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Some Identities of the Degenerate Poly-Frobenius-Genocchi Polynomials of Complex Variables

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Abstract The main of this paper is to define and investigate a new class of the degenerate poly-Frobenius-Genocchi polynomials with the help of the polyexponential functions. In this paper, we define the degenerate poly-Frobenius-Genocchi polynomials of complex variables arising from the modified polyexponential functions, and establish some explicit expressions for these polynomials. Meanwhile, some interesting connections between these polynomials and some other special polynomials are also showed.

Keywords: *Frobenius-Euler numbers and polynomials, Genocchi numbers and polynomials, Frobenius-Genocchi numbers and polynomials, The degenerate Stirling numbers of both kind, The degenerate Stirling polynomials of the second kind, The Bernoulli polynomials of the second kind, The polyexponential functions, The degenerate poly-Frobenius-Genocchi polynomials of complex variables*

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1. Introduction

Throughout this paper, N denotes the set of natural numbers, N_0 denotes the set of nonnegative integeres, R denotes the set of real numbers and C denotes the set of complex numbers. We begin by introducing the following definitions and notaitions [1-18].

The Frobenius-Euler polynomials $H_n(x;u)$ are defined by [1-18];

$$\sum_{n=0}^{\infty} H_n(x;u)\frac{t^n}{n!} = \frac{1-u}{e^t-u}e^{xt}, \tag{1.1}$$

where $u \neq 1$ and $e^t \neq u$.

When x=0, $H_n(u) := H_n(0;u)$ are called the Frobenius-Euler numbers.

The Genocchi polynomials are defined by [11,12,14])

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, |t| < \pi \tag{1.2}$$

When x=0, $G_n(0) := G_n$ are called the Genocchi numbers.

The Frobenius-Genocchi polynomials are defined by [18]

$$\sum_{n=0}^{\infty} FG_n(x,u)\frac{t^n}{n!} = \frac{(1-u)t}{e^t-u}e^{t}. \tag{1.3}$$

For u=-1, $FG_n(x,-1) = G_n(x)$ and x=0, $FG_n(u) := FG_n(0,u)$ are called the Frobenius-Genocchi numbers.

The degenerate exponential function is defined by [3-11] with λ∈ $R\backslash\{0\}$

$$e_\lambda^x(t) = (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x)_{n,\lambda}\frac{t^n}{n!}$$

and

$$e_\lambda(t) = e_\lambda^1(t) = (1+\lambda t)^{1/\lambda} \tag{1.4}$$

where $(x)_{0,1} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-n-1\lambda)$, $n\geq 1$.

For $x \in R$ and k nonnegative integer, the degenerate λ-Stirling polynomials of the second kind are defined by [5]

$$\frac{(e_\lambda(t)-1)^k}{k!}e_\lambda^x(t) = \sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!}. \tag{1.5}$$

Note that

$$\lim_{\lambda\to 0}\sum_{n=k}^{\infty} S_{2,\lambda}^{(x)}(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}e^{xt}.$$

From (1.4), we get

$$(t+x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(t)_k, n>0, \tag{1.6}$$

where $(t)_0 = 1$, $(t)_n = t(t-1)(t-2)\cdots(t-(n-1))$, $n \geq 1$.

Using (1.4) and (1.6), we note that

$$e_\lambda^{(x+y)}(t) = \sum_{n=0}^{\infty}(x+y)_{n,\lambda}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_{2,\lambda}^{(x)}(n,k)(y)_k\frac{t^n}{n!}.$$

The degenerate Stirling numbers of the first kind are defined by [3-10]

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!},\ k \geq 0. \quad (1.7)$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(n,l) = S_1(n,l)$ where $S_1(n,l)$ are the Stirling numbers of the first kind given by [5]

$$\frac{\left(\log(1+t)\right)^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, k \geq 0 \quad (1.8)$$

The degenerate Stirling numbers of the second kind are defined by [3-10]

$$\frac{\left(e_\lambda(t)-1\right)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, k \geq 0 \quad (1.9)$$

