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Hermite-Hadamard Type Inequalities for Multiplicatively *h*-Preinvex Functions

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Abstract In this paper, we establish integral inequalities of Hermite-Hadamard type for multiplicatively *h*-preinvex functions. We also obtain some new inequalities involving multiplicative integrals by using some properties of preinvex and multiplicatively *h*-preinvex functions.

Keywords: *invex sets, preinvex functions, multiplicatively h-preinvex functions, Multiplicative calculus, Hermite-Hadamard inequalities*

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1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated in [1,2,3] as:

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $I = [a_1, a_2]$ of real numbers with $a_1 < a_2$, then

$$f\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(x)dx \le \frac{f(a_1) + f(a_2)}{2}. \quad (1)$$

Both the inequalities hold in the reversed direction if f is concave. For some results which generalize, extend and improve the inequality (1), we refer the interested reader [4,5,6,7,8].

A significant generalization of convex functions is that of preinvex functions. In recent years, lots of efforts have been made by many researchers to generalize Hermite-Hadamard inequality for preinvex functions [9-15]. These studies include among others the work of Hanson [16], Ben-Israel and Mond [17], Pini [18], Weir and Mond [19], Noor [20] and Yang and Li [21] have studied the basic properties of the preinvex function and their role in optimization, variational inequalities and equilibrium problems. Hanson [16] introduced a significant class of generalized convex functions, which is called invex functions. Ben-Israel and Mond [19] introduced the notions of invex sets and preinvex functions. Yang and Li [21] studied the basic properties of the preinvex function and their role in optimization, variational inequalities and equilibrium problems. Let us recall some definitions and known results concerning invexity and preinvexity.

Definition 1.1 [21] *A set $\mathfrak{I} \subseteq \mathbb{R}$ is said to be invex if there exist a function $\eta: \mathfrak{I} \times \mathfrak{I} \to \mathbb{R}$ such that*

$$a_1 + \mu\eta(a_2, a_1) \in \mathfrak{I}, \ \forall a_1, a_2 \in \mathfrak{I}, \ \mu \in [0,1].$$

The invex set $\mathfrak{I}$ is also called a η-connected set.

Definition 1.2 [19] *Let f be a function on the invex set $\mathfrak{I}$. Then, f is said to be preinvex with respect to η, if*

$$f(a_1 + \mu\eta(a_2, a_1)) \le (1-\mu) f(a_1) + \mu f(a_2),$$
$$\forall a_1, a_2 \in \mathfrak{I}, \ \mu \in [0,1].$$

It is to be noted that every convex function is preinvex with respect to the map $\eta(a_2, a_1) = a_2 - a_1$, but the converse is not true, see for example [19,22].

In [23] Noor has obtained the following Hermite-Hadamard inequalities for the preinvex functions.

Theorem 1.1 *Let $f: \mathfrak{I} = [a_1, a_1 + \eta(a_2, a_1)] \to (0, \infty)$ be a preinvex function on the interval of real numbers $\mathfrak{I}^{\circ}$ and $a_1, a_2 \in \mathfrak{I}^{\circ}$ with $a_1 < a_1 + \eta(a_2, a_1)$. Then the following inequality holds:*

$$f\left(\frac{2a_1 + \eta(a_2, a_1)}{2}\right)$$
$$\le \frac{1}{\eta(a_2, a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))dx \le \frac{f(a_1) + f(a_2)}{2}.$$

Definition 1.3 [24] Let $h: (0,1) \subseteq I \to \mathbb{R}$ be an interval in $\mathbb{R}$, and let $\mathfrak{I}$ be an invex set with respect to η. A nonnegative function $f: \mathfrak{I} \to \mathbb{R}$ is called h-preinvex with respect to η if

$$f(a_1 + \mu\eta(a_2, a_1)) \le h(1-\mu) f(a_1) + h(\mu) f(a_2),$$
$$a_1, a_2 \in \mathfrak{I}, \ \mu \in (0,1).$$

Definition 1.4 [25] *A nonnegative function $f: \mathfrak{I} \to (0, \infty)$ is said to be multiplicatively (or logarithmically) h-preinvex with respect to η, if*

$$f\left(a_1+\mu\eta\left(a_2,a_1\right)\right)\le\left[f\left(a_1\right)\right]^{h(1-\mu)}\left[f\left(a_2\right)\right]^{h(\mu)},$$
$$a_1,a_2\in\Im,\ \mu\in(0,1).$$

