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Unique Lacunary interpolations with Estimate Errors Bound

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Abstract This paper presents a formulation of a Lacunary approximation for the class ninth of spline function at uniform mesh points and the function values at the end points of the interval. Error bounds for the function and its derivatives are derived. Finally, efficiency estimation and convergence orders are also illustrate errors derivations.

Keywords: *lacunary interpolations function, convergence analysis, differential equations*

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1. Introduction

Consider the initial value problem

$$y^{(q)}(x) = f\begin{pmatrix} x, y(x), y'(x), \\ ..., y^{(q)}(x) \end{pmatrix}, \; x \in [0, 1], \tag{1}$$

$$y(x_0) = y_1 \; , y'(x_1) = y'_2, ..., y^{(q-1)}(a) = y_n^{(q-1)}(x_n)$$

With the help of lacunary spline functions of type (0, 3, 5, 7) see) [8], by using that $f \in C^{n-1}([0,1] \times R^2)$, $n \geq 2$ and that it satisfies the Lipchitz continuous

$$\begin{aligned} &\left| f^{(q)}(x, y_1, y'_1) - f^{(q)}(x, y_2, y'_2) \right| \\ &\leq L\left\{ \left| y_1 - y_2 \right| + \left| y'_1 - y'_2 \right| \right\}, q = 0,1,...,n-1. \end{aligned} \tag{2}$$

Also initial value problems are satisfied, and for all $x \in [0,1]$ and for all real $y(x_0) = y_1, y'(x_1) = y'_2, ..., y^{(n-1)}(a) = y_n^{(n-1)}(x_n)$ from [5]. These conditions ensure the existence of unique solution of the problem (1).

Many phenomena in physics, engineering, and other sciences can be described very successfully by model using Mathematical tools from interpolation polynomials. The theory of interpolations polynomial and their applications are relatively recent development, classes of spline functions possess many nice structural properties as well as excellent approximation powers, since they are easy to store and the lacunary interpolation can be designed of curves and surfaces see [3,4,7]. Many researchers used different degree of spline functions of the type cubic, quadratic, quantic, and sixtic for different constructions, and also they obtained the error bounds for each case [1,6,9]. The purpose of this paper is continuous of the work [8], that he used new technique for ninth degree spline but in the article for seven degree spline.

2. Description of the Method

We present a ninth spline interpolation approximate for one dimensional and for a given sufficiently smooth $f(x)$ define on the interval $I = [a,b]$, and $\Delta_n : a = x_0 < x_1 < x_2 < ... < x_n = b$, denote the uniform partition of I with knots $x_i = a + ih$, where $i = 1, 2,.., n-1$ and $h = \frac{b-a}{n}$ is the length of each subintervals, and d the ninth spline is denoted by $p_\Delta(x)$ and defined on I as:

$$\begin{aligned} p(x) = y_0 + hy'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{6} y'''_0 + h^4 a_{0,4} + \frac{h^5}{120} y_0^{(5)} \\ + h^6 a_{0,6} + \frac{h^7}{5040} y_0^{(7)} + h^8 a_{0,8} + h^9 a_{0,9} \end{aligned} \tag{3}$$

On the subinterval $[x_0, x_1]$ where $a_{0,j}, j = 4, 6, 8$ and 9 are unknowns to be determined. Let as examine subintervals $[x_i, x_{i+1}], i = 1, 2,.., n-2$. By taking into account the interpolating conditions, form [8] provided that construction has been unique and the expression, for $p_i(x)$ in the follow form:

$$\begin{aligned} p_i(x) = y_i + ha_{i,1} + h^2 a_{i,2} + \frac{h^3}{6} y'''_i + h^4 a_{i,4} \\ + \frac{h^5}{120} y_i^{(5)} + h^6 a_{i,6} + \frac{h^7}{5040} y_i^{(7)} + h^8 a_{i,8} + h^9 a_{i,9} \end{aligned} \tag{4}$$

Where $a_{i,j}, i=1,2,...,n-1, j=1,2, 4, 6, 8$ and 9, which are determined. Now we define the new approximate polynomial on the subinterval $[x_0, x_1]$, as

$$\overline{p}_0(x) = \overline{y}_0 + h\overline{y}_0' + \frac{h^2}{2}\overline{y}_0'' + \frac{h^3}{6}\overline{y}_0''' + h^4\overline{a}_{0,4} + \frac{h^5}{120}\overline{y}_0^{(5)} + h^6\overline{a}_{0,6} + \frac{h^7}{5040}\overline{y}_0^{(7)} + h^8\overline{a}_{0,8} + h^9\overline{a}_{0,9};$$

$$\overline{p}_0(x) = \overline{y}_1;\ \overline{p}_0'''(x) = \overline{y}_1''';\ \overline{p}_0^{(5)}(x) = \overline{y}_1^{(5)} \text{ and } \overline{p}_0^{(7)}(x) = \overline{y}_1^{(7)} \quad (5)$$