Observe that $\lim_{\lambda\to 0} S_{2,\lambda}(n,l) = S_2(n,l)$ where $S_2(n,l)$ are the Stirling numbers of the second kind given by [5]

$$\frac{\left(e^t-1\right)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, k \geq 0. \quad (1.10)$$

The degenerate Bernoulli polynomials of the second kind are given by [6,8]

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}(x)\frac{t^n}{n!} \quad (1.11)$$

Note that $\lim_{\lambda\to 0} b_{n,\lambda}(x) = b_n(x)$ where $b_n(x)$ are the Bernoulli polynomials of the second kind given by [6]

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \quad (1.12)$$

2. Degenerate Poly-Frobenius-Genocchi Numbers and Polynomials

In this section, we introduce and investigate the modified polyexponential functions. We give some identities and explicit relations for the modified degenerate polyexponential functions. We define the degenerate poly-Frobenius-Genocchi polynomials. Also, we give some relations and identities for these polynomials.

In [2], Boyadzhiev introduced the polyexponential function, Kim et al. in [6,7] considered and investigated the polyexponential functions and the degenerate polyexponential functions.

The polyexponential functions are defined by [3-11,14]

$$Ei_k(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^k(n-1)!}, k \in Z. \quad (2.1)$$

For k=1, $Ei_1(x) = e^x - 1$.

The modified degenerate polyexponential functions are given by [3-11,14]

$$Ei_{k,\lambda}(x) = \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n^k(n-1)!}x^n,\ \lambda \in R. \quad (2.2)$$

Note that

$$Ei_{1,\lambda}(x) = \sum_{n=1}^{\infty}(1)_{n,\lambda}\frac{x^n}{n!} = e_\lambda(x)-1.$$

For $k \in Z$ and by means of the modified degenerate polyexponential functions. We define the degenerate poly-Frobenius-Genocchi polynomials by the following generating functions.

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(k)}(x,u)\frac{t^n}{n!} = \frac{(1-u)Ei_{k,\lambda}\left(\log_\lambda(1+t)\right)}{e_\lambda(t)-u}e_\lambda^x(t). \quad (2.3)$$

When x=0, $FG_{n,\lambda}^{(k)}(u) := FG_{n,\lambda}^{(k)}(0,u)$ are called the degenerate poly-Frobenius-Genocchi numbers, where $\log_\lambda(t) = \frac{1}{\lambda}\left(t^\lambda - 1\right)$ is the compositional inverse of $e_\lambda(t)$ satisfying

$$\log_\lambda\left(e_\lambda(t)\right) = e_\lambda\left(\log_\lambda(1+t)\right) = t.$$

For k=1 and u=-1, we get the degenerate Genocchi polynomials

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(1)}(x,-1)\frac{t^n}{n!} = \frac{2Ei_{1,\lambda}\left(\log_\lambda(1+t)\right)}{e_\lambda(t)+1}e_\lambda^x(t)$$

$$= \frac{2t}{e_\lambda(t)+1}e_\lambda^x(t) = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}.$$

From (2.3), we can write the following equations

$$FG_{n,\lambda}^{(k)}(x,u) = \sum_{m=0}^{n}\binom{n}{m} FG_{m,\lambda}^{(k)}(x)_{n-m,\lambda}, \quad (i)$$

$$FG_{n,\lambda}^{(k)}(x+y,u) = \sum_{m=0}^{n}\binom{n}{m} FG_{m,\lambda}^{(k)}(x,u)(y)_{n-m,\lambda}$$
$$= \sum_{m=0}^{n}\binom{n}{m} FG_{m,\lambda}^{(k)}(y,u)(x)_{n-m,\lambda}. \quad (ii)$$

$$FG_{n,\lambda}^{(k)}(x+y,u) = \sum_{m=0}^{n}\binom{n}{m} FG_{m,\lambda}^{(k)}(x+y)_{n-m,\lambda}, \quad (iii)$$

