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Generalized Contractions in Partially Ordered Metric Space with Rational Expressions and Related Fixed Point Results

Muhammed Raji[1], Laxmi Rathour[2,*], Vinay Singh[2], Mutalib Sadiq[3], Lakshmi Narayan Mishra[4], Vishnu Narayan Mishra[5]

[1]Department of Mathematics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria
[2]Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India
[3]Department of Mathematics, Federal University Lokoja, Nigeria
[4]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
[5]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India
*Corresponding author: laxmirathour817@gmail.com

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Abstract This paper aims to prove the existence and uniqueness of some fixed point for nonlinear contractive mappings in the setting of metric spaces and partially ordered metrics spaces satisfying a contraction condition of rational type. These contributions extend the existing literature on metric spaces and fixed point theory. Through illustrative examples, we showcase the practical applicability of our proposed notions and results, demonstrating their effectiveness in real-world scenarios.

Keywords: *Partially ordered metric space, Fixed point, Rational contractions, Monotone property*

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1. Introduction

Metric fixed point theory is widely recognized to have been originated in the work of Stefan Banach in 1922 called the classical Banach contraction principle (BCP) [1] which is one of the most notable results which has played a vital role in the development of a metric fixed point theory. The principle has been generalized by numerous authors in the different directions by improving the underlying contraction conditions, enhancing the number of involved mappings, weakening the involved metrical notions, and enlarging the class of ambient spaces [2,3,4,5,6,7].

In 2004, Ran and Reurings [8] obtained a new variant of the classical Banach contraction principle to a complete metric space endowed with partial order relation, which was slightly modified by Nieto and Rodriguez-Lopez [9] in 2005 and established fixed point results. Other generalizations of Banach contraction principle can be found in [10-21] [28-30].

Recently, Raji et. al. [22] obtained the existence and uniqueness of fixed points for rational type contraction mappings in a metric space that is equipped with a partial order.

Based on the above insight, we prove the existence and uniqueness of some fixed point for nonlinear contractive mappings in the setting of metric spaces and partially ordered metrics spaces satisfying a contraction condition of rational type. These contributions extend the existing literature on metric spaces and fixed point theory. To bolster our findings, we showcase the practical applicability of our proposed notions and results.

2. Preliminaries

The following are some of the definitions that are relevant in our study.

Definition 2.1 [23] The triple $(X, d, \preccurlyeq)$ is called partially ordered metric spaces, if $(X, \preccurlyeq)$ is a partial ordered set and (X, d) is a metric space.

Definition 2.2 [24,25] If (X, d) is a complete metric space, then the triple $(X, d, \preccurlyeq)$ is called complete partially ordered metric space.

Definition 2.3 [26,27] Let $(X, \preccurlyeq)$ be a partial ordered set and let $T : X \to X$ be a mapping. Then

(1). elements $x, y \in X$ are comparable, if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds;

(2). a non empty set X is called well ordered set, if every two elements of it are comparable.

Definition 2.4 [7] A partially ordered metric space $(X,d,\preccurlyeq)$ is called ordered complete if for each convergent sequence $\{x_n\}_{n=0}^{\infty}\subset X$, if x_n is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \preccurlyeq x$, for all $n\in\mathbb{N}$ that is, $x=\sup\{x_n\}$.

3. Main Results

Definition 3.1 Suppose $(X,\preccurlyeq)$ is a partially ordered set and $T:X\to X.T$ is said to be monotone non-decreasing if for all $x,y\in X$,

$$x\preccurlyeq y \text{ implies } Tx\preccurlyeq Ty \tag{3.1}$$

Theorem 3.2 Let $(X,d,\preccurlyeq)$ be a complete partially ordered metric space. Suppose that $T:X\to X$ is a continuous self-mapping on X,T is monotone non-decreasing mapping satisfying

$$\begin{aligned} d(Tx,Ty) &\le a_1\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \\ &+a_2\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}+a_3\frac{d(x,Tx)d(y,Ty)}{d(x,y)} \\ &+a_4\left[d(x,Tx)+d(y,Ty)\right]+a_5d(x,y) \end{aligned} \tag{3.2}$$

for all $x,y\in X, x\succcurlyeq y, x\neq y$ and for some $a_1,a_2,a_3,a_4,a_5\in[0,1)$ with $a_1+a_2+a_3+2a_4+a_5<1$. If there exists $x_0\in X$ with $x_0\preccurlyeq Tx_0$, then T has a fixed point.