From the above definition, we have

$$\begin{aligned}&\ln f\left(a_1+\mu\eta\left(a_2,a_1\right)\right)\\&\le\ln\left\{\left[f\left(a_1\right)\right]^{h(1-\mu)}\left[f\left(a_2\right)\right]^{h(\mu)}\right\}\\&=\ln\left[f\left(a_1\right)\right]^{h(1-\mu)}+\ln\left[f\left(a_2\right)\right]^{h(\mu)}\\&=h(1-\mu)\ln f\left(a_1\right)+h(\mu)\ln f\left(a_2\right)\end{aligned}$$

1.1. Multiplicative Calculus

Recall that the notion of multiplicative integral is denoted by $\int_u^v\left(f(x)\right)^{dx}$ while the ordinary integral is denoted by $\int_u^v\left(f(x)\right)dx$. This comes from the fact that the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[u,v]$, the product of terms raised to certain powers is used in the definition of multiplicative integral of f on $[u,v]$.

There is the following relation between Riemann integral and multiplicative integral [26].

Proposition 1.1 *If f is Riemann integrable on $[u,v]$, then f is multiplicative integrable on $[u,v]$ and*

$$\int_u^v\left(f(x)\right)^{dx}=e^{\int_u^v\ln\left(f(x)\right)dx}.$$

In [26], Bashirov et al. show that multiplicative integral has the following results:

Proposition 1.2 *If f is positive and Riemann integrable on $[u,v]$, then f is multiplicative integrable on $[u,v]$ and*

1. $\int_u^v\left(\left(f(x)\right)^r\right)^{dx}=\int_u^v\left(\left(f(x)\right)^{dx}\right)^r$,
2. $\int_u^v\left(f(x)g(x)\right)^{dx}=\int_u^v\left(f(x)\right)^{dx}.\int_u^v\left(g(x)\right)^{dx}$,
3. $\int_u^v\left(\frac{f(x)}{g(x)}\right)^{dx}=\frac{\int_u^v\left(f(x)\right)^{dx}}{\int_u^v\left(g(x)\right)^{dx}}$,
4. $\int_u^v\left(f(x)\right)^{dx}=\int_u^w\left(f(x)\right)^{dx}.\int_w^v\left(f(x)\right)^{dx},\ u\le w\le v.$
5. $\int_u^u\left(f(x)\right)^{dx}=1$ and $\int_u^v\left(f(x)\right)^{dx}=\left(\int_v^u\left(f(x)\right)^{dx}\right)^{-1}$.

2. Main Results

In this section we establish some Hermite-Hadamard type inequalities for multiplicatively h-preinvex functions. We also obtain integral inequalities of Hermite-Hadamard type for product and quotient of multiplicatively h-preinvex and preinvex positive functions.

Theorem 2.1 *$\Im\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1,a_2\in\Im$ with $a_1<a_1+\eta\left(a_2,a_1\right)$. If f is a positive and multiplicatively h-preinvex function on the interval $\left[a_1,a_1+\eta\left(a_2,a_1\right)\right]$ such that $h\left(\frac{1}{2}\right)\ne 0$ and η satisfies Condition C, then*

$$\begin{aligned}&\left[f\left(\frac{2a_1+\eta\left(a_2,a_1\right)}{2}\right)\right]^{\frac{1}{2h(1/2)}}\\&\le\left(\int_{a_1}^{a_1+\eta\left(a_2,a_1\right)}\left(f(x)\right)^{dx}\right)^{\frac{1}{\eta\left(a_2,a_1\right)}}\qquad(2)\\&\le\left(f\left(a_1\right).f\left(a_2\right)\right)^{\int_0^1h(\mu)d\mu}.\end{aligned}$$

Proof Since f is a multiplicatively h-preinvex function, we have for every $\alpha,\beta\in\left[a_1,a_1+\eta\left(a_2,a_1\right)\right]$ with $\mu=\frac{1}{2}$

$$f\left(\frac{2\alpha+\eta(\beta,\alpha)}{2}\right)=f\left(\alpha+\frac{\eta(\beta,\alpha)}{2}\right)\le\left(f(\alpha)\right)^{\frac{1}{2}}\left(f(\beta)\right)^{\frac{1}{2}}.$$