By (1.8) and (2.2), we get

$$Ei_{k,\lambda}\left(\log_\lambda(1+t)\right) = \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}\left(\log_\lambda(1+t)\right)^n}{n^k(n-1)!}$$
$$= \sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{(n-1)!}\sum_{m=n}^{\infty} S_{1,\lambda}(m,n)\frac{t^n}{n!} \quad (2.4)$$
$$= t\sum_{m=0}^{\infty}\sum_{n=1}^{m+1}\frac{(1)_{n,\lambda}}{n^k-1}\frac{S_{1,\lambda}(m+1,n)}{m+1}\frac{t^m}{m!}$$

Using (2.3) and (2.4), we get

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(k)}(x,u)\frac{t^n}{n!}=\frac{(1-u)e_q^x(t)}{e_\lambda(t)-u}Ei_{k,\lambda}(\log_\lambda(1+t))$$

$$=t\sum_{l=0}^{\infty} FG_{l,\lambda}(x,u)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{1}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\frac{t^m}{m!}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$ the above equations, we have the following theorem.

Theorem 1. For $n \geq 0$, we have

$$FG_{n,\lambda}^{(k)}(x,u)=n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}$$

$$\times\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)FG_{n-1-m,\lambda}(x,u).$$

From (2.3), we write as

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(k)}(x,u)\frac{t^n}{n!}\left(e_q(t)-u\right)$$

$$=(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))e_\lambda^x(t) \tag{2.5}$$

$$=(1-u)\sum_{n=0}^{\infty}\left(\begin{array}{l}\sum_{m=0}^{n}\binom{n}{m}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}\\ \times\frac{S_{1,\lambda}(m+1,j)}{m+1}(x)_{n-m,\lambda}\end{array}\right)\frac{t^{n+1}}{n!}$$

Comparing the coefficients of both sides in (2.5).

We have the following theorem.

Theorem 2. For $n \geq 0$, we have

$$\sum_{m=0}^{n}\binom{n}{m}FG_{m,\lambda}^{(k)}(x,u)(1)_{n-m,\lambda}-uFG_{n,\lambda}^{(k)}(x,u)$$

$$=(1-u)n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=1}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)(x)_{n-1-m,\lambda}.$$

From (2.2), we note that

$$\frac{d}{dx}Ei_{k,\lambda}(x)=\frac{d}{dx}\sum_{n=1}^{\infty}\frac{(1)_{j,\lambda}}{(n-1)!n^k}x^n=\frac{1}{x}Ei_{k-1,\lambda}(x). \tag{2.6}$$

Thus, by (2.5), we get

$$Ei_{k,\lambda}(x)=\int_0^x\frac{1}{t}Ei_{k-1,\lambda}(t)dt$$

$$=\underbrace{\int_0^x\frac{1}{t}\int_0^t\cdots\frac{1}{t}\int_0^t\frac{1}{t}}_{(k-2)times}Ei_{1,\lambda}(x)\underbrace{dt\cdots dt}_{(k-2)times}$$

$$=\underbrace{\int_0^x\frac{1}{t}\int_0^t\cdots\frac{1}{t}\int_0^t\frac{1}{t}}_{(k-2)times}(e_\lambda(t)-1)\underbrace{dt\cdots dt}_{(k-2)times}$$

where $k \in Z^+$ with $k \geq 2$.

From (1.11), (2.3) and (2.6), for k=2

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(2)}(x,u)\frac{t^n}{n!}$$

$$=\frac{1-u}{e_\lambda(t)-u}\int_0^t\frac{t}{\log_\lambda(1+t)}(1+t)^{\lambda-1}dt$$

$$=\frac{1-u}{e_\lambda(t)-u}\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{t^m}{m!}$$

$$=\sum_{l=0}^{\infty}H_{l,\lambda}(0,u)\frac{t^l}{l!}\sum_{m=0}^{\infty}\frac{b_{m,\lambda}(\lambda-1)}{m+1}\frac{t^m}{m!}.$$

From the last equations, we have the following theorem.

