

Existence and Uniqueness Theorem for Set-Valued Volterra Integral Equations

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Abstract The space of nonempty compact sets of \mathbb{R}^n is well-known to be a nonlinear space. This fact essentially complicates the research of set-valued differential and integral equations. In this article we consider set-valued Volterra integral equations and prove the existence and uniqueness theorem.

Keywords: set-valued integral equation, existence, uniqueness, set-valued differential equation

1. Introduction

The majority of the theories describing the world we live in, are based on differential and integral equations. Such equations appear not only in the physical sciences, but in biology, medicine, chemistry, sociology, and all scientific disciplines that attempt to understand the world surrounding us. Recently the set-valued differential or integral equations have attracted the increasing interest of scientists. The development of the theory of set-valued differential equations (SDEs) begun from works of F.S. de Blasi and F. Iervolino [1]. After that the properties of solutions of SDEs [2-14], the set-valued integro-differential equations (SIDEs) [15], the impulse set-valued differential equations [9,11,16], set-valued control systems [12,17,18,19], the set-valued integral equations (SIEs) [20,21] and asymptotic methods [9,11,12,19] were considered. On the other hand, SDEs and SIEs are useful in other areas of mathematics. For example, SDEs and SIEs are used as an auxiliary tool to prove the existence results for differential and integral inclusions [9,13,14]. Also, one can employ SDEs and SIEs in the investigation of fuzzy differential and integral equations [5,7,8,11,23,24,25]. Moreover, SDEs and SIEs are a natural generalization of usual ordinary differential and integral equations in finite (or infinite) dimensional Banach spaces. Therefore, in this article we consider set-valued Volterra integral equations and prove the existence and uniqueness theorem.

2. Main Result

Let $\text{conv}(\mathbb{R}^n)$ be the family of all nonempty convex compact subsets of \mathbb{R}^n with the Hausdorff metric

$$h(A, B) = \min_{r \geq 0} \{B \subset S_r(A), A \subset S_r(B)\}$$

where $S_r(A)$ be a r -neighborhood of the set A .

Lemma 2.1 [26]. The following properties hold:

1. $(\text{conv}(\mathbb{R}^n), h)$ is a complete metric space,
2. $h(A + C, B + C) = h(A, B)$,
3. $h(\lambda A, \lambda B) = |\lambda| h(A, B)$ for all $A, B, C \in \text{conv}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$.

Definition 2.1 [27]. Let $X, Y \in \text{conv}(\mathbb{R}^n)$. A set $Z \in \text{conv}(\mathbb{R}^n)$ such that $X = Y + Z$ is called a Hukuhara difference of the sets X and Y and is denoted by $X \overset{H}{-} Y$.

Consider the set-valued Volterra integral equation

$$X(t) + \int_0^t F(t, s, X(s)) ds = A \quad (2.1)$$

where $t, s \in \mathbb{R}_+^1$, $0 \leq s \leq t$, $X \in \text{conv}(\mathbb{R}^n)$, $F(\cdot, \cdot, \cdot): \mathbb{R}_+^1 \times \mathbb{R}_+^1 \times \text{conv}(\mathbb{R}^n) \rightarrow \text{conv}(\mathbb{R}^n)$ is a set-valued mapping, $A \in \text{conv}(\mathbb{R}^n)$. The integral is understood in the sense of [27].

Definition 2.2. A set-valued mapping $X(\cdot): [0, T] \rightarrow \text{conv}(\mathbb{R}^n)$ is called a solution of integral equation (2.1) if it is continuous and satisfies integral equation (2.1) on interval $[0, T]$.

Remark 2.1. There exist $A, B \in \text{conv}(\mathbb{R}^n)$ such that $\text{diam}(A) < \text{diam}(B)$ but there exists no such $C \in \text{conv}(\mathbb{R}^n)$ that $A + C = B$. For example, $A = \{a \in \mathbb{R}^2 : \|a\| \leq 1\}$, $B = \{b \in \mathbb{R}^2 : |b_i| \leq 2, i = 1, 2\}$. In this case there exists such $C \in \text{conv}(\mathbb{R}^2)$ that $A + C \subset B$, but does not exist such $C \in \text{conv}(\mathbb{R}^2)$ that $A + C = B$.

Let $CC(R^n)$, ($n \geq 2$) be a space of all nonempty strictly convex closed sets of R^n and all element of R^n [28].

Remark 2.2. If $A, B \in CC(R^n)$ and $A + C = B$ then $C \in CC(R^n)$ [28].

Definition 2.3. It is said that the set $B \in CC(R^n)$ can be embedded in the set $C \in CC(R^n)$ if there exists $\zeta(B, C) \in R^n$ such that $B + \zeta(B, C) \subset C$.

