Turkish Journal of Analysis and Number Theory, 2021, Vol. 9, No. 3, 42-47
Available online at http://pubs.sciepub.com/tjant/9/3/2
Published by Science and Education Publishing
DOI:10.12691/tjant-9-3-2

Schur *m*-Power Convexity of a New Class of Symmetric Functions with Applications

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Received October 02, 2021; Revised November 05, 2021; Accepted November 12, 2021

Abstract In the paper, by using the properties of Schur *m*-power convex function, we discuss Schur *m*-power convexity of a new class of symmetric functions $\Phi^*(x,r,p)=\prod_{1\le i_1<i_2<\cdots<i_r\le n}\sum_{j=1}^{r}x_{i_j}^p$, $r\in\{1,2,\cdots,n\}$, where $i_1,i_2,\cdots,i_r$ are non-negative integers, $x\in R_{++}^n$, and $p\in N^+$. We obtain that $\Phi_n^*(x,r,p)$ is Schur *m*-power convex for $m\le 0$ and Schur *m*-power concave for $m\ge p$. We also give a counter example to illustrate $\Phi_n^*(x,r,p)$ is neither Schur convex nor Schur concave for $p>1$. As applications, a Klamkin-Newman type inequality and some analytic inequalities are derived.

Keywords: *Schurm-power convexity, symmetric function, mean, majorization, inequality*

Cite This Article: Shuhong Wang, Hui Wang, and Haiyan Yu, "Schur *m*-Power Convexity of a New Class of Symmetric Functions with Applications." *Turkish Journal of Analysis and Number Theory*, vol. 9, no. 3 (2021): 42-47. doi: 10.12691/tjant-9-3-2.

1. Introduction

Throughout this paper, let

$$R_{++}^n=\{x=(x_1,x_2,\cdots,x_n)\in R^n : x_i>0, i=1,2,\cdots,n\},$$

$R=R^1$ and $N^+=\{1,2,\cdots,\infty\}$.

In 1923, Schur had firstly introduced the concept of Schur convexity ([1]). In 1970, Schur convexity was generalized and the notion of Schur harmonic convexity was brought in [2]. In 2004, Zhang had defined the Schur geometrical convexity as a parallel one to Schur convex theory ([3]). In 2012, Yang generalized the notion of Schur convexity to Schur *f*-convexity, which contains the Schur convexity, the Schur geometrical convexity, Schur harmonic convexity and so on ([4,5,6]). Further, Schur *m*-power convexity of some special means have been discussed in [4,5,6]. The Schur convex theory have played an important role in the investigation of Mathematics and other disciplines. In recent years, the study on the properties of the symmetric functions and the properties of the means is very active by using theory of majorization and Schur convexity. Subsequently, lots of new analytic inequalities were obtained and many classical inequalities were generalized and improved. Related work please see References [2,3,6,7-32].

In [15], the Hamy symmetric function was defined as following:

$$F_n(x,r)=F_n(x_1,x_2,\cdots,x_n;r)=\sum_{1\le i_1<i_2<\cdots<i_r\le n}\left(\prod_{j=1}^{r}x_{i_j}\right)^{\frac{1}{r}},$$

where $i_1,i_2,\cdots,i_r$ are non-negative integers, $x\in R_{++}^n$, and $r\in N^+$. Its properties and applications can be found in [8].

In [11], Guan generalized the Hamy symmetric function and defined the generalized Hamy symmetric function:

$$F_n^*(x,r)=\sum_{i_1+i_2+\cdots+i_n=r}\left(x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}\right)^{\frac{1}{r}}$$

where $i_1,i_2,\cdots,i_r$ are non-negative integers, $x\in R_{++}^n$, and $r\in N^+$.

Guan also proved that Hamy symmetric function $F_n(x,r)$ and generalized Hamy symmetric function $F_n^*(x,r)$ are Schur concave and Schur geometrically convex in R_{++}^n, and established some analytic inequalities by use of the theory of majorization [11].

In [9], Chu and Sun proved the generalized Hamy symmetric function $F_n^*(x,r)$ is Schur harmonically convex in R_{++}^n and established some analytic inequalities as its applications.