Now, let $\alpha=a_1+(1-t)\eta\left(a_2,a_1\right),\ \ \beta=a_1+t\eta\left(a_2,a_1\right)$. From Condition C, we have

$$\begin{aligned}&f\left(\begin{array}{l}a_1+(1-\mu)\eta\left(a_2,a_1\right)\\+\dfrac{\eta\left(u_1+\mu\eta\left(a_2,a_1\right),u_1+(1-\mu)\eta\left(a_2,a_1\right)\right)}{2}\end{array}\right)\\&=f\left(a_1+(1-\mu)\eta\left(a_2,a_1\right)+\frac{(2\mu-1)\eta\left(a_2,a_1\right)}{2}\right)\\&=f\left(\frac{2a_1+\eta\left(a_2,a_1\right)}{2}\right)\\&\le\left(f\left(a_1+\mu\eta\left(a_2,a_1\right)\right)\right)^{h(1/2)}\\&\quad\cdot\left(f\left(a_1+(1-\mu)\eta\left(a_2,a_1\right)\right)\right)^{h(1/2)}.\end{aligned}$$

Taking logarithms of both sides of the above inequality leads to

$$\begin{aligned}&\ln f\left(\frac{2a_1+\eta\left(a_2,a_1\right)}{2}\right)\\&\le\ln\left(\begin{array}{l}\left(f\left(a_1+\mu\eta\left(a_2,a_1\right)\right)\right)^{h(1/2)}\\.\left(f\left(a_1+(1-\mu)\eta\left(a_2,a_1\right)\right)\right)^{h(1/2)}\end{array}\right)\\&=h\left(\frac{1}{2}\right)\ln\left(f\left(a_1+\mu\eta\left(a_2,a_1\right)\right)\right)\\&\quad+h\left(\frac{1}{2}\right)\ln\left(f\left(a_1+(1-\mu)\eta\left(a_2,a_1\right)\right)\right).\end{aligned}$$

Integrating the above inequality with respect to μ on $[0,1]$,

$$\ln f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)$$
$$\le h\left(\frac{1}{2}\right)\int_0^1 \ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu$$
$$+h\left(\frac{1}{2}\right)\int_0^1 \ln\left(f\left(a_1+(1-\mu)\eta(a_2,a_1)\right)\right)d\mu$$
$$=h\left(\frac{1}{2}\right)\left[\begin{array}{l}\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\\ +\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\end{array}\right]$$
$$=h\left(\frac{1}{2}\right)\left[\begin{array}{l}\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\\ +\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\end{array}\right]$$
$$=\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx.$$

Thus,

$$f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\le e^{\left(\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\right)}$$
$$=\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}.$$

Hence, we have

$$f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\le\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}},\quad(3)$$

which completes the proof of the first inequality in (2).

Now consider the second inequality in (2).

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=\left(e^{\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\right)}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$=e^{\frac{1}{\eta(a_2,a_1)}\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx\right)}$$
$$=e^{\left(\int_0^1\ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu\right)}$$
$$\le e^{\left(\int_0^1\ln\left(\left(f(a_1)\right)^{h(1-\mu)}\left(f(a_2)\right)^{h(\mu)}\right)d\mu\right)}$$
$$=e^{\left(\int_0^1\left(h(1-\mu)\ln f(a_1)+h(\mu)\ln f(a_2)\right)d\mu\right)}$$
$$=e^{\left(\ln\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}\right)}=\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}.$$

Hence, we get the inequality

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}\le\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}.\quad(4)$$

Combining (3) and (4) gives the desired result.

Corollary 2.1 *Let $\Im\subseteq\mathbb{R}$ an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1,a_2\in\Im$ with $a_1<a_1+\eta(a_2,a_1)$. If f and g are positive and multiplicatively h-preinvex functionson $\left[a_1,a_1+\eta(a_2,a_1)\right]$ such that $h\left(\frac{1}{2}\right)\neq 0$ and η satisfies Condition C, then*

$$\left[f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)\right]^{\frac{1}{2h(1/2)}}$$
$$\le\left(\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}.\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$\le\left[\left(f(a_1).f(a_2)\right).\left(g(a_1).g(a_2)\right)\right]^{\int_0^1 h(\mu)d\mu}.$$

Since f and g are multiplicatively h-preinvex functions, fg. is a multiplicatively h-preinvex function. Thus, if we apply Theorem 2.1 to the function fg, then we obtain the required result.

