\beta}};\ 0\le x\le 1, \tag{1.8}$$
$\lambda,\beta>0$ and $\bar{\alpha}=1-\alpha$

$$\bar{G}(x)=\frac{\alpha\left(1-x^{\lambda}\right)^{\beta}}{1-\bar{\alpha}\left(1-x^{\lambda}\right)^{\beta}};\ 0\le x\le 1, \tag{1.9}$$
$\lambda,\beta>0$ and $\bar{\alpha}=1-\alpha$

and

$$r(x)=\frac{\lambda\beta x^{\lambda-1}}{\left(1-x^{\lambda}\right)\left[1-\bar{\alpha}\left(1-x^{\lambda}\right)^{\beta}\right]}; \tag{1.10}$$
$0\le x\le 1,\ \lambda,\theta>0$ and $\bar{\alpha}=1-\alpha$

In view of (1.7) and (1.9), we have

$$\bar{G}(x)=\frac{g(x)}{\lambda\beta}\left[\left(x^{1-\lambda}-x\right)-\bar{\alpha}\sum_{i=1}^{\infty}\binom{\beta}{i}(-1)^{i}\left(x^{1-\lambda(1-i)}-x^{1+\lambda i}\right)\right] \tag{1.11}$$

The inverse of the distribution function (1.8) yields a very simple quantile function

$$Q(y)=\left[1-\left(\frac{1-y}{1-\bar{\alpha}y}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}};y\in(0,1) \tag{1.12}$$

which facilitates ready quantile based statistical modeling. In addition, $Q(y)$ gives a trivial random variable generation. If $U\sim u(0,1)$, then $X\sim M\text{-}OEKD(\alpha,\beta,\lambda)$ is given by

$$X=\left[1-\left(\frac{1-U}{1-\bar{\alpha}U}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}. \tag{1.13}$$

The mode for *M-OEKD* can be obtained as the root of the following equation:

$$(\lambda-1)-x^{\lambda}(\lambda\beta-1)-\bar{\alpha}\left(1-x^{\lambda}\right)^{\beta}\left[x^{\lambda}(\lambda\beta+1)+(\lambda-1)\right]=0 \tag{1.14}$$

and the median for *M-OEKD* is

$$X=\left[1-\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\beta}}\right]^{\frac{1}{\lambda}}. \tag{1.15}$$

This distribution can be applied on some real percentage data. Carrasco et al [16] applied Generalized Kumaraswamy Distribution on the observed percentage of children living in households with per capita income less than R$ 75.50 in 1991 in 5509 Brazilian municipal districts.

The concept of generalized order statistics (*gos*) was introduced by Kamps [13]. A variety of order models of random variables is contained in this concept.

Let, for simplicity, F throughout denote an absolutely continuous distribution function with density function *f*. The random variables $X(1,n,\tilde{m},k),\ldots,\ X(n,n,\tilde{m},k)$ are called generalized order statistics based on F, if their joint *pdf* of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i}f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1}f(x_n)$$

for $F^{-1}(1)>x_1\ge x_2\ge\ldots\ge x_n>F^{-1}(0)$, with parameters $n\in N,\ n\ge 2,\ k>0,\quad \tilde{m}=(m_1,m_2,\ldots,m_{n-1})\in R^{n-1}$, $M_r=\sum_{i=r}^{n-1}m_i$, such that $\gamma_r=k+n-r+M_r>0$, for all $r\in\{1,\ 2,\ \ldots,\ n-1\}$.

For $\gamma_i\ne\gamma_j, i\ne j$ for all $i,j\in(1,2,\ldots,n)$ the *pdf* of $X(r,n,\tilde{m},k)$ is given by Kamps and Cramer [14] in the following way

$$f_{X(r,n,\tilde{m},k)}(x)=C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1} \tag{1.16}$$

The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1\le r<s\le n$ is given as

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)=C_{s-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \tag{1.17}$$

where $x<y$ and

$$a_i(r)=\prod_{\substack{j=1\\ j\ne i}}^{r}\frac{1}{\gamma_j-\gamma_i},\quad 1\le i\le r\le n,$$