In [24], the following symmetric function was defined by Wang:

$$\Phi_n(x,r)=\prod_{i_1+i_2+\cdots+i_n=r}\left(x_1^{i_1}+x_2^{i_2}+\cdots+x_n^{i_n}\right),\quad r\in N^+$$

where $x\in R_{++}^n$ and $i_1,i_2,\cdots,i_r$ are non-negative integers. Wang proved that $\Phi_n(x,r)$ is Schur geometrically convex and Schur *m*-power convex for *m*≤0 and listed a counter example to illustrate $\Phi_n(x,r)$ is neither Schur convex nor Schur concave.

Now, we define the following new symmetric function:

Definition 1.1. Let $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$. For fixed $p\in N^+$, define the symmetric function as following

$$\Phi^*(x,r,p)=\prod_{1\le i_1<i_2<\cdots<i_r\le n}\sum_{j=1}^{r}x_{i_j}^p,\ r\in\{1,2,\cdots,n\},\quad (1.1)$$

where $i_1,i_2,\cdots,i_r$ are non-negative integers.

Obviously, $\sum_{i=1}^n x_i^p$ is a factor of $\Phi_n^*(x,r,p)$.

In the paper, by using the properties of Schur *m*-power convex function, we discuss Schur *m*-power convexity of a new class of symmetric functions $\Phi_n^*(x,r,p)$. We obtain that $\Phi_n^*(x,r,p)$ is Schur *m*-power convex for $m\le 0$ and Schur *m*-power concave for $m\ge p$. We also give a counter example to illustrate $\Phi_n^*(x,r,p)$ is neither Schur convex nor Schur concave for $p>1$. As applications, a Klamkin-Newman type inequality and some analytic inequalities are derived.

2. Preliminary

Firstly, we recall some necessary definitions and lemmas.

Definition 2.1 ([1]). Assume that $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)\in R^n$ are two n-tuples real numbers.

(1) *y* majorizes *x* (in symbols $x\prec y$), if

$$\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]},\ (k=1,2,\cdots,n-1)$$

and

$$\sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]},$$

where $x_{[1]}\ge x_{[2]}\ge\cdots\ge x_{[n]}, y_{[1]}\ge y_{[2]}\ge\cdots\ge y_{[n]}$ are rearrangements of *x* and *y* in a descending order.

(2) Let $I^n\subset R^n$, then I^n is said to be a convex set if

$$(\lambda x_1+(1-\lambda)y_1,\cdots,\lambda x_n+(1-\lambda)y_n)\in I^n,$$

for any $x=(x_1,x_2,\cdots,x_n)$ and $y=(y_1,y_2,\cdots,y_n)\in R^n$, where $\lambda\in[0,1]$.

A function $f:I^n\subset R^n\to R$ is said to be Schur convex on I^n if

$$x\prec y \text{ on } I^n\Rightarrow f(x)\le f(y)$$

A function f is said to be Schur concave on I^n if and only if $-f$ is a Schur convex function.

Lemma 2.1 ([1]). Let I^n be a convex set of R^n and has a nonempty interior set. Assume that $f:I^n\to R$ is a symmetric function which is continuous on I^n and differentiable in I^n. Then *f* is a Schur convex function (or a Schur concave function) if and only if

$$(x_1-x_2)\left(\frac{\partial f}{\partial x_1}-\frac{\partial f}{\partial x_2}\right)\ge 0\ (\le 0)\text{ for all }x\in I_n.\quad (2.1)$$

Definition 2.2. Let I^n be a subset of R_{++}^n,

$$x=(x_1,x_2,\cdots,x_n)\text{ and }y=(y_1,y_2,\cdots,y_n)\in I^n.$$

(1) I^n is said to be a harmonically convex set if

$$\left(\frac{x_1y_1}{\lambda x_1+(1-\lambda)y_1},\cdots,\frac{x_ny_n}{\lambda x_n+(1-\lambda)y_n}\right)\in I^n,$$

where $\lambda\in[0,1]$.

Let $f:I^n\to(0,\infty)$ be a symmetric function and has continuous partial derivatives on I^n. Then f is a Schur harmonically convex function if

$$x\prec y \text{ on } I^n\Rightarrow f\left(\frac{1}{x}\right)\le f\left(\frac{1}{y}\right),$$

where $\frac{1}{x}=\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n}\right)$. f is called Schur harmonically concave if $-f$ is Schur harmonically convex.