Form the above boundary conditions, and [8] found the coefficients of $p_i(x)$ on $[x_0, x_1]$, as follows

$$\overline{a}_{0,4} = \frac{18}{13h^4}[\overline{y}_1 - \overline{y}_0] - \frac{18}{13h^3}\overline{y}_0' - \frac{9}{13h^2}\overline{y}_0'' - \frac{1}{312h}[5\overline{y}_1''' + 67\overline{y}_0'''] + \frac{h}{9360}[7\overline{y}_1^{(5)} - 40\overline{y}_0^{(5)}] - \frac{h^3}{157248}[4\overline{y}_1^{(7)} - 7\overline{y}_0^{(7)}],$$

$$\overline{a}_{0,6} = -\frac{6}{13h^6}[\overline{y}_1 - \overline{y}_0] + \frac{6}{13h^5}\overline{y}_0' + \frac{3}{13h^4}\overline{y}_0'' + \frac{1}{52h^3}[\overline{y}_1''' + 3\overline{y}_0'''] - \frac{1}{9360}[11\overline{y}_1^{(5)} + 43\overline{y}_0^{(5)}] + \frac{h}{786240}[37\overline{y}_1^{(7)} - 133\overline{y}_0^{(7)}],$$

$$\overline{a}_{0,8} = \frac{9}{91h^8}[\overline{y}_1 - \overline{y}_0] - \frac{9}{91h^7}\overline{y}_0' - \frac{9}{182h^6}\overline{y}_0'' - \frac{3}{728h^5}[\overline{y}_1''' + 3\overline{y}_0'''] + \frac{1}{36400h^3}[20\overline{y}_1^{(5)} + 2\overline{5}_0^{(5)}] - \frac{1}{1834560h}[64\overline{y}_1^{(7)} + 161\overline{y}_0^{(7)}],$$

and

$$\overline{a}_{0,9} = -\frac{2}{91h^9}[\overline{y}_1 - \overline{y}_0] + \frac{2}{91h^8}\overline{y}_0' + \frac{1}{91h^7}\overline{y}_0'' + \frac{1}{1092h^6}[\overline{y}_1''' + 3\overline{y}_0'''] - \frac{1}{163800h^4}[20\overline{y}_1^{(5)} - 25\overline{y}_0^{(5)}] + \frac{1}{5503680h^2}[73\overline{y}_1^{(7)} - 77\overline{y}_0^{(7)}].$$

The difference between polynomials $p_i(x)$ and $\overline{p}_i(x)$ obtain the new polynomial denoted by $s_i(x)$ and defined on the interval $[x_0, x_1]$, putting the value of $a_{0,j}$ and $\overline{a}_{0,j}$ where j=4,6,8 and 9 in $s_0(x)$ and $s_0^{(n)}(x)$, n=1,2,…,9.Also for $s_i(x)$ on the interval $[x_i, x_{i+1}], i=1, 2,.., n-2$, and satisfy the boundary conditions, we obtain the following theorem:

Theorem1. Let $\overline{y}_k^{(r)}$ $(r = 0,3, 5, 7; k = 0,1,2..., n)$ be the approximate values defined before. Then the following estimates of the spline function $\overline{s}_\Delta(x)$ are valid:

$$\left|P_k^{(q)}(x) - \overline{P}_k^{(q)}(x)\right| \le C_k\, h^{9-q}\, \omega_9(h);$$

for $q = 0,1,....,9$, $k = 0, 1,..., n-2$, where $x \in [x_0, x_1]$ and C_k denote the constants dependent of h, and $\omega_9(h) = \omega(h, y^{(9)})$ is the modulus continuity.

Proof The first construction polynomial from [8] and (3), in the first interval $[x_0, x_1]$, we have

$$\begin{aligned} s_0(x) &= p_0(x) - \overline{p}_0(x) \\ s_0(x) &= x^4(a_{0,4} - \overline{a}_{0,4}) + x^6(a_{0,6} - \overline{a}_{0,6}) + x^8(a_{0,8} - \overline{a}_{0,8}) + x^9(a_{0,9} - \overline{a}_{0,9}) \end{aligned} \quad (6)$$

$$\left|s_0(x)\right| \le x^4\left|a_{0,4} - \overline{a}_{0,4}\right| + x^6\left|a_{0,6} - \overline{a}_{0,6}\right| + x^8\left|a_{0,8} - \overline{a}_{0,8}\right| + x^9\left|a_{0,9} - \overline{a}_{0,9}\right| \le C_0\,\omega_9(h),$$