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\ne i}}^{s}\frac{1}{\gamma_j-\gamma_i},\quad r+1\le i\le s\le n.$$

It may be noted that for $m_1=m_2=\ldots=m_{n-1}=m\ne -1$,

$$a_i(r)=\frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i}, \tag{1.18}$$

and

$$a_i^{(r)}(s)=\frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}. \tag{1.19}$$

Therefore *pdf* of $X(r,n,\tilde{m},k)$ given in (16) reduces to

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right], \tag{1.20}$$

and joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ given in (1.17) reduces to

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y)=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^{m} f(x)g_m^{r-1}\left[F(x)\right] \left\{h_m\left[F(y)\right]-h_m\left[F(x)\right]\right\}^{s-r-1} \left[\bar{F}(y)\right]^{\gamma_s-1} f(y),\quad x<y \tag{1.21}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\gamma_i=k+(n-i)(m+1)$$

$$h_m(x)=\begin{cases}\frac{-1}{m+1}x^{m+1}, & m\neq -1\\ -\ln x, & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(1),\quad x\in[0,\ 1).$$

We shall also take $X(0,n,m,k)=0$. If m = 0, k = 1, then $X(r,n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1,X_2,...,X_n$ and when m = -1, then $X(r,n,m,k)$ reduces to the k^{th} record values (Pawlas and Szynal [21]).

Many authors utilized the *gos* in their work, such as Kamps and Gather [15], Athar et al [7], Cramer and Kamps [8], Ahsanullah [4], Pawlas and Szynal [21], Ahmed [2], Ahmed and Fawzy [3], Khan et al. [17], AL-Hussaini et al. [5] and Kumar [18]. Abdul-Moniem [1] obtained recurrence relations for moments of lower *gos* from exponentiated Lomax distribution and its characterization.

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of *gos* from MOEKD family of life distributions. Further its various deductions and particular cases are discussed. Characterization of M-OEKD has been obtained on using a recurrence relation for single moments.

2. Single Moments of *gos*

Theorem 2.1. let X be a random variable has *pdf* (7). Then for integer j such that $j>0$, the follow-ing recurrence relation is satisfied.

$$E\left[X^{j}(r,n,\tilde{m},k)\right]=\frac{1}{\lambda\beta\gamma_r+j}\left\{\lambda\beta\gamma_r E\left[X^{j}(r-1,n,\tilde{m},k)\right]+jE\left[X^{j-\lambda}(r,n,\tilde{m},k)\right]\right\}-\frac{j\bar{\alpha}}{\lambda\beta\gamma_r+j}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\left\{E\left[X^{j-\lambda(1-i)}(r,n,\tilde{m},k)\right]+E\left[X^{j+\lambda i}(r,n,\tilde{m},k)\right]\right\} \tag{2.1}$$

Proof. We have from Lemma 2.3 (Athar and Islam [6]) that

$$E\left[\xi\left\{X(r,n,\tilde{m},k)\right\}\right]-E\left[\xi\left\{X(r-1,n,\tilde{m},k)\right\}\right]=C_{r-2}\int_{\theta}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}dx$$

If we let $\xi(x)=x^j$, then

$$E\left[X^{j}(r,n,\tilde{m},k)\right]-E\left[X^{j}(r-1,n,\tilde{m},k)\right]=jC_{r-2}\int_{\theta}^{\beta}X^{j-1}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}dx \tag{2.2}$$

On using (1.11) in (2.2), we get

$$E\left[X^{j}(r,n,\tilde{m},k)\right]-E\left[X^{j}(r-1,n,\tilde{m},k)\right]=\frac{j}{\lambda\beta\gamma_r}C_{r-1}\int_0^1 X^{j-1}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}\left[\left(x^{1-\lambda}-x\right)-\bar{\alpha}\sum_{i=1}^{\infty}\binom{\beta}{i}(-1)^i\left(x^{1-\lambda(1-i)}-x^{1+\lambda i}\right)\right]f(x)dx$$

$$=\frac{j}{\lambda\beta\gamma_r}C_{r-1}\int_0^1\left(x^{j-\lambda}-x^{j}\right)\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}f(x)dx-\frac{j\bar{\alpha}}{\lambda\beta\gamma_r}C_{r-1}\sum_{i=1}^{\infty}\binom{\beta}{i}(-1)^i\int_0^1\left(x^{j-\lambda(1-i)}-x^{j+\lambda i}\right)\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}f(x)dx$$

Which after simplification leads to (1.2).