Lemma 2.2 ([2]). Let $I^n\subset R_{++}^n$ be a symmetric harmonically convex set with a nonempty interior. Assume that $f:I^n\to(0,\infty)$ is a symmetric function which is continuous on I^n and differentiable in I^n. Then *f* is a Schur harmonically convex function (or a Schur harmonically concave function) if and only if

$$(x_1-x_2)\left(x_1^2\frac{\partial f}{\partial x_1}-x_2^2\frac{\partial f}{\partial x_2}\right)\ge 0\ (\le 0)\text{ for all }x\in I^n.\quad (2.2)$$

Definition 2.3 ([3]). Let I^n be a subset of R_{++}^n,

$$x=(x_1,x_2,\cdots,x_n)\text{ and }y=(y_1,y_2,\cdots,y_n)\in I^n.$$

(1) The *n*-tuple *x* is said to be geometrically majorized by *y* (in symbols $\ln x\prec\ln y$), if

$$\prod_{i=1}^k x_{[i]}\le\prod_{i=1}^k y_{[i]},\ k=1,2,\cdots,n-1,$$

and

$$\prod_{i=1}^n x_{[i]}=\prod_{i=1}^n y_{[i]}.$$

(2) I^n is said to be a geometrically convex set if

$$(x_1^{\lambda} y_1^{1-\lambda}, \cdots, x_n^{\lambda} y_n^{1-\lambda}) \in I^n,$$

where $\lambda \in [0,1]$.

(2) A function $f: I^n \to (0,\infty)$ is Schur geometrically convex function if

$$\ln x \prec \ln y \text{ on } I^n \Rightarrow f(x) \le f(y).$$

A function f is called Schur geometrically concave if $-f$ is Schur geometrically convex.

Lemma 2.3 ([3]). Let $I^n \subset R_{++}^n$ be a symmetric geometrically convex set with a nonempty interior. Assume that $f: I^n \to (0,\infty)$ is a symmetric function which is continuous on I^n and differentiable in I^n. Then f is a Schur geometrically convex function (or a Schur geometrically concave function) if and only if

$$(x_1 - x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \ge 0 \ \ (\le 0) \text{ for all } x \in I^n. \quad (2.3)$$

Definition 2.4 ([4,5,6]). Let $f:(0,\infty) \to R$ be defined by

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \ne 0, \\ \ln x, & m = 0. \end{cases} \quad (2.4)$$

Then function $\psi: I^n \subseteq R_{++}^n \to R$ is said to be Schur m-power convex on I^n if

$$(f(x_1), f(x_2), \cdots, f(x_n)) \prec (f(y_1), f(y_2), \cdots, f(y_n)) \text{ on } I^n \Rightarrow \psi(x) \le \psi(y). \quad (2.5)$$

A function ψ is said to be Schur m-power concave if $-\psi$ is Schur m-power convex.

If taking $f(x) = x, \ln x, \frac{1}{x}$ in Definition 2.4, then the concepts of Schur-convex, Schur-geometrically convex and Schur-harmonically convex functions can be deduced respectively.

Lemma 2.4 ([4,5,6]). Let $\Omega \subseteq R_{++}^n$ be a symmetric set with a nonempty interior Ω^0. Assume that $\psi: \Omega \subseteq R_{++}^N \to R$ is a symmetric function which is continuous on Ω and differentiable in Ω^0. Then ψ is a Schur m-power convex (m-power concave) on Ω if and only if

$$(x_1 - x_2)\left(x_1^{1-m} \frac{\partial \psi}{\partial x_1} - x_2^{1-m} \frac{\partial \psi}{\partial x_2}\right) \ge 0 \quad (\le 0) \quad (2.6)$$

hold for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$ with $x_1 \ne x_2$.

The relation of different orders of Schur m-power convex function have been studied by Zhang in [30] and the following result were obtained.

Lemma 2.5 ([22,30]). Let $p > q, I \subset (0,\infty)$, and a real function $f: I^n \to R$. If f is a increasing and Schur p-power convex function, then f must be Schur q-power convex function.