Theorem 3. For $n \geq 0$, we have

$$FG_{n,\lambda}^{(2)}(x,u)=\sum_{m=0}^{n}\binom{n}{m}H_{n-m,\lambda}(0,u)\frac{b_{m,\lambda}(\lambda-1)}{m+1},$$

where $H_{n,\lambda}(0,u)$ is degenerate Frobenius-Euler numbers.

Recently, Masjed-Jamai et al. in [13] and Srivastava et al. in [15,16] introduced a new type parametric Euler numbers and polynomials as

$$\frac{2}{e^t+1}e^{pt}\cos(qt)=\sum_{n=0}^{\infty}E_n^{(c)}(p,q)\frac{t^n}{n!}$$

and

$$\frac{2}{e^t+1}e^{pt}\sin(qt)=\sum_{n=0}^{\infty}E_n^{(s)}(p,q)\frac{t^n}{n!}$$

where

$$e^{pt}\cos(qt)=\sum_{n=0}^{\infty}C_n(p,q)\frac{t^n}{n!}$$

and

$$e^{pt}\sin(qt)=\sum_{n=0}^{\infty}S_n(p,q)\frac{t^n}{n!}.$$

3. Degenerate Poly-Frobenius-Genocchi Polynomials of Complex Variables

In this section, we define the Frobenius-Genocchi polynomials of the complex variables. We consider the degenerate cosine function and the degenerate sine function. Using the degenerate cosine function and the degenerate sine function, we introduce the cosine degenerate poly-Frobenius-Genocchi polynomials and the sine degenerate poly-Frobenius-Genocchi polynomials.

From (2.3), we write as

$$\sum_{n=0}^{\infty} FG_{n,\lambda}^{(k)}(x+iy;u)\frac{t^n}{n!}$$

$$=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_\lambda^{(x+iy)}(t)$$

$$=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_\lambda^x(t)\begin{bmatrix}\cos_\lambda^{(y)}(t)\\+i\sin_\lambda^{(y)}(t)\end{bmatrix}. \quad (3.1)$$

and

$$\sum_{n=0}^{\infty}FG_{n,\lambda}^{(k)}(x-iy;u)\frac{t^n}{n!}=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_\lambda^x(t)\begin{bmatrix}\cos_\lambda^{(y)}(t)\\-i\sin_\lambda^{(y)}(t)\end{bmatrix}. \quad (3.2)$$

By (3.1) and (3.2), we get

$$\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_\lambda^x(t)\cos_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\frac{FG_{n,\lambda}^{(k)}(x+iy;u)+FG_{n,\lambda}^{(k)}(x-iy;u)}{2}\frac{t^n}{n!} \quad (3.3)$$

and

$$\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_\lambda^x(t)\sin_\lambda^{(y)}(t)=\sum_{n=0}^{\infty}\frac{FG_{n,\lambda}^{(k)}(x+iy;u)-FG_{n,\lambda}^{(k)}(x-iy;u)}{2i}\frac{t^n}{n!}. \quad (3.4)$$

Using (1.4), we define the degenerate cosine-functions and the degenerate sine-functions as

$$\cos_\lambda^{(y)}(t)=\frac{e_\lambda^{(iy)}(t)+e_\lambda^{(-iy)}(t)}{2}=\cos\left(\frac{y}{\lambda}\log(1+\lambda t)\right) \quad (3.5)$$

and

$$\sin_\lambda^{(y)}(t)=\frac{e_\lambda^{(iy)}(t)-e_\lambda^{(-iy)}(t)}{2i}=\sin\left(\frac{y}{\lambda}\log(1+\lambda t)\right) \quad (3.6)$$

where $\lim_{\lambda\to 0}\cos_\lambda^{(y)}(t)=\cos(yt)$ and

$$\lim_{\lambda\to 0}\sin_\lambda^{(y)}(t)=\sin(yt).$$

Now, we define the cosine degenerate poly-Frobenius-Genocchi polynomials and the sine degenerate poly-Frobenius-Genocchi polynomials, respectively;

$$\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,c]}(x,y;u)\frac{t^n}{n!}=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-u}e_\lambda^x(t)\cos_\lambda^{(y)}(t) \quad (3.7)$$

and

$$\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,s]}(x,y;u)\frac{t^n}{n!}=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_\lambda(t)-u}e_\lambda^x(t)\sin_\lambda^{(y)}(t). \quad (3.8)$$