Where C_0 constant is depend of h, similarly form equation (3), we have

$$\left|s_0'''(x)\right| \le C_3\,\omega_9(h),$$

$$\left|s_0^{(5)}(x)\right| \le C_5\omega_9(h)$$

and

$$\left|s_0^{(7)}(x)\right| \le C_7\omega_9(h)$$

Where C_3, C_5 and C_7 constant is depend of h.

$$\left|s_0'(x)\right| = \left|P_0'(x) - \overline{P}_0'(x)\right| \le \frac{1}{1100736h}\left[3701376\left|y_1 - \overline{y}_1\right| + 29232h^3\left|y_1''' - \overline{y}_1'''\right| - 840h^5\left|y_1^{(5)} - \overline{y}_1^{(5)}\right| + 23h^7\left|y_1^{(7)} - \overline{y}_1^{(7)}\right|\right]$$

$$\left|s_0'(x)\right| \le \frac{1}{1100736h}\begin{bmatrix} 3701376C_1^* + 29232h^3C_2^* \\ -840h^5C_3^* + 23h^7C_4^* \end{bmatrix} \le \frac{1}{1100736h}C_1\,\omega_9(h)$$

Where $\begin{matrix} C_1 = 3701376C_1^* + 29232h^3C_2^* \\ -840h^5C_3^* + 23h^7C_4^* \end{matrix}$ constant is depend of h,

$$\left|s_0''(x)\right| \le \frac{1}{305760\,h^2}\begin{bmatrix} 2056320\,C_5^* + 67200h^3C_6^* \\ -1316h^5C_7^* + 33x^7C_8^* \end{bmatrix} \le \frac{1}{305760\,h^2}\,C_2\,\omega_9(h)$$

Where $\begin{matrix} C_2 = 2056320C_5^* + 67200h^3C_6^* \\ -1316h^5C_7^* + 33x^7C_8^* \end{matrix}$ constant is depend of h, $\left|s_0^{(4)}(x)\right| \le \frac{1}{32760h^4}C_4\,\omega_9(h)$

Where $\begin{array}{l} C_4 = 1088640 C_9^* + 78120 h^3 C_{10}^* \\ + 4872 h^5 C_{11}^* - 71 h^7 C_{12}^* \end{array}$ constant is depend of h, $\left|s_0^{(6)}(x)\right| \le \frac{1}{1092 h^6} C_6\, \omega_9(h)$

Where $\begin{array}{l} C_6 = 362880 C_{13}^* - 15120 h^3 C_{14}^* \\ + 3108 h^5 C_{15}^* + 145 h^7 C_{16}^* \end{array}$ constant is depend of h, $\left|s_0^{(8)}(x)\right| \le \frac{1}{1092 h^8} C_8\, \omega_9(h)$

Where $\begin{array}{l} C_8 = -362880 C_{17}^* + 15120 h^3 C_{18}^* \\ - 2016 h^5 C_{19}^* + 310 h^7 C_{20}^* \end{array}$ constant is depend of h, $\left|s_0^{(9)}(x)\right| \le C_9\, \omega_9(h)$

Where $\begin{array}{l} C_9 = -\frac{725760}{91 h^9} C_{21}^* + \frac{362880}{1092 h^6} C_{22}^* \\ - \frac{725760}{163800 h^4} C_{23}^* + \frac{26490240}{5503680 h^2} C_{24}^* \end{array}$ constant is depend of h, similarly on the interval $[x_i, x_{i+1}]$ can obtain the following:

$$\left|s_i(x)\right| \le C_{10}\, \omega_9(h), \left|s_i'''(x)\right| \le C_{13}\, \omega_9(h),$$

$$\left|s_i^{(5)}(x)\right| \le C_{15}\, \omega_9(h), and \left|s_i^{(7)}(x)\right| \le C_{17}\, \omega_9(h).$$

and for the other derivatives can be find as follows

$$\left|s_i'(x)\right| \le \frac{1}{1100736 h} C_{11}\, \omega_9(h),$$

where $\begin{array}{l} C_{11} = 3701376 C_{1i}^* - 2600640 h C_{2i}^* \\ -1499904 h^2 C_{3i}^* + 29232 h^3 C_{4i}^* \\ -840 h^5 C_{5i}^* + 23 h^7 C_{6i}^* \end{array}$ constant is depend of h,

$$\left|s_i''(x)\right| \le \frac{1}{305760 h^2} C_{12} \omega_9(h),$$

$$\left|s_i^{(4)}(x)\right| \le \frac{1}{32760 h^4} C_{14}\, \omega_9(h)$$

$$\left|s_i^{(6)}(x)\right| \le \frac{1}{1092 h^6} C_{16}\, \omega_9(h),$$

$$\left|s_i^{(8)}(x)\right| \le \frac{1}{91 h^8} C_{18}\, \omega_9(h),$$

and finally

$$\left|s_i^{(9)}(x)\right| \le -\frac{725760}{91 h^9} C_{19}\, \omega_9(h),$$

where C_{12}, C_{14}, C_{16}, C_{18} and C_{19} are constants depend of h.