Corollary 2.2 *Let $\Im\subseteq\mathbb{R}$ an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1,a_2\in\Im$ with $a_1<a_1+\eta(a_2,a_1)$. If f and g are positive and multiplicatively h-preinvex functions on $\left[a_1,a_1+\eta(a_2,a_1)\right]$ such that $h\left(\frac{1}{2}\right)\neq 0$ and η satisfies Condition C, then*

$$\left[\frac{f\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}{g\left(\frac{2a_1+\eta(a_2,a_1)}{2}\right)}\right]^{\frac{1}{2h(1/2)}}$$
$$\le\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$
$$\le\left(\frac{f(a_1).f(a_2)}{g(a_1).g(a_2)}\right)^{\int_0^1 h(\mu)d\mu}.$$

Proof Since f and g are multiplicatively h-preinvex functions, $\frac{f}{g}$ is a multiplicatively h-preinvex function. Thus, if we apply Theorem 2.1 to the function $\frac{f}{g}$, then we obtain the desired result.

Theorem 2.2 *Let $\Im\subseteq\mathbb{R}$ an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1,a_2\in\Im$ with $a_1<a_1+\eta(a_2,a_1)$.*

Let f and g be preinvex and multiplicatively h-preinvex positive functions, respectively, on the interval $\left[a_1, a_1+\eta\left(a_2,a_1\right)\right]$. Then, we have

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{\left(\frac{\left(f(a_2)\right)^{f(a_2)}}{\left(f(a_1)\right)^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}}{e.\left(g(a_1).g(a_2)\right)^{\int_0^1 h(\mu)d\mu}}.$$

Proof Note that,

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=\left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(g(x)\right)dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx-\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(g(x)\right)dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=e^{\int_0^1\ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu-\int_0^1\ln\left(g\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu}$$

$$\le e^{\left[\begin{array}{l}\int_0^1\ln\left(f(a_1)+\mu\left(f(a_2)-f(a_1)\right)\right)d\mu\\ -\int_0^1\ln\left(\left(g(a_1)\right)^{h(1-\mu)}\left(g(a_2)\right)^{h(\mu)}\right)d\mu\end{array}\right]}$$

$$=e^{\ln\left(\left(\frac{\left(f(a_2)\right)^{f(a_2)}}{\left(f(a_1)\right)^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}\right)-1-\ln\left(g(a_1).g(a_2)\right)^{\int_0^1 h(\mu)d\mu}}$$

$$=\frac{\left(\frac{\left(f(a_2)\right)^{f(a_2)}}{\left(f(a_1)\right)^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}}{e.\left(g(a_1).g(a_2)\right)^{\int_0^1 h(\mu)d\mu}}.$$

Thus, we have

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{\left(\frac{\left(f(a_2)\right)^{f(a_2)}}{\left(f(a_1)\right)^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}}{e.\left(g(a_1).g(a_2)\right)^{\int_0^1 h(\mu)d\mu}},$$

which completes the proof.

Theorem 2.3 *Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta:\Im\times\Im\to\mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1+\eta\left(a_2,a_1\right)$. Let f and g be multiplicatively h-preinvex and preinvex positive functions, respectively, on the interval $\left[a_1, a_1+\eta\left(a_2,a_1\right)\right]$. Then, we have*

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{e.\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}}{\left(\frac{\left(g(a_2)\right)^{g(a_2)}}{\left(g(a_1)\right)^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}}$$

Proof Note that

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=\left(\frac{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx}}{e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(g(x)\right)dx}}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=\left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(f(x)\right)dx-\int_{a_1}^{a_1+\eta(a_2,a_1)}\ln\left(g(x)\right)dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$=e^{\int_0^1\ln\left(f\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu-\int_0^1\ln\left(g\left(a_1+\mu\eta(a_2,a_1)\right)\right)d\mu}$$

$$\le e^{\left[\begin{array}{l}\int_0^1\ln\left(\left(f(a_1)\right)^{h(1-\mu)}\left(f(a_2)\right)^{h(\mu)}\right)d\mu\\ -\int_0^1\ln\left(g(a_1)+\mu\left(g(a_2)-g(a_1)\right)\right)d\mu\end{array}\right]}$$

$$=e^{\ln\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}-\ln\left(\left(\frac{\left(g(a_2)\right)^{g(a_2)}}{\left(g(a_1)\right)^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}\right)+1}$$

$$=\frac{e.\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}}{\left(\frac{\left(g(a_2)\right)^{g(a_2)}}{\left(g(a_1)\right)^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}}.$$