Theorem 2.1. Let in the domain $Q = \{(t, s, X) \in R_+^1 \times R_+^1 \times CC(R^n)\}$ the following conditions hold:

- i) for any fixed (s, X) the set-valued mapping $F(\cdot, s, X)$ is continuous;
- ii) for any fixed (t, X) the set-valued mapping $F(t, \cdot, X)$ is continuous;
- iii) there exists a positive constant L such that

$$h(F(t, s, X'), F(t, s, X'')) \leq Lh(X', X'')$$

for all $(t, s, X'), (t, s, X'') \in Q$;

- iv) there exists $K > 0$ such that

$$h(F(t, s, X), \{0\}) \leq K(1 + h(X, \{0\}))$$

for all $(t, s, X) \in Q$;

- v) $A \in CC(R^n)$ and $\text{int } A \neq \emptyset$;

vi) if the set $B \in CC(R^n)$ can be embedded in the set $C \in CC(R^n)$, then the set $F(t, s, B)$ can be embedded in the set for all $(t, s) \in R_+^1 \times R_+^1$;

- vii) $F(\cdot, \cdot, \cdot): Q \rightarrow CC(R^n)$.

Then equation (2.1) has a unique solution on the interval $[0, d]$.

Proof. According to Definition 2.1. we associate with set integral equation (2.1) the following set integral equation

$$X(t) = A \frac{H}{0} \int_0^t F(t, s, X(s)) ds \tag{2.2}$$

Let us prove the existence of a solution of equation (2.2) on interval $[0, d]$.

- a) As $F(t, s, X) \in CC(R^n)$ for all $(t, s, X) \in Q$, then

$$\int_0^t F(t, s, X) ds \in CC(R^n) \text{ for all } (t, s, X) \in Q \text{ [28].}$$

Therefore, as $A \in CC(R^n)$ and $\text{int } A \neq \emptyset$, then there exists $d > 0$ such that the Hukuhara difference

$$A \frac{H}{0} \int_0^t F(t, s, A) ds \text{ exists for all } t \in [0, d] \text{ [28].}$$

- b) Let us build the successive approximations of the solution

$$\begin{aligned} X^0(t) &= A \text{ for } t \in [0, d], \\ X^k(t) &= A \frac{H}{0} \int_0^t F(t, s, X^{k-1}(s)) ds \text{ for } t \in [0, d]. \end{aligned} \tag{2.3}$$

By a), $X^1(t)$ exists and $X^1(t) \in CC(R^n)$ for all $t \in [0, d]$. Also by conditions i) ,ii) and iii) of the theorem $X^1(t)$ is continuous on $[0, d]$ and the set $X^1(t)$ can be embedded in the set A for all $t \in [0, d]$, i.e. there exists

$$\zeta_1(\cdot): [0, d] \rightarrow R^n \text{ such that } X^1(t) + \zeta_1(t) \subset A, \text{ where } \zeta_1(t) \in \int_0^t F(t, s, A) ds \text{ for } t \in [0, d].$$

By condition vi), the set $F(t, s, X^1(s))$ can be embedded in the set $F(t, s, A)$ for all $t \in [0, d]$ and $s \leq t$.

Then the set $\int_0^t F(t, s, X^1(s)) ds$ can be embedded in the set

$$\int_0^t F(t, s, A) ds \text{ for all } t \in [0, d], \text{ i.e. there exists}$$

$\zeta_1(\cdot): [0, d] \rightarrow R^n$ such that

$$\int_0^t F(t, s, X^1(s)) ds + \zeta_1(t) \subset \int_0^t F(t, s, A) ds$$

Therefore, $X^2(t) = A \frac{H}{0} \int_0^t F(t, s, X^1(s)) ds$ exists and

$X^2(t) \in CC(R^n)$ for all $t \in [0, d]$. Also by conditions i), ii) and iii) of the theorem $X^2(t)$ is continuous on $[0, d]$ and the set $X^2(t)$ can be embedded in the set A for all $t \in [0, d]$, i.e. there exists $\zeta_2(\cdot): [0, d] \rightarrow R^n$ such that

$$X^2(t) + \zeta_2(t) \subset A, \text{ where } \zeta_2(t) \in \int_0^t F(t, s, X^1(s)) ds \text{ for}$$

$t \in [0, d]$.

By condition vi), the set $F(t, s, X^2(s))$ can be embedded in the set $F(t, s, A)$ for all $t \in [0, d]$ and $s \leq t$.