Proof If $Tx_0=x_0$, then we have the result. Suppose that $x_0\prec Tx_0$. Since T is a monotone non-decreasing mapping, we obtain by induction that

$$x_0\prec Tx_0\preccurlyeq T^2x_0\preccurlyeq\cdots\preccurlyeq T^nx_0\preccurlyeq T^{n+1}x_0\preccurlyeq\cdots \tag{3.3}$$

Inductively, we can construct a sequence a sequence $\{x_n\}$ in X such that $x_{n+1}=Tx_n$, for every $n\ge 0$. Since T is monotone non-decreasing mapping, we obtain

$$x_0\preccurlyeq x_1\preccurlyeq x_2\preccurlyeq\cdots\preccurlyeq x_n\preccurlyeq x_{n+1}\preccurlyeq\cdots$$

If there exists $n\ge 1$ such that $x_n=x_{n+1}$, then from $x_{n+1}=Tx_n=x_n$, x_n is a fixed point and the proof is finish. Suppose that $x_n\neq x_{n+1}$, for all $n\ge 1$.

Since $x_n\succ x_{n-1}$, for all $n\ge 1$, from (3.2) we have

$$\begin{aligned} d(x_{n+2},x_{n+1}) &= d(Tx_{n+1},Tx_n) \\ &\le a_1\frac{d(x_{n+1},Tx_{n+1})d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1})d(x_n,Tx_n)}{d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)} \\ &+a_2\frac{d(x_{n+1},Tx_{n+1})d(x_n,Tx_n)}{d(x_{n+1},x_n)+d(x_{n+1},Tx_n)+d(x_n,Tx_{n+1})} \\ &+a_3\frac{d(x_{n+1},Tx_{n+1})d(x_n,Tx_n)}{d(x_{n+1},x_n)} \\ &+a_4\left[d(x_{n+1},Tx_{n+1})+d(x_n,Tx_n)\right]+a_5d(x_{n+1},x_n) \\ &=a_1\frac{d(x_{n+1},x_{n+2})d(x_{n+1},x_{n+1})+d(x_n,x_{n+2})d(x_n,x_{n+1})}{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})} \\ &+a_2\frac{d(x_{n+1},x_{n+2})d(x_n,x_{n+1})}{d(x_{n+1},x_n)+d(x_n,x_{n+2})} \\ &+a_3\frac{d(x_{n+1},x_{n+2})d(x_n,x_{n+1})}{d(x_{n+1},x_n)} \\ &+a_4\left[d(x_{n+1},x_{n+2})+d(x_n,x_{n+1})\right]+a_5d(x_{n+1},x_n) \end{aligned}$$

By triangular inequality $\begin{aligned}&(x_{n+1},x_{n+2})\\ &\le d(x_{n+1},x_n)+d(x_n,x_{n+2})\end{aligned}$, we have

$$\begin{aligned} &=a_1d(x_n,x_{n+1})+a_2d(x_n,x_{n+1}) \\ &+a_3d(x_{n+1},x_{n+2})+a_4d(x_{n+1},x_{n+2}) \\ &+a_4d(x_n,x_{n+1})+a_5d(x_{n+1},x_n) \\ &=\frac{a_1+a_2+a_4+a_5}{1-a_3-a_4}d(x_{n+1},x_n) \end{aligned} \tag{3.4}$$

which implies that

$$\begin{aligned} &d(x_{n+2},x_{n+1}) \\ &\le\frac{a_1+a_2+a_4+a_5}{1-a_3-a_4}d(x_{n+1},x_n) \end{aligned} \tag{3.5}$$