Hence,

$$\left(\frac{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(f(x)\right)^{dx}}{\int_{a_1}^{a_1+\eta(a_2,a_1)}\left(g(x)\right)^{dx}}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{e.\left(f(a_1).f(a_2)\right)^{\int_0^1 h(\mu)d\mu}}{\left(\frac{\left(g(a_2)\right)^{g(a_2)}}{\left(g(a_1)\right)^{g(a_1)}}\right)^{\frac{1}{g(a_2)-g(a_1)}}},$$

which is the desired result.

Theorem 2.4 Let $\Im \subseteq \mathbb{R}$ an open invex subset with respect to $\eta : \Im \times \Im \to \mathbb{R}$ and $a_1, a_2 \in \Im$ with $a_1 < a_1 + \eta(a_2, a_1)$. Let f and g be preinvex and multiplicatively h -preinvex positive functions, respectively, on the interval $\left[a_1, a_1 + \eta(a_2, a_1)\right]$. Then, we have

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} . \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}} .(g(a_1).g(a_2))^{\int_0^1 h(\mu)d\mu}}{e}.$$

Proof Note that

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} . \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$= \left(e^{\int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(f(x))dx + \int_{a_1}^{a_1+\eta(a_2,a_1)} \ln(g(x))dx} \right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$= \left(e^{\eta(a_2,a_1)\left(\begin{array}{l}\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu \\ +\int_0^1 \ln(a_1+\mu\eta(a_2,a_1))d\mu\end{array}\right)} \right)^{\frac{1}{\eta(a_2,a_1)}}$$

$$= e^{\int_0^1 \ln(f(a_1+\mu\eta(a_2,a_1)))d\mu + \int_0^1 \ln(g(a_1+\mu\eta(a_2,a_1)))d\mu}$$

$$\le e^{\left(\begin{array}{l}\int_0^1 \ln(f(a_1)+\mu(f(a_2)-f(a_1)))d\mu \\ -\int_0^1 \ln\left((g(a_1))^{h(1-\mu)}(g(a_2))^{h(\mu)}\right)d\mu\end{array}\right)}$$

$$= e^{\ln\left(\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}}\right) - 1 + \ln(g(a_1).g(a_2))^{\int_0^1 h(\mu)d\mu}}$$

$$= \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}} .(g(a_1).g(a_2))^{\int_0^1 h(\mu)d\mu}}{e}.$$

Consequently,

$$\left(\int_{a_1}^{a_1+\eta(a_2,a_1)} (f(x))^{dx} . \int_{a_1}^{a_1+\eta(a_2,a_1)} (g(x))^{dx}\right)^{\frac{1}{\eta(a_2,a_1)}} \le \frac{\left(\frac{(f(a_2))^{f(a_2)}}{(f(a_1))^{f(a_1)}}\right)^{\frac{1}{f(a_2)-f(a_1)}} .(g(a_1).g(a_2))^{\int_0^1 h(\mu)d\mu}}{e}.$$

This completes the proof.

Remark 2.1 Now we point out some special cases which are included in our main results.

1. If $h(\mu) = \mu$, then our results reduce to the results for multiplicatively preinvex functions given in [15].
2. If $h(\mu) = \mu$ and $\eta(a_2, a_1) = a_2 - a_1$, then our results reduce to the results for multiplicatively convex functions given in [4].
3. If $h(\mu) = \mu^s$ and $\eta(a_2, a_1) = a_2 - a_1$, then our results reduce to the results for multiplicatively *s*-convex functions given in [6].
4. If $h(\mu) = 1$ and $\eta(a_2, a_1) = a_2 - a_1$, then our results reduce to the results for multiplicatively *P*-functions given in [7].

Availability of Data and Materials

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Competing Interests

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