From (1.4), we write

$$e_\lambda^{(iy)}(t)=\sum_{n=0}^{\infty}(iy)_{n,\lambda}\frac{t^n}{n!}$$

and

$$e_\lambda^{(-iy)}(t)=\sum_{n=0}^{\infty}(-iy)_{n,\lambda}\frac{t^n}{n!}.$$

Using (3.5) and (3.6), we get

$$\cos_\lambda^{(y)}(t)=\frac{1}{2}\sum_{n=0}^{\infty}\left((iy)_{n,\lambda}+(-iy)_{n,\lambda}\right)\frac{t^n}{n!} \quad (3.9)$$

and

$$\sin_\lambda^{(y)}(t)=\frac{1}{2i}\sum_{n=0}^{\infty}\left((iy)_{n,\lambda}-(-iy)_{n,\lambda}\right)\frac{t^n}{n!}. \quad (3.10)$$

By (1.4), (3.9) and (1.4), (3.10), we have the following equations, respectively,

$$e_\lambda^x(t)\cos_\lambda^{(y)}(t)=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}\left((iy)_{n,\lambda}+(-iy)_{n,\lambda}\right)\frac{t^n}{n!} \quad (3.11)$$

and

$$e_\lambda^x(t)\sin_\lambda^{(y)}(t)=\frac{1}{2i}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}\left((iy)_{n,\lambda}-(-iy)_{n,\lambda}\right)\frac{t^n}{n!} \quad (3.12)$$

From (3.7) and (3.11), we write

$$\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,c]}(x,y;u)\frac{t^n}{n!}=\frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u}e_q^x(t)\cos_\lambda^{(y)}(t)$$
$$=\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,c]}\frac{t^n}{n!}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}\begin{pmatrix}(iy)_{n,\lambda}\\+(-iy)_{n,\lambda}\end{pmatrix}\frac{t^n}{n!}$$

Using the Cauchy product and comparing the coefficients, we have

$$FG_{n,\lambda}^{[k,c]}(x,y;u)=\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}FG_{n-j,\lambda}^{(k)}\sum_{k=0}^{j}\binom{j}{k}(x)_{j-k,\lambda}\begin{pmatrix}(iy)_{n,\lambda}\\+(-iy)_{n,\lambda}\end{pmatrix}. \quad (3.13)$$

From (3.8) and (3.11), similarly, we have

$$FG_{n,\lambda}^{[k,s]}(x,y;u)=\frac{1}{2i}\sum_{j=0}^{n}\binom{n}{j}FG_{n-j,\lambda}^{(k)}\sum_{k=0}^{j}\binom{j}{k}(x)_{j-k,\lambda}\begin{pmatrix}(iy)_{n,\lambda}\\-(-iy)_{n,\lambda}\end{pmatrix}. \quad (3.14)$$

From (3.13) and (3.14), we have the following theorems.

Theorem 4. The following relations hold true:

$$FG_{n,\lambda}^{[k,c]}(x,y;u) = \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}FG_{n-j,\lambda}^{(k)}\sum_{k=0}^{j}\binom{j}{k}(x)_{j-k,\lambda}\begin{pmatrix}(iy)_{n,\lambda}\\+(-iy)_{n,\lambda}\end{pmatrix}$$

and

$$FG_{n,\lambda}^{[k,s]}(x,y;u) = \frac{1}{2i}\sum_{j=0}^{n}\binom{n}{j}FG_{n-j,\lambda}^{(k)}\sum_{k=0}^{j}\binom{j}{k}(x)_{j-k,\lambda}\begin{pmatrix}(iy)_{n,\lambda}\\-(-iy)_{n,\lambda}\end{pmatrix}.$$