Using mathematical induction, we have

$$d(x_{n+2},x_{n+1})\le r^{n+1}d(x_1,x_0) \tag{3.6}$$

where $r=\dfrac{a_1+a_2+a_4+a_5}{1-a_3-a_4}<1$. We shall now prove that $\{x_n\}$ is a Cauchy sequence. For $m\ge n$, we have

$$\begin{aligned} d(x_m,x_n) &\le d(x_m,x_{m-1})+d(x_{m-1},x_{m-2}) \\ &+\ldots+d(x_{n+1},x_n) \\ &\le\left(r^{m-1}+r^{m-2}+\ldots+r^n\right)d(x_1,x_0) \\ &\le\left(\frac{r^n}{1-r}\right)d(x_1,x_0), \end{aligned} \tag{3.7}$$

which implies that $d(x_m,x_n)\to 0$, as $m,n\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence in a complete metric space X. Therefore, there exists $\mu\in X$ such that $\lim_{n\to\infty}x_n=\mu$. By the continuity of T, we have

$$\begin{aligned} T\mu &= T\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} Tx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \mu. \end{aligned}$$

Hence μ is a fixed point of T.

The prove of Theorem 3.2 is still valid for T, not necessarily continuous, assuming the following hypothesis in X.

If $\{x_n\}$ is a non-decreasing sequence in X. such that $x_n \to x$, then $x = \sup\{x_n\}$.

Theorem 3.3 Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a continuous self-mapping on X, T is monotone non-decreasing mapping satisfying

$$\begin{aligned} &d(Tx,Ty) \\ &\le a_1 \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \\ &+a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ &+a_3 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_4\left[d(x,Tx)+d(y,Ty)\right] \\ &+a_5 d(x,y) \end{aligned} \tag{3.8}$$

for all $x, y \in X, x \succcurlyeq y, x \neq y$and for some $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ with $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$.

Assume that $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

Proof From the proof of Theorem 3.2, $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to \mu$, then $\mu = \sup\{x_n\}$. Particularly, $x_n \preccurlyeq \mu$ for all $n \in \mathbb{N}$.

Since T is monotone non-decreasing mapping $Tx_n \le T\mu$ for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \preccurlyeq T\mu$ for all $n \in \mathbb{N}$. Moreover, as $x_n \prec x_{n+1} \preccurlyeq T\mu$ and $\mu = \sup\{x_n\}$, we have $\mu \preccurlyeq T\mu$

Now, we construct $\{y_n\}$ as $y_0 = \mu, y_{n+1} = Ty_n$, for all $n \ge 0$. Since $y_0 \preccurlyeq Ty_0$. Similarly, we have $\{y_n\}$ is a non-decreasing sequence and $\lim_{n\to\infty} y_n = y$ for certain $y \in X$, so we have $y = \sup\{y_n\}$. $y = \sup\{y_n\}$. Since $x_n \prec \mu = y_0 \preccurlyeq T\mu = Ty_0 \preccurlyeq y_n \preccurlyeq y$, for all n, using (3.8), we have