Lemma 2.6 ([1]). Let $\Omega \in R^n$ be open and convex set, and a real function $f(x)$ is differentiable on Ω. Then $f(x)$ is increasing if and only if $\nabla f(x) \ge 0$, where

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n}\right) \in R^n.$$

3. Main Results

In this section, the Schur m-power convexity of $\Phi_n^*(x,r,p)$ for $r \in \{1,2,\cdots,n\}$, $x \in R_{++}^n$ and $p \in N^+$ is discussed.

Theorem 3.1. Let $x = (x_1, x_2, \cdots, x_n) \in R_{++}^n$. For fixed $p \in N^+$,

$$\Phi_n^*(x,r,p) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j}^p, \ r \in \{1,2,\cdots,n\}, \quad (3.1)$$

where $i_1, i_2, \cdots, i_r$ are non-negative integers.

(1) when $m \le 0$, $\Phi_n^*(x,r,p)$ is Schur m-power convex.

(2) when $m \ge p$, $\Phi_n^*(x,r,p)$ is Schur m-power concave.

Proof. Obviously, the function $\Phi_n^*(x,r,p)$ is symmetric and continuously differentiable.

By calculation, it follows that

$$\begin{aligned} \Phi_n^*(x,r,p) &= \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j}^p \\ &= \prod_{3 \le i_1 < \cdots < i_{r-2} \le n} \left(x_1^p + x_2^p + \sum_{j=1}^{r-2} x_{i_j}^p\right) \\ &\times \prod_{3 \le i_1 < \cdots < i_{r-1} \le n} \left(x_1^p + \sum_{j=1}^{r-1} x_{i_j}^p\right) \\ &\times \prod_{3 \le i_1 < \cdots < i_{r-1} \le n} \left(x_2^p + \sum_{j=1}^{r-1} x_{i_j}^p\right) \\ &\times \prod_{3 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j}^p. \end{aligned} \quad (3.2)$$

Make

$$\begin{aligned} L &= \prod_{3 \le i_1 < \cdots < i_{r-2} \le n} \left(x_1^p + x_2^p + \sum_{j=1}^{r-2} x_{i_j}^p\right), \\ Q &= \prod_{3 \le i_1 < \cdots < i_{r-1} \le n} \left(x_1^p + \sum_{j=1}^{r-1} x_{i_j}^p\right), \\ R &= \prod_{3 \le i_1 < \cdots < i_{r-1} \le n} \left(x_2^p + \sum_{j=1}^{r-1} x_{i_j}^p\right), \\ S &= \prod_{3 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} x_{i_j}^p. \end{aligned} \quad (3.3)$$

By calculating the partial derivatives of L, Q, R and S on x_1 and x_2, respectively, it is easy to obtain

$$\frac{\partial L}{\partial x_i}=L\prod_{3\le i_1<\cdots<i_{r-2}\le n}\frac{px_i^{p-1}}{\left(x_1^p+x_2^p+\sum_{j=1}^{r-2}x_{i_j}^p\right)},i=1,2,$$

$$\frac{\partial Q}{\partial x_1}=Q\prod_{3\le i_1<\cdots<i_{r-1}\le n}\frac{px_1^{p-1}}{\left(x_1^p+\sum_{j=1}^{r-1}x_{i_j}^p\right)} \quad (3.4)$$

$$\frac{\partial R}{\partial x_2}=R\prod_{3\le i_1<\cdots<i_{r-1}\le n}\frac{px_2^{p-1}}{\left(x_2^p+\sum_{j=1}^{r-1}x_{i_j}^p\right)}$$

$$\frac{\partial S}{\partial x_i}=0,\quad i=1,2.$$

Thus by differentiating $\Phi_n^*(x,r,p)$ with respect to $x_i\,(i=1,2)$, we can derive

$$\frac{\partial\Phi_n^*(x,r,p)}{\partial x_i}=QRS\frac{\partial L}{\partial x_i}+LRS\frac{\partial Q}{\partial x_i}+LQS\frac{\partial R}{\partial x_i}+LQR\frac{\partial S}{\partial x_i},i=1,2. \quad (3.5)$$