Theorem 2: Consider $y(x)$ is the exact solution of problem (1) and $\bar{P}_\Delta(x)$ be the approximate value of the ninth degree spline function approximation then

$$\left|y_k^{(q)}(x) - \bar{P}_k^{(q)}(x)\right| \le D_k h^{9-q}\, \omega_9(h); \text{for } q = 0,1,....,9,$$

where $x \in [x_i, x_{i+1}]$, $i = 1, 2,.., n-2$ and D_k^* denote the difference constants dependent of h, and $\omega_9(h) = \omega(h, y^{(9)})$.

Proof: since

$$\left|y^{(q)}(x) - \bar{P}_\Delta^{(q)}(x)\right| \le \left|y^{(q)}(x) - P_\Delta^{(q)}(x)\right| + \left|P_\Delta^{(q)}(x) - \bar{P}_\Delta^{(q)}(x)\right|$$

From theorem 2 of [8], the following estimates are valid

$$\left|y^{(q)}(x) - \bar{P}_\Delta^{(q)}(x)\right| \le T_q h^{9-q} \omega_9(h) \qquad (7)$$

Using equation (7) and estimate in theorem1, we have

$$\begin{aligned} \left|y^{(q)}(x) - \bar{P}_\Delta^{(q)}(x)\right| &\le C_k h^{9-q} \omega_9(h) + T_k h^{9-q} \omega_9(h) \\ &= (C_k + T_k)\, h^{9-q} \omega_9(h) \\ &= D_k h^{9-q} \omega_9(h). \end{aligned}$$

Where T_q is a constant depending of h.

Theorem 3: If the function f in initial value problem (1) satisfies conditions (2) and (3), then the following inequalities are hold:

$$\left\|\bar{P}_0^{(r)}(x) - f\begin{bmatrix} x, \bar{P}_0\ (x), \\ \bar{P}_0'\ (x),.., \bar{P}_0^{(r)}(x) \end{bmatrix}\right\|_{Lp} \le H_{0,r}^*\ \omega_9(h),$$

where $H_{0,2}^*$ is constants dependent of h, $x \in [x_0, x_1]$ and $r = 0,1,,...,n-1$.

$$\left\|\bar{P}_K^{(r)}(x) - f\begin{bmatrix} x, \bar{P}_0\ (x), \\ \bar{P}_0'\ (x),.., \bar{P}_0^{(r)}(x) \end{bmatrix}\right\|_{Lp} \le H_{i,r}^*\ \omega_9(h)$$

where $H_{i,2}^*$ is constants dependent of h, $x \in [x_{i-1}, x_i]$ and $r = 0,1,,...,n-1$.

$$\left\|\bar{P}_{m-1}^{(r)}(x) - f\begin{bmatrix} x, \bar{P}_0\ (x), \\ \bar{P}_0'\ (x),.., \bar{P}_0^{(r)}(x) \end{bmatrix}\right\|_{Lp} \le H_{m-1,r}^*\ \omega_9(h),$$

where $H_{m-1,q}^*$ is constants dependent of h, $x \in [x_{m-1}, x_m]$ and $r = 0,1,,...,n-1$.

Proof: Using condition (1), (2) and (3), we have

$$\left\|D^r(f(x) - y(x))\right\|_{Lp} \le C_1 \omega_r(f; b-a)$$

and $\left\|D^r y(x)\right\|_{Lp} \le C_2\, \omega_r(f;1)$

by the Taylor expanssion of y about zero, then

$$\begin{aligned} \left|D^r(y(x) - \bar{P}_\Delta(x))\right| &\le \int_0^u \left|D^{r+1}(y - \bar{P}_\Delta)(u)\right| du \\ &\le \left\|D^{r+1}(y - \bar{S}_\Delta)\right\|_{Lp} \\ &\le \left\|D^r y\right\|_{Lp} \le C_3\, \omega_r(f; h) \end{aligned}$$

$$\left\| D^r (\bar{S}_0(x) - f(x)) \right\|_{Lp}$$
$$\leq \left\| D^r (\bar{S}_0 - y) \right\|_{Lp} + \left\| D^r y - f \right\|_{Lp},$$
$$\leq H^*_{0,r} \omega_9 (f; h)$$

Where $r = 0,1,,...,n-1$

Similarly for each the intervals can be proving it.

3. Conclusion

A new approximate polynomial is constructed which converts a errors estimations to its interpolation by a ninth spline model with error bound. The principal difference between the two spline interpolations showed slight superiority over the ninth spline model, the continuity of derivatives across element edges improves convergence for all coefficients. In this construct of approximate polynomial is established that reduces the total errors and order convergence also compared with that developed by [1], [2] and [9], the new methods enable us to the optimal minimize errors with exact solution.

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