$$\begin{aligned} &d(x_{n+1}, y_{n+1}) = d(Tx_n, Ty_n) \\ &\le a_1 \frac{d(x,Tx_n)d(x,Ty_n)+d(y,Tx_n)d(y,Ty_n)}{d(y,Tx_n)+d(x,Ty_n)} \\ &+a_2 \frac{d(x,Tx_n)d(y,Ty_n)}{d(x,y)+d(x,Ty_n)+d(y,Tx_n)} \\ &+a_3 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_4\left[d(x_n,Tx_n)+d(y_n,Ty_n)\right] \\ &+a_5 d(x,y) \\ &= a_1 \frac{d(x_n,x_{n+1})d(x_n,y_{n+1})+d(y_n,x_{n+1})d(y_n,y_{n+1})}{d(y_n,x_{n+1})+d(x_n,y_{n+1})} \\ &+a_2 \frac{d(x_n,x_{n+1})d(y_n,y_{n+1})}{d(x_n,y_n)+d(x_n,y_{n+1})+d(y_n,x_{n+1})} \\ &+a_3 \frac{d(x_n,x_{n+1})d(y_n,y_{n+1})}{d(x_n,y_n)} \\ &+a_4\left[d(x_n,x_{n+1})+d(y_n,y_{n+1})\right]+a_5 d(x_n,y_n) \end{aligned} \tag{3.9}$$

Letting $n \to \infty$, we have $d(\mu, y) \le a_5 d(\mu, y)$. As $a_5 < 1$, we have $d(\mu, y) = 0$. Particularly, $\mu = y = \sup\{y_n\}$, and consequence, $\mu \le T\mu \le \mu$.

Hence μ is a fixed point of T.

We will now present example to illustrate where it can be appreciated that hypotheses in Theorem 3.2 and 3.3 do not guarantee uniqueness of fixed point.

Example 3.4. Let $X = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$, and define the usual order

$$(x, y) \preccurlyeq (z, t) \Leftrightarrow x \preccurlyeq z \text{ and } y \preccurlyeq t. \tag{3.10}$$

Consider $(X, \preccurlyeq)$, a partially ordered set, whose different elements are not comparable. Beside, (X, d_2) is a complete metric space considering d_2, the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and nondecreasing and condition (3.2) holds for any $a_1, a_2, a_3, a_4, a_5, a_6 \in [0,1)$, since elements in X are only comparable to themselves. Moreover, $(1,0) \preccurlyeq T(1,0) = (1,0)$. In this case, there are two fixed points in X, the hypotheses in Theorem 3.2 holds. Theorem 3.3 is also applicable since $\{(x_n, y_n)\} \subseteq X$ is a monotone nondecreasing sequence converging to $(x, y) \in X$.

We now present a sufficient condition for the uniqueness of the fixed point in Theorem 3.2 and 3.3. The condition that guarantee uniqueness of fixed point can be found in [10], such that

for every pair $x, y \in X$ has a lower bound or upper bound. (3.11)

Furthermore, [9] proved the condition (3.11) is equivalent,

for every $x, y \in X$, there exists $z \in X$ which is comparable to x and y. (3.12)

Consequently,

Theorem 3.5. Adding condition (3.12) to the hypotheses of Theorem 3.2 (or Theorem 3.3), then T has a unique fixed point.

Proof Suppose that for every $x, y \in X$, there exists $z \in X$ that is comparable to x and y. From Theorem 3.2 (or Theorem 3.3), the set of fixed points of T is non-empty. Suppose that $x, y \in X$, are two fixed points of T

We have the following two cases:

Case 1. Suppose x and y are comparable and $x \neq y$, then using (3.2) we get

$$\begin{aligned} d(x,y) &= d(Tx,Ty) \\ &\le a_1 \frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)} \\ &+a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} \\ &+a_3 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_4\left[d(x,Tx)+d(y,Ty)\right] \\ &+a_5 d(x,y) = a_5 d(x,y) \end{aligned}$$

which implies that $d(x,y)=0$, as $a_5 < 1$. Hence $x = y$.