Corollary 2.2. For $m_1=m_2=...=m_{n-1}=m\neq -1,$ the recurrence relations for single moment of *gos* for M-OEKD is given as

$$E\left[X^{j}(r,n,m,k)\right]=\frac{1}{\lambda\beta\gamma_r+j}\left\{\lambda\beta\gamma_r E\left[X^{j}(r-1,n,m,k)\right]+jE\left[X^{j-\lambda}(r,n,m,k)\right]\right\}-\frac{j\bar{\alpha}}{\lambda\beta\gamma_r+j}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\left\{E\left[X^{j-\lambda(1-i)}(r,n,m,k)\right]+E\left[X^{j+\lambda i}(r,n,m,k)\right]\right\} \tag{2.3}$$

Proof. This can easy be deduced from (2.1) in view of the relation (1.18).

Note that: We can obtain the recurrence relations for single moment of *gos* for Kumaraswamy distribution by taking $\alpha=1$ in (2.3), established by Kumar [18].

Remark 2.3 Putting $m=0,\ k=1$ in (2.3), we obtain recurrence relations for single moments of order statistics as

$$E\left[X_{r:n}^{j}\right]=\frac{\lambda\beta(n-r+1)}{\lambda\beta(n-r+1)+j}$$
$$E\left[X_{r-1:n}^{j}\right]+\frac{j}{\lambda\beta(n-r+1)+j}$$
$$\left\{E\left[X_{r:n}^{j-\lambda}\right]-\bar{\alpha}\sum_{l=0}^{\infty}\binom{\beta}{l}(-1)^{l}\right. \quad (2.4)$$
$$\left.\left[E\left[X_{r:n}^{j-\lambda(1-l)}\right]-E\left[X_{r:n}^{j+\lambda l}\right]\right]\right\}$$

Remark 2.4 Setting $m=-1,\ k\geq 1$ in (2.3), we obtain the recurrence relations of upper K-record values as

$$E\left[\left(X_{U(n)}^{j}\right)^{k}\right]=\frac{\lambda\beta k}{\lambda\beta k+j}E\left[\left(X_{U(n-1)}^{j}\right)^{k}\right]$$
$$+\frac{j}{\lambda\beta k+j}\left\{E\left[\left(X_{U(n)}^{j-\lambda}\right)^{k}\right]-\bar{\alpha}\sum_{l=0}^{\infty}\binom{\beta}{l}(-1)^{l}\right. \quad (2.5)$$
$$\left.\left[E\left[\left(X_{U(n)}^{j-\lambda(1-l)}\right)^{k}\right]-E\left[\left(X_{U(n)}^{j+\lambda l}\right)^{k}\right]\right]\right\}$$

3. Product Moments of *gos*

Theorem 3.1 let X be a random variable has *pdf* (1.7). Then for integer i,j such th-at $i,j>0$, the following recurrence relation is satisfied.

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right]=$$
$$\frac{\gamma_s\lambda\beta}{\gamma_s\lambda\beta+j}E\left[X^{i}(r,n,\tilde{m},k)X^{j}(s-1,n,\tilde{m},k)\right]+\frac{j}{\gamma_s\lambda\beta+j}$$
$$\{E\left[X^{i}(r,n,\tilde{m},k)X^{j-\lambda}(s,n,\tilde{m},k)\right] \quad (3.1)$$
$$-\bar{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!}E\begin{bmatrix}X^{i}(r,n,\tilde{m},k)\\ X^{j-\lambda(1-l)}(s,n,\tilde{m},k)\end{bmatrix}$$
$$\left.-E\left[X^{i}(r,n,\tilde{m},k).X^{j+\lambda l}(s,n,\tilde{m},k)\right]\right\}$$