Then the set $\int_0^t F(t, s, X^2(s)) ds$ can be embedded in the

$$\text{set } \int_0^t F(t, s, A) ds \text{ for all } t \in [0, d], \text{ i.e. there exists}$$

$\zeta_2(\cdot): [0, d] \rightarrow R^n$ such that

$$\int_0^t F(t, s, X^2(s)) ds + \zeta_2(t) \subset \int_0^t F(t, s, A) ds$$

and so on.

Similarly, $X^k(t) = A \int_0^t F(t,s, X^{k-1}(s))ds$ exists and $X^k(t) \in CC(R^n)$ for all $k \in N$ and $t \in [0, d]$. Also by conditions i), ii) and iii) of the theorem $X^k(t)$ is continuous on $[0, d]$ and the set $X^k(t)$ can be embedded in the set A for all $t \in [0, d]$ for all $k \in N$.

Besides

$$\begin{aligned} h(X^1(t), X^0(t)) &= h\left(A \int_0^t F(t,s, A)ds, A\right) \\ &\leq h\left(\int_0^t F(t,s, A)ds, \{0\}\right) \\ &\leq \int_0^t h(F(t,s, A), \{0\})ds \\ &\leq \int_0^t K(1+h(A, \{0\}))ds \\ &= K(1+h(A, \{0\}))t \end{aligned}$$

$$\begin{aligned} h(X^2(t), X^1(t)) &= h\left(A \int_0^t F(t,s, X^1(s))ds, A \int_0^t F(t,s, X^0(s))ds\right) \\ &= h\left(\int_0^t F(t,s, X^1(s))ds, \int_0^t F(t,s, X^0(s))ds\right) \\ &\leq \int_0^t h(F(t,s, X^1(s)), F(t,s, X^0(s)))ds \\ &\leq \int_0^t Lh(X^1(s), X^0(s))ds \leq \int_0^t LK(1+h(A, \{0\}))s ds \\ &= LK(1+h(A, \{0\}))\frac{t^2}{2!} \end{aligned}$$

$$\begin{aligned} h(X^3(t), X^2(t)) &\leq \int_0^t Lh(X^2(s), X^1(s))ds \\ &\leq \int_0^t L^2 K(1+h(A, \{0\}))\frac{s^2}{2!} ds \\ &= L^2 K(1+h(A, \{0\}))\frac{t^3}{3!} \end{aligned}$$

and so on.

Therefore,

$$\begin{aligned} h(X^{n+1}(t), X^n(t)) &\leq \int_0^t Lh(X^n(s), X^{n-1}(s))ds \\ &\leq \int_0^t L^n K(1+h(A, \{0\}))\frac{s^n}{n!} ds \\ &= L^n K(1+h(A, \{0\}))\frac{t^{n+1}}{(n+1)!} \end{aligned}$$

Then

$$\begin{aligned} &\max_{t \in [0, d]} h(X^{n+1}(t), A) \\ &\leq \max_{t \in [0, d]} h(X^{n+1}(t), X^n(t)) + \dots + \max_{t \in [0, d]} h(X^2(t), X^1(t)) \\ &\quad + \max_{t \in [0, d]} h(X^1(t), X^0(t)) + \max_{t \in [0, d]} h(X^0(t), A) \\ &\leq L^n K(1+h(A, \{0\}))\frac{d^{n+1}}{(n+1)!} + \dots + K(1+h(A, \{0\}))d \\ &= \frac{K(1+h(A, \{0\}))}{L} \sum_{i=1}^{n+1} \frac{(Ld)^i}{i!} \\ &\leq \frac{K(1+h(A, \{0\}))}{L} e^{Ld} = b \end{aligned}$$

Hence, it follows that the sequence of the set-valued mappings $\{X^k(t)\}_{k=0}^\infty$ in uniformly bounded:

$$h(X^k(t), \{0\}) \leq b + h(A, \{0\})$$

for all $t \in [0, d]$.

Let us show that the sequence of the set-valued mappings $\{X^k(t)\}_{k=0}^\infty$ is a Cauchy sequence. For any $m, p \in N$ we have

$$\begin{aligned} h(X^{m+p}(t), X^p(t)) &= h\left(\int_0^t F(t,s, X^{m+p-1}(s))ds, \int_0^t F(t,s, X^{p-1}(s))ds\right) \\ &\leq \int_0^t h(F(t,s, X^{m+p-1}(s)), F(t,s, X^{p-1}(s)))ds \\ &\leq L \int_0^t h(X^{m+p-1}(s), X^{p-1}(s))ds \end{aligned}$$

Hence,

$$\begin{aligned} h(X^{m+p}(t), X^p(t)) &\leq L^p \int_0^t \dots \int_0^{t_{p-1}} h(X^m(t_p), A) dt_p \dots dt_1 \\ &\leq \frac{bL^p t^p}{p!} \leq \frac{bL^p d^p}{p!} \end{aligned}$$

Therefore, the sequence $\{X^k(t)\}_{k=0}^\infty$ is a Cauchy sequence. Its limit is a continuous set-valued mapping that we will denote by $X(t)$. Owing to the theorem conditions in (2.3) it is possible to pass to the limit under the sign of the integral. We receive that the set-valued mapping $X(t)$ satisfies equation (2.2), i.e. $X(t)$ is the solution of (2.1) on the interval $[0, d]$.