Therefore, it deduce that

$$\begin{aligned}&(x_1-x_2)\left(x_1^{1-m}\frac{\partial\Phi_n^*}{\partial x_1}-x_2^{1-m}\frac{\partial\Phi_n^*}{\partial x_2}\right)\\
&=(x_1-x_2)LQRS\left(\sum_{3\le i_1<\cdots<i_{r-2}\le n}\frac{px_1^{p-m}}{x_1^p+x_2^p+\sum_{j=1}^{r-2}x_{i_j}^p}\right.\\
&+\sum_{3\le i_1<\cdots<i_{r-1}\le n}\frac{px_1^{p-m}}{x_1^p+\sum_{j=1}^{r-1}x_{i_j}^p}\\
&-\sum_{3\le i_1<\cdots<i_{r-1}\le n}\frac{px_2^{p-m}}{x_2^p+\sum_{j=1}^{r-1}x_{i_j}^p}\\
&\left.-\sum_{3\le i_1<\cdots<i_{r-2}\le n}\frac{px_2^{p-m}}{x_1^p+x_2^p+\sum_{j=1}^{r-2}x_{i_j}^p}\right)\\
&=p\,(x_1-x_2)\\
&\times LQRS\left(\sum_{3\le i_1<\cdots<i_{r-2}\le n}\frac{x_1^{p-m}-x_2^{p-m}}{x_1^p+x_2^p+\sum_{j=1}^{r-2}x_{i_j}^p}\right.\\
&\left.+\sum_{3\le i_1<\cdots<i_{r-1}\le n}\frac{\left[\begin{array}{l}x_1^px_2^p(x_1^{-m}-x_2^{-m})\\+\sum_{j=1}^{r-1}x_{i_j}^p(x_1^{p-m}-x_2^{p-m})\end{array}\right]}{\left(x_1^p+\sum_{j=1}^{r-1}x_{i_j}^p\right)\left(x_2^p+\sum_{j=1}^{r-1}x_{i_j}^p\right)}\right).\end{aligned} \quad (3.6)$$

Because the function x^k is increasing for $k>0$ and decreasing for $k<0$ in $(0,\infty)$, then for $m\le 0$,

$$(x_1-x_2)(x_1^{1-m}\frac{\partial\Phi_n^*}{\partial x_1}-x_2^{1-m}\frac{\partial\Phi_n^*}{\partial x_2})\ge 0,$$

for $m>p$,

$$(x_1-x_2)(x_1^{1-m}\frac{\partial\Phi_n^*}{\partial x_1}-x_2^{1-m}\frac{\partial\Phi_n^*}{\partial x_2})\le 0.$$

By means of Lemma 2.5, the Theorem is proved.

Remark 3.1. When $p>1$, $\Phi_n^*(x,r,p)$ is neither Schur convex nor Schur concave.

In fact, if $\Phi_n^*(x,r,p)$ is Schur convex for $x\in R_{++}^n$, $p\in N^+$, and $r\in\{1,2,\cdots,n\}$, then $\Phi_2^*(x,1,p)=x_1^px_2^p$ is also Schur convex.

For $a>b>0$,

$$\left(\frac{a+b}{2},\frac{a+b}{2}\right)\prec(a,b)\prec(a+b,0).$$

Thus

$$a^pb^p\le(a+b)^p\times 0^p=0.$$

which contradicts with $a>b>0$.

If $\Phi_n^*(x,r,p)$ is Schur concave forr $x\in R_{++}^n$, $p\in N^+$, and $r\in\{1,2,\cdots,n\}$, then

$$\Phi_2^*(x,2,p)=x_1^p+x_2^p$$

is also Schur concave.

That is

$$a^p+b^p\ge(a+b)^p+0^p=(a+b)^p,$$

which contradicts with $a^p+b^p\le(a+b)^p$ for $p>1$.

Corollary 3.1.1. For $x\in R_{++}^n$, $n,p\in N^+$, and $r=\{1,2,\cdots,n\}$, we have

(1) $\Phi_n^*(x,r,p)$ is Schur harmonically convex.

(2) $\Phi_n^*(x,r,p)$ is Schur geometrically convex.

Proof. In Theorem 3.2, m takes -1 and 0, respectively, we get the results.

Corollary 3.1.2. For $x\in R_{++}^n$ and $r=\{1,2,\cdots,n\}$, $\Phi_n^*(x,r,1)$ is Schur concave.