Proof. We have from Lemma 3.2 (Athar and Islam [6]) that

$$E\left[\xi\{X(r,n,\tilde{m},k).X(s,n,\tilde{m},k)\}\right]-$$
$$E\left[\xi\{X(r,n,\tilde{m},k).X(s-1,n,\tilde{m},k)\}\right]=$$
$$C_{s-2}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma l}\frac{f(x)}{\bar{F}(x)}dydx$$

If we let $\xi(x,y)=x^{i}y^{j}$, then

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right]-$$
$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s-1,n,\tilde{m},k)\right]=$$
$$\frac{jC_{s-1}}{\gamma_s}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j-1}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma l}\frac{f(x)}{\bar{F}(x)}dydx$$

In view of (1.11), note that

$$\frac{\bar{F}(y)}{f(y)}=\frac{1}{\lambda\beta}\left[\left(y^{1-\lambda}-y\right)-\right.$$
$$\left.\bar{\alpha}\sum_{l=1}^{\infty}\binom{\beta}{l}(-1)^{l}\left(y^{1-\lambda(1-l)}-y^{1+\lambda l}\right)\right]$$

Therefore,

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right]-$$
$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s-1,n,\tilde{m},k)\right]=$$
$$\frac{jC_{s-1}}{\gamma_s\lambda\beta}\left\{\int_{0}^{1}\int_{x}^{1}x^{i}y^{j-\lambda}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma l}\right.$$
$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$
$$-\int_{0}^{1}\int_{x}^{1}x^{i}y^{j}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma l}$$
$$\left.\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx\right\}$$
$$-\frac{jC_{s-1}\bar{\alpha}}{\gamma_s\lambda\beta}\sum_{l=1}^{\infty}\binom{\beta}{l}(-1)^{l}\{\int_{0}^{1}\int_{x}^{1}x^{i}y^{j-\lambda(1-l)}$$
$$\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{ri}\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{ri}$$
$$\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx-\int_{0}^{1}\int_{x}^{1}x^{i}y^{j-\lambda l}$$
$$\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{ri}$$
$$\left.\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx\right\}$$

Which after simplification leads to (3.1).

Corollary 3.2. For $m_1=m_2=\ldots=m_{n-1}=m\neq -1$, the recurrence relations for product moments of *gos* for M-OEKD is given as

$$E\left[X^{i}(r,n,m,k).X^{j}(s,n,m,k)\right]=$$
$$\frac{\gamma_s\lambda\beta}{\gamma_s\lambda\beta+j}E\left[X^{i}(r,n,m,k).X^{j}(s-1,n,m,k)\right]$$
$$+\frac{j}{\gamma_s\lambda\beta+j}\{E[X^{i}(r,n,m,k)$$
$$X^{j-\lambda}(s,n,m,k)-\bar{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^{l}}{l!} \quad (3.2)$$
$$E\left[X^{i-\lambda}(r,n,m,k)X^{j-\lambda(1-l)}(s,n,m,k)\right]$$
$$\left.-E\left[X^{i}(r,n,m,k).X^{j+\lambda l}(s,n,m,k)\right]\right\}$$

Proof. This can easy be deduced from (3.1) in view of the relation (1.19).

Note that: We can obtain the recurrence relations for product moments of *gos* for Kumaraswamy distribution by taking $\alpha = 1$ in (3.2), established by Kumar [18].

Remark 3.3 Putting $m = 0$, $k = 1$ in (3.2), we obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right] = \frac{\lambda\beta(n-s-1)}{\lambda\beta(n-s-1)+j} E\left[X_{r,s-1:n}^{i,j}\right] + \frac{j}{\lambda\beta(n-s-1)+j}\left\{E\left[X_{r,s:n}^{i,j-\lambda}\right] - \bar{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[E\left[X_{r,s:n}^{i,j-\lambda(1-l)}\right] - E\left[X_{r,s:n}^{i,j+\lambda l}\right]\right]\right\} \quad (3.3)$$

Remark 3.4 Setting $m = -1$ in (3.2), we obtain the recurrence relations for product moments of k^{th} record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right] = \frac{\lambda\beta k}{\lambda\beta k + j} E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right] + \frac{j}{\lambda\beta k + j}\left\{E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-\lambda}\right] - \bar{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left[E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-\lambda(1-l)}\right] - E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j+\lambda l}\right]\right]\right\} \quad (3.4)$$