Now, we define the degenerate two parametric $C_{n,\lambda}(x,y)$ and $S_{n,\lambda}(x,y)$ polynomials, respectively,

$$e_{\lambda}^{(x)}(t)\cos_{\lambda}^{(y)}(t) = \sum_{n=0}^{\infty}C_{n,\lambda}(x,y)\frac{t^n}{n!} \tag{3.15}$$

and

$$e_{\lambda}^{(x)}(t)\sin_{\lambda}^{(y)}(t) = \sum_{n=0}^{\infty}S_{n,\lambda}(x,y)\frac{t^n}{n!}. \tag{3.16}$$

From (1.4) and (3.9), we get

$$C_{n,\lambda}(x,y) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}\left((iy)_{k,\lambda}+(-iy)_{k,\lambda}\right).$$

Similarly, (1.4) and (3.10), we get

$$S_{n,\lambda}(x,y) = \frac{1}{2i}\sum_{k=0}^{n}\binom{n}{k}(x)_{n-k,\lambda}\left((iy)_{k,\lambda}-(-iy)_{k,\lambda}\right).$$

From (2.4), (3.7) and (3.11), we write

$$\left(e_q(t)-u\right)\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,c]}(x,y;u)\frac{t^n}{n!} = (1-u)Ei_{k,\lambda}\left(\log_{\lambda}(1+t)\right)e_q^{(x)}(t)\cos_{\lambda}^{(y)}(t).$$

The left hand side of this equation is

$$\sum_{n=0}^{\infty}\begin{Bmatrix}\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\,FG_{l,\lambda}^{[k,c]}(x,y;u)\\ -uFG_{n,\lambda}^{[k,c]}(x,y;u)\end{Bmatrix}\frac{t^n}{n!}. \tag{3.17}$$

The right hand side of this equation is

$$\frac{1-u}{2}\sum_{n=0}^{\infty}n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=0}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j) \sum_{l=0}^{n-1-m}\binom{n-1-m}{l}(x)_{n-1-m,\lambda}\begin{pmatrix}(iy)_{n-1-m,\lambda}\\+(-iy)_{n-1-m,\lambda}\end{pmatrix}\frac{t^n}{n!}. \tag{3.18}$$

From (3.17) and (3.18), we get

$$2\begin{Bmatrix}\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\,FG_{l,\lambda}^{[k,c]}(x,y;u)\\ -uFG_{n,\lambda}^{[k,c]}(x,y;u)\end{Bmatrix} = (1-u)\begin{Bmatrix}n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=0}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\\ \sum_{l=0}^{n-1-m}\binom{n-1-m}{l}\\ \times(x)_{n-1-m,\lambda}\begin{pmatrix}(iy)_{n-1-m,\lambda}\\+(-iy)_{n-1-m,\lambda}\end{pmatrix}\end{Bmatrix} \tag{3.19}$$

Similarly, (2.4), (3.8) and (3.12)

$$2i\begin{Bmatrix}\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\,\mathrm{FG}_{l,\lambda}^{[k,s]}(x,y;u)\\ -uFG_{n,\lambda}^{[k,s]}(x,y;u)\end{Bmatrix} = (1-u)\begin{Bmatrix}n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=0}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\\ \sum_{l=0}^{n-1-m}\binom{n-1-m}{l}\\ \times(x)_{n-1-m,\lambda}\begin{pmatrix}(iy)_{n-1-m,\lambda}\\+(-iy)_{n-1-m,\lambda}\end{pmatrix}\end{Bmatrix}. \tag{3.20}$$

Theorem 5. The following relations hold true:

$$2\begin{Bmatrix}\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\,FG_{l,\lambda}^{[k,c]}(x,y;u)\\ -uFG_{n,\lambda}^{[k,c]}(x,y;u)\end{Bmatrix} = (1-u)\begin{Bmatrix}n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=0}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\\ \sum_{l=0}^{n-1-m}\binom{n-1-m}{l}(x)_{n-1-m,\lambda}\\ \times\begin{pmatrix}(iy)_{n-1-m,\lambda}\\+(-iy)_{n-1-m,\lambda}\end{pmatrix}\end{Bmatrix}$$