Case 2. If x is not comparable to y, there exists $z \in X$ that is comparable to x and y, By monotonicity $T^n z$ is comparable to $T^n x = x$ and $T^n y = y$ for $n = 0,1,2,\ldots$. If there exists $n_0 \ge 1$ such that $T^{n_0} x = x$, then as x is a fixed point, the sequence $\left\{T^n z : n \ge n_0\right\}$ is constant, and, consequently, $\lim_{n\to\infty} T^n z = x.$ On the other hand, if $T^n z \neq x$ for $n \ge 1$, using the contractive condition, we have, for $n \ge 2$,

$$\begin{aligned} d\left(T^n z, x\right) &= d\left(T^n z, T^n x\right) \\ &\le a_1 \frac{d\left(T^{n-1}x,T^n x\right)d\left(T^{n-1}x,T^{n-1}z\right)+d\left(T^{n-1}z,T^n x\right)d\left(T^{n-1}z,T^n z\right)}{d\left(T^{n-1}z,T^n x\right)+d\left(T^n x,T^n z\right)} \\ &+a_2 \frac{d\left(T^{n-1}x,T^n x\right)d\left(T^{n-1}z,T^n z\right)}{d\left(T^{n-1}x,T^{n-1}z\right)+d\left(T^{n-1}x,T^n z\right)+d\left(T^{n-1}z,T^n x\right)} \\ &+a_2 \frac{d\left(T^{n-1}x,T^n x\right)d\left(T^{n-1}z,T^n z\right)}{d\left(T^{n-1}x,T^{n-1}z\right)} \\ &+a_4\left[d\left(T^{n-1}x,T^n x\right)+d\left(T^{n-1}z,T^n z\right)\right]+a_5 d\left(T^{n-1}x,T^{n-1}z\right) \\ &= a_1 \frac{d(x,x)d\left(x,T^{n-1}z\right)+d\left(T^{n-1}z,x\right)d\left(T^{n-1}z,T^n z\right)}{d\left(T^{n-1}z,x\right)+d\left(x,T^n z\right)} \\ &+a_2 \frac{d(x,x)\left(T^{n-1}z,T^n z\right)}{d\left(x,T^{n-1}z\right)+d\left(x,T^n z\right)+d\left(T^{n-1}z,x\right)} \\ &+a_3 \frac{d(x,x)\left(T^{n-1}z,T^n z\right)}{d\left(x,T^{n-1}z\right)} \\ &+a_4\left[d(x,x)+d\left(T^{n-1}z,T^n z\right)\right]+a_5 d\left(T^{n-1}z,x\right) \\ &+a_5 d\left(T^{n-1}x,T^{n-1}z\right) \end{aligned}$$

By triangular inequality

$d\left(T^{n-1}z,T^n z\right) \le d\left(T^{n-1}z,x\right)+d\left(x,T^n z\right)$, we have

$$\begin{aligned} d\left(T^n z,x\right) &\le a_1 d\left(T^{n-1}z,x\right)+a_4 d\left(T^{n-1}z,x\right) \\ &+a_4 d\left(x,T^n z\right)+a_5 d\left(x,T^{n-1}z\right) \end{aligned}$$

which implies that

$$d\left(T^n z,x\right) \le \frac{a_1+a_4+a_5}{1-a_4} d\left(T^{n-1}z,x\right).$$

Using mathematical induction, we have

$d\left(T^n z,x\right) \le \left(\frac{a_1+a_4+a_5}{1-a_4}\right)^n d\left(T^{n-1}z,x\right)$, for $n \ge 2$, and as $\frac{a_1+a_4+a_5}{1-a_4} < 1$, we have $\lim_{n\to\infty} T^n z = x.$

Similarly, we can prove that $\lim_{n\to\infty} T^n z = y.$ Now, the uniqueness of the limit implies $x = y.$

Hence T has a unique fixed point.

Example 3.6 Suppose $X = \left\{(0,1),(1,0),(1,1)\right\} \subset \mathbb{R}^2$ and let X be a partial order given by $R = \left\{(x,x) : x \in X\right\}$. The elements in X are only comparable to themselves with (X, d_2), a complete metric space where d_2 is the Euclidean distance. Let $T : X \to X$ be defined as

$$T(0,1)=(1,0), T(1,0)=(0,1), T(1,1)=(1,1) \quad (3.13)$$

Observe that T is trivially continuous and nondecreasing, and assumption in (3.2) of Theorem 3.2 satisfied. Since elements in X are only comparable to themselves. Observe also, $(1,1) \le T(1,1) = (1,1)$ and by Theorem 3.2, T has a fixed point (1.1).