4. Characterization

Theorem 4.1 Let X be a non-negative random variable having an absolutely continuous distribution function *F* (*x*) with *F* (0) = 0 and 0 < *F* (*x*) < 1 for all *x* > 0, then

$$E\left[X^j(r,n,m,k)\right] = \frac{1}{\lambda\beta\gamma_r + j}\left\{\lambda\beta\gamma_r E\left[X^j(r-1,n,m,k)\right] + jE\left[X^{j-\lambda}(r,n,m,k)\right]\right\} - \frac{j\bar{\alpha}}{\lambda\beta\gamma_r + j}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\left\{E\left[X^{j-\lambda(1-i)}(r,n,m,k)\right] + E\left[X^{j+\lambda i}(r,n,m,k)\right]\right\} \quad (4.1)$$

if and only if $\bar{F}(x) = \dfrac{\alpha\left(1-x^\lambda\right)^\beta}{1-\bar{\alpha}\left(1-x^\lambda\right)^\beta}$.

Proof The necessary part follows immediately from equation (2.3). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.20), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 x^j\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx = \frac{1}{\lambda\beta\gamma_r + j}\left\{\frac{\lambda\beta\gamma_r C_{r-2}}{(r-2)!}\int_0^1 x^j\left[\bar{F}(x)\right]^{\gamma_r+m} f(x) g_m^{r-2}\left[F(x)\right]dx + \frac{jC_{r-1}}{(r-1)!}\int_0^1 x^{j-\lambda}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx\right\}$$
$$- \frac{j\bar{\alpha}C_{r-1}}{(\lambda\beta\gamma_r + j)(r-1)!}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\left\{\int_0^1 x^{j-\lambda(1-i)}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx - \int_0^1 x^{j+\lambda l}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx\right\}$$

Integrating the first integral on the right hand side of the above equation, by parts, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 x^j\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx = \frac{C_{r-1}}{(\lambda\beta\gamma_r + j)(r-1)!}\left\{\lambda\beta\gamma_r\int_0^1 x^j\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx - j\lambda\beta\int_0^1 x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}\left[F(x)\right]dx + j\int_0^1 x^{j-\lambda}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx\right\}$$
$$- \frac{j\bar{\alpha}}{\lambda\beta\gamma_r + j}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\left\{\frac{C_{r-1}}{(r-1)!}\int_0^1 x^{j-\lambda(1-i)}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx - \frac{C_{r-1}}{(r-1)!}\int_0^1 x^{j+\lambda l}\left[\bar{F}(x)\right]^{\gamma_r-1} f(x) g_m^{r-1}\left[F(x)\right]dx\right\}$$

Which is implies that

$$\frac{j\,C_{r-1}}{(\lambda\beta\gamma_r + j)(r-1)!}\int_0^1 x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1} g_m^{r-1}\left[F(x)\right]\left\{xf(x) + \lambda\beta\bar{F}(x) - x^{1-\lambda}f(x) + \bar{\alpha}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i f(x)\left[x^{1-\lambda(1-i)} - x^{1+\lambda i}\right]\right\}dx = 0 \quad (4.2)$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [12]) to equation (4.2), we get

$$\frac{\overline{F}(x)}{f(x)} = \left[\left(x^{1-\lambda} - x\right) - \overline{\alpha}\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^{i} \left(x^{1-\lambda(1-i)} - x^{1+\lambda i}\right)\right]$$

which prove that

$$\overline{F}(x) = \frac{\alpha\left(1-x^{\lambda}\right)^{\beta}}{1-\overline{\alpha}\left(1-x^{\lambda}\right)^{\beta}}$$

5. Conclusion

Here we propose a new model, the so-called the Marshall – Olkin exte-nded Kumaraswamy distribution which extends the Kumaraswamy distribution. Some properties for this distribution have been obtained. We give recurrence relations for single and product moments of generalized order statistics from Marshall – Olkin extended Kumaraswamy distribution. Specializations to order statistics and records have been made. We obtain characterization of Marshall – Olkin extended Kumaraswamy distribution using a recurrence relation for single mom-ents.

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