and

$$2i\left\{\begin{array}{l}\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\, FG_{l,\lambda}^{[k,s]}(x,y;u)\\ -uFG_{n,\lambda}^{[k,s]}(x,y;u)\end{array}\right\}$$

$$=(1-u)\left\{\begin{array}{l} n\sum_{m=0}^{n-1}\frac{\binom{n-1}{m}}{m+1}\sum_{j=0}^{m+1}\frac{(1)_{j,\lambda}}{j^{k-1}}S_{1,\lambda}(m+1,j)\\ \sum_{l=0}^{n-1-m}\binom{n-1-m}{l}(x)_{n-1-m,\lambda}\\ \times\left((iy)_{n-1-m,\lambda}+(-iy)_{n-1-m,\lambda}\right)\end{array}\right\}$$

From (1.6) and (3.7),

$$\begin{aligned}&\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,c]}(x_1+x_2,y;u)\frac{t^n}{n!}\\ &=\frac{e_{\lambda}^{(x_1+x_2)}(t)(1-u)Ei_{k,\lambda}(\log_{\lambda}(1+t))}{e_q(t)-u}\cos_{\lambda}^{(y)}(t)\\ &=\sum_{m=0}^{\infty}\sum_{k=0}^{m}S_{2,\lambda}^{(x_1)}(m,k)(x_2)_k\frac{t^m}{m!}\sum_{l=0}^{\infty}FG_{l,\lambda}^{[k,c]}(0,y;u)\frac{t^l}{l!}\\ &=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}S_{2,\lambda}^{(x_1)}(m,k)(x_2)_k\,FG_{n-m,\lambda}^{[k,c]}(0,y;u)\right\}\end{aligned}\tag{3.21}$$

From (1.6) and (3.8), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}FG_{n,\lambda}^{[k,s]}(x_1+x_2,y;u)\frac{t^n}{n!}\\ &=e_{\lambda}^{(x_1+x_2)}(t)\frac{(1-u)Ei_{k,\lambda}(\log_{\lambda}(1+t))}{e_q(t)-u}\sin_{\lambda}^{(y)}(t)\\ &=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}S_{2,\lambda}^{(x_1)}(m,k)(x_2)_k\,FG_{n-m,\lambda}^{[k,s]}(0,y;u)\right\}\end{aligned}\tag{3.22}$$

Comparing the coefficients of $\frac{t^n}{n!}$ both sides the equations (3.21) and (3.22), we have the following theorem.

Theorem 6. The following relations hold true:

$$FG_{n,\lambda}^{[k,c]}(x_1+x_2,y;u)=\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}S_{2,\lambda}^{(x_1)}(m,k)(x_2)_k\,FG_{n-m,\lambda}^{[k,c]}(0,y;u)$$

and

$$FG_{n,\lambda}^{[k,s]}(x_1+x_2,y;u)=\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{m}S_{2,\lambda}^{(x_1)}(m,k)(x_2)_k\,FG_{n-m,\lambda}^{[k,s]}(0,y;u).$$

Now, for our use in the present investigation, we find the expressions of $\cos_{\lambda}^{(x_1+x_2)}(t)$ and $\sin_{\lambda}^{(x_1+x_2)}(t)$.

From (3.5), we get

$$\begin{aligned}\cos_{\lambda}^{(x_1+x_2)}(t)&=\cos\left(\frac{x_1+x_2}{\lambda}\log(1+\lambda t)\right)\\ &=\cos\left(\frac{x_1}{\lambda}\log(1+\lambda t)\right)\cos\left(\frac{x_2}{\lambda}\log(1+\lambda t)\right)\\ &\quad-\sin\left(\frac{x_1}{\lambda}\log(1+\lambda t)\right)\sin\left(\frac{x_2}{\lambda}\log(1+\lambda t)\right)\\ &=\cos_{\lambda}^{(x_1)}(t)\cos_{\lambda}^{(x_2)}(t)-\sin_{\lambda}^{(x_1)}(t)\sin_{\lambda}^{(x_2)}(t).\end{aligned}\tag{3.23}$$