To prove the uniqueness, suppose that there exist at least two different solutions $X(\cdot)$ and $Y(\cdot)$ of (2.1) on $[0, d]$, then $\delta = \max_{t \in [0, d]} h(X(t), Y(t)) > 0$.

$$\text{As } X(t) = A \int_0^t F(t,s, X(s))ds \text{ and } Y(t) = A \int_0^t F(t,s, Y(s))ds$$

then

$$\begin{aligned}
 h(X(t), Y(t)) &= h\left(\int_0^t F(t, s, X(s))ds, \int_0^t F(t, s, Y(s))ds\right) \\
 &\leq \int_0^t h(F(t, s, X(s)), F(t, s, Y(s)))ds \\
 &\leq L \int_0^t h(X(s), Y(s))ds \leq L\delta t \leq L\delta d
 \end{aligned}$$

Similarly,

$$h(X(t), Y(t)) \leq L \int_0^t L\delta s ds \leq \frac{L^2\delta t^2}{2!} \leq \frac{L^2\delta d^2}{2!}, \dots,$$

$$h(X(t), Y(t)) \leq \frac{L^k \delta d^k}{k!}$$

Then $\delta = \max_{t \in [0, d]} h(X(t), Y(t)) \leq \frac{L^k \delta d^k}{k!}$. Therefore,

$$1 \leq \frac{L^k d^k}{k!} \text{ for any } k \in \mathbb{N} \text{ that contradicts } \lim_{k \rightarrow \infty} \frac{L^k d^k}{k!} = 0.$$

This concludes the proof.

Finally we consider example for case $CC(R^2)$.

Example 3.1. Consider the following set-valued integral equation

$$X(t) + \int_0^t X(s)ds = S_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.4}$$

where $X(\cdot) : R^1 \rightarrow CC(R^2)$. It is obvious that

$$X(t) = S_{e^{-t}} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

is the solution of equation (2.4) (see Figure 2.1)

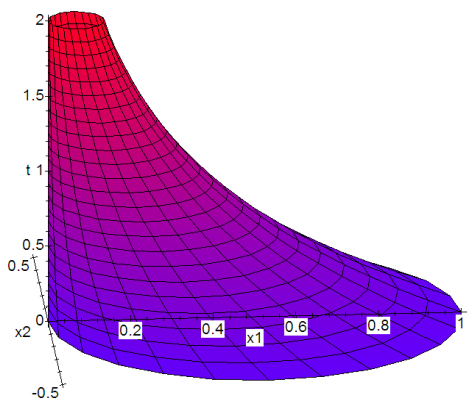


Figure 2.1. The graph of a solution of system (2.4)

3. Conclusion

Also it is possible to prove the similar results if instead of $CC(R^n)$ we consider a space of all nonempty M -strongly convex closed sets of R^n and all elements of R^n [29], i.e. $MCC(R^n)$.

Example 3.1. Consider the example 2.1 if $A = \{x \in R^2 : |x_1| + |x_2| \leq 1\} \in MCC(R^n)$. It is obvious

$$X(t) = \{x \in R^2 : |x_1| + |x_2| \leq e^{-t}\}$$

is the solution (see Figure 3.1).

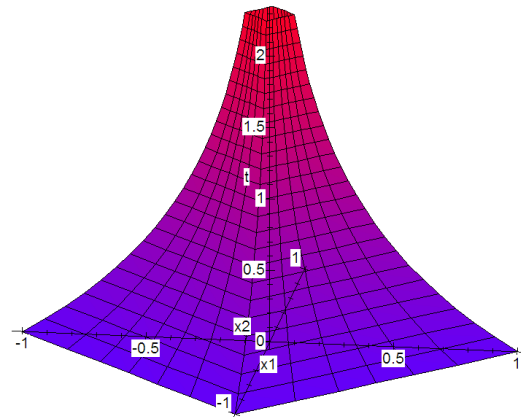


Figure 3.1. The graph of a solution of example 3.1

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