Proof. In Theorem 3.2, m and p take 1, the results are obtained.

4. Applications

The following lemmas are useful for establishing the inequalities.

Lemma 4.1 ([12]). Let $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$ and $\sum_{i=1}^n x_i=s$. If $c\ge s$, then

$$\left(\frac{c-x_1}{\frac{nc}{s}-1},\frac{c-x_2}{\frac{nc}{s}-1},\cdots,\frac{c-x_n}{\frac{nc}{s}-1}\right)\prec(x_1,x_2,\cdots,x_n)=x. \quad (4.1)$$

Lemma 4.2 ([12]). Let $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$ and $\sum_{i=1}^n x_i = s$. If $c\le 0$, then

$$\left(\frac{c+x_1}{\frac{nc}{s}+1},\frac{c+x_2}{\frac{nc}{s}+1},\cdots,\frac{c+x_n}{\frac{nc}{s}+1}\right)\prec(x_1,x_2,\cdots,x_n)=x. \quad (4.2)$$

Let $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$ and $p\in R$, then the mean

$$M_p(x)=\begin{cases}\left(\dfrac{x_1^p+x_2^p+\cdots+x_n^p}{n}\right)^{\frac{1}{p}}, & p\ne 0;\\ \sqrt{x_1x_2\cdots x_n}, & p=0.\end{cases}$$

is famously the p-th power mean of order p of $x_i(i=1,2,\cdots,n)$. Specially, taking $p=1$, $p=0$, and $p=-1$, respectively, the arithmetic, the geometric and the harmonic means of $x_i(i=1,2,\cdots,n)$ are derived as following:

$$A_n(x)=\frac{1}{n}\sum_{i=1}^n x_i, G_n(x)=\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}, H_n(x)=\frac{n}{\sum_{i=1}^n \frac{1}{x_i}}.$$

Lemma 4.3 ([28]). Assume that $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$, then

$$(A_n(x),A_n(x),\cdots,A_n(x))\prec(x_1,x_2,\cdots,x_n), \quad (4.3)$$

$$(\ln(G_n(x),G_n(x),\cdots,G_n(x))\prec\ln(x_1,x_2,\cdots,x_n), \quad (4.4)$$

$$\left(\frac{1}{H_n(x)},\frac{1}{H_n(x)},\cdots,\frac{1}{H_n(x)}\right)\prec(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n}). \quad (4.5)$$

By making use of above Lemmas and Corollary 3.1.1 and 3.1.2, the following results are easy to be get.

Theorem 4.1. Assume that $x=(x_1,x_2,\cdots,x_n)\in R_{++}^n$, and $\sum_{i=1}^n x_i = s$. If $c\ge s$ and fixed $r,p\in N^+$, then

$$\Phi_n^*\left(\frac{\frac{nc}{s}-1}{c-x_1},\frac{\frac{nc}{s}-1}{c-x_2},\cdots,\frac{\frac{nc}{s}-1}{c-x_n},r,p\right)\le\Phi_n^*\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n},r,p\right); \quad (4.6)$$

$$\Phi_n^*\left(\frac{\frac{nc}{s}+1}{c+x_1},\frac{\frac{nc}{s}+1}{c+x_2},\cdots,\frac{\frac{nc}{s}+1}{c+x_n},r,p\right)\le\Phi_n^*\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n},r,p\right); \quad (4.7)$$

$$\Phi_n^*\left(\frac{1}{A_n(x)},\frac{1}{A_n(x)},\cdots,\frac{1}{A_n(x)},r,p\right)\le\Phi_n^*\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n},r,p\right); \quad (4.8)$$

$$\Phi_n^*\left(G_n(x),G_n(x),\cdots,G_n(x),r,p\right)\le\Phi_n^*(x_1,x_2,\cdots,x_n,r,p); \quad (4.9)$$

$$\Phi_n^*\left(H_n(x),H_n(x),\cdots,H_n(x),r,p\right)\le\Phi_n^*(x_1,x_2,\cdots,x_n,r,p); \quad (4.10)$$