We present additional example to illustrate or support our result.

Example 3.7 Consider $(\mathbb{R}^2, \preccurlyeq)$, where $\preccurlyeq$ represents the usual order relation in hypotheses (3.8) is valid. Indeed, if

$$(x,y),(z,t) \in \mathbb{R}^2 \quad (3.14)$$

then $\left(\max\{x,z\}, \max\{y,t\}\right) \in \mathbb{R}^2$ and $\left(\min\{x,z\}, \min\{y,t\}\right) \in \mathbb{R}^2$ are respectively, upper and lower bound of (x,y) and (z,t). If $\left\{(x_n, y_n)\right\}$ is a monotone nondecreasing sequence in $\mathbb{R}^2$, converging to (x,y), then $\{x_n\}$ and $\{y_n\}$ are monotone nondecreasing sequences which converge, respectively, to x and y in $\mathbb{R}$, then $x_n \preccurlyeq x$ and $y_n \preccurlyeq y$ for all n, and (x,y) is an upper bound of all terms in the sequence $\left\{(x_n, y_n)\right\}$.

4. Application

The Some application of the main results to a self mapping involving an integral type contraction.

Let us consider the set of all functions χ defined on $[0,\infty)$ satisfying the following conditions:

1. Each χ is Lebesque integrable mapping on each compact subset of $[0,\infty)$.

2. For any $\epsilon > 0$, we have $\int_0^\epsilon \chi(t)dt > 0$.

Corollary 4.1 Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a continuous self-mapping on X, T is monotone non-decreasing mapping satisfying

$$\begin{aligned}\int_0^{d(Tx,Ty)} \phi(t)dt \leq\ & a_1 \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} \phi(t)dt \\ & + a_2 \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} \phi(t)dt \\ & + a_3 \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} \phi(t)dt \\ & + a_4 \int_0^{d(x,Tx)+d(y,Ty)} \phi(t)dt \\ & + a_5 \int_0^{d(x,y)} \phi(t)dt \end{aligned} \quad (4.1)$$

for all $x, y \in X, x \succcurlyeq y, x \neq y$ and for some $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ with $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

Corollary 4.2 Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Suppose that $T : X \to X$ is a continuous self-mapping on X, T is monotone non-decreasing mapping satisfying

$$\begin{aligned}\int_0^{d(Tx,Ty)} \phi(t)dt \leq\ & a_1 \int_0^{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)d(y,Ty)}{d(y,Tx)+d(x,Ty)}} \phi(t)dt \\ & + a_2 \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)}} \phi(t)dt \\ & + a_3 \int_0^{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}} \phi(t)dt \\ & + a_4 \int_0^{d(x,Tx)+d(y,Ty)} \phi(t)dt \\ & + a_5 \int_0^{d(x,y)} \phi(t)dt \end{aligned} \quad (4.2)$$

for all $x, y \in X, x \succcurlyeq y, x \neq y$ and for some $a_1, a_2, a_3, a_4, a_5 \in [0,1)$ with $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$.

Assume that $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x = \sup\{x_n\}$. If there exists $x_0 \in X$ with $x_0 \preccurlyeq Tx_0$, then T has a fixed point.

5. Conclusion

The main findings of this study demonstrate the existence and uniqueness of some fixed point for nonlinear contractive mappings in the setting of metric spaces and partially ordered metrics spaces satisfying a contraction condition of rational type. These contributions extend the existing literature on metric spaces and fixed point theory. Through illustrative examples, we showcased the practical applicability of our proposed notions and results. This study provides significant advancements in the understanding of metric spaces, with potential applications in differential equations.

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