$$\Phi_n^*\left(\frac{c-x_1}{\frac{nc}{s}-1},\frac{c-x_2}{\frac{nc}{s}-1},\cdots,\frac{c-x_n}{\frac{nc}{s}-1},r,1\right)\ge\Phi_n^*(x_1,x_2,\cdots,x_n,r,1); \quad (4.11)$$

$$n^n\left(\frac{c+x_1}{\frac{nc}{s}+1},\frac{c+x_2}{\frac{nc}{s}+1},\cdots,\frac{c+x_n}{\frac{nc}{s}+1},r,1\right)\ge(x_1,x_2,\cdots,x_n,r,1); \quad (4.12)$$

$$\Phi_n^*\left(A_n(x),A_n(x),\cdots,A_n(x),r,1\right)\ge(x_1,x_2,\cdots,x_n,r,1); \quad (4.13)$$

$$\Phi_n^*\left(\frac{1}{H_n(x)},\frac{1}{H_n(x)},\cdots,\frac{1}{H_n(x)},r,1\right)\ge\left(\frac{1}{x_1},\frac{1}{x_2},\cdots,\frac{1}{x_n},r,1\right). \quad (4.14)$$

The following inequality

$$\prod_{i=1}^n (x_i^{-1}+1)\ge(n+1)^n, \quad (4.15)$$

where $x_i>0$, $i=1,2,\cdots,n$ and $\sum_{i=1}^n x_i=1$ is well known Klamkin-Newman inequality ([31]).

Putting $r=1$ and $r=n$ respectively in (4.8), the Klamkin-Newman type inequalities are also derived.

Corollary 4.1.1. For $\sum_{i=1}^n x_i=1$ for $0<x_i<1$ $(i=1,2,\cdots,n)$, then

$$\prod_{i=1}^n x_i^{-p}\ge n^{np} \textit{ and } \prod_{i=1}^n x_i^{-p}\ge n^{1+p}.$$

Specially, when $p=1$, we have

$$\prod_{i=1}^n x_i^{-1}\ge n^n\ge n^2.$$

For $n=2$ in $\Phi_n^*(x,r,p)$, it follows that

$$\Phi_2^*(x,r,p)=x_1^p x_2^p(x_1^p+x_2^p).$$

For simplicity, assume that

$$M_p^p=M_p^p(x_1,x_2)=\frac{x_1^p+x_2^p}{2},$$

$$A=A(x_1,x_2)=\frac{x_1+x_2}{2}, \textit{ and } G=G(x_1,x_2)=\sqrt{x_1x_2}.$$

Since $\Phi_2^*(x,r,p)$ is harmonically convex function on $(x_1,x_2)\in(0,\infty)\times(0,\infty)$ for $r\in\{1,2,\cdots,n\}$ and $p\in N^+$, it is easy to derive the following inequality by means of the inequality (4.8).

Corollary 4.1.2. For $(x_1,x_2)\in(0,\infty)\times(0,\infty)$, then

$$M_p^p A^{3p} \ge G^{4p}.$$

Further,

$$M_p A \ge G^2.$$

5. Conclusion

To sum up, we define a new class of symmetric functions in this paper

$$\Phi^*(x,r,p)=\prod_{1\le i_1<i_2<\cdots<i_r\le n}\sum_{j=1}^{r} x_{i_j}^p,\ r\in\{1,2,\cdots,n\},$$

where $i_1,i_2,\cdots,i_r$ are non-negative integers, $x\in R_{++}^n$, and $p\in N^+$, and discuss Schur m-power convexity of $\Phi_n^*(x,r,p)$ by using the properties of Schur m-power convex function. We obtain that $\Phi_n^*(x,r,p)$ is Schur m-power convex for $m\le 0$ and Schur m-power concave for $m\ge p$. We also give a counter example to illustrate $\Phi_n^*(x,r,p)$ is neither Schur convex nor Schur concave for $p>1$. As applications, a Klamkin-Newman type inequality and some analytic inequalities are derived.

Acknowledgements

This work was supported by PhD Research Foundation of Inner Mongolia University for Nationalities (No. BS401 and No. BS402) and Science Research Foundation of Inner Mongolia University for Nationalities (No. NMDYB1777) and Natural Science Foundation of Inner Mongolia Autonomous Region (No. 2019MS01007).

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