Putting (3.23), $x_1=x_2=x$, we get

$$\cos_{\lambda}^{(2x)}(t)=\left(\cos_{\lambda}^{(x)}(t)\right)^2-\left(\sin_{\lambda}^{(x)}(t)\right)^2.$$

By (3.6), we get

$$\begin{aligned}\sin_{\lambda}^{(x_1+x_2)}(t)&=\sin\left(\frac{x_1+x_2}{\lambda}\log(1+\lambda t)\right)\\ &=\sin_{\lambda}^{(x_1)}(t)\cos_{\lambda}^{(x_2)}(t)+\cos_{\lambda}^{(x_1)}(t)\sin_{\lambda}^{(x_2)}(t).\end{aligned}\tag{3.24}$$

Setting (3.24), $x_1=x_2=x$, we get

$$\sin_{\lambda}^{(2x)}(t)=2\cos_{\lambda}^{(x)}(t)\sin_{\lambda}^{(x)}(t).$$

From (3.15) and (3.23), we write

$$\begin{aligned}&\sum_{n=0}^{\infty}C_{n,\lambda}(x_1+x_2,y_1+y_2)\frac{t^n}{n!}\\ &=e_{\lambda}^{(x_1+x_2)}(t)\cos_{\lambda}^{(y_1+y_2)}(t)\\ &=e_{\lambda}^{(x_1)}(t)e_{\lambda}^{(x_2)}(t)\left(\begin{array}{l}\cos_{\lambda}^{(y_1)}(t)\cos_{\lambda}^{(y_2)}(t)\\ -\sin_{\lambda}^{(y_1)}(t)\sin_{\lambda}^{(y_2)}(t)\end{array}\right)\\ &=\sum_{n=0}^{\infty}C_{n,\lambda}(x_1,y_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}C_{n,\lambda}(x_2,y_2)\frac{t^n}{n!}\\ &\quad-\sum_{n=0}^{\infty}S_{n,\lambda}(x_1,y_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_{n,\lambda}(x_2,y_2)\frac{t^n}{n!}.\end{aligned}\tag{3.25}$$

Using (3.16) and (3.24), we write

$$\begin{aligned}&\sum_{n=0}^{\infty}S_{n,\lambda}(x_1+x_2,y_1+y_2)\frac{t^n}{n!}\\ &=e_{\lambda}^{(x_1+x_2)}(t)\sin_{\lambda}^{(y_1+y_2)}(t)\\ &=\sum_{n=0}^{\infty}S_{n,\lambda}(x_1,y_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}C_{n,\lambda}(x_2,y_2)\frac{t^n}{n!}\\ &\quad+\sum_{n=0}^{\infty}C_{n,\lambda}(x_1,y_1)\frac{t^n}{n!}\sum_{n=0}^{\infty}S_{n,\lambda}(x_2,y_2)\frac{t^n}{n!}.\end{aligned}\tag{3.26}$$

By using Cauchy product above the equations (3.25) and (3.26), we have the following theorem.

Theorem 7. The following relations hold true:

$$C_{n,\lambda}(x_1+x_2,y_1+y_2)=\sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l}C_{n-k,\lambda}(x_1,y_1)C_{k,\lambda}(x_2,y_2)\\ -S_{n-k,\lambda}(x_1,y_1)S_{k,\lambda}(x_2,y_2)\end{array}\right\}\tag{3.27}$$

and

$$S_{n,\lambda}(x_1+x_2,y_1+y_2) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} S_{n-k,\lambda}(x_1,y_1)C_{k,\lambda}(x_2,y_2) \\ +C_{n-k,\lambda}(x_1,y_1)S_{k,\lambda}(x_2,y_2) \end{array}\right\}. \quad (3.28)$$

Setting $x_1=x_2=x$ and $y_1=y_2=y$ in (3.27) and (3.28), we have respectively,

$$C_{n,\lambda}(2x,2y) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} C_{n-k,\lambda}(x,y)C_{k,\lambda}(x,y) \\ -S_{n-k,\lambda}(x,y)S_{k,\lambda}(x,y) \end{array}\right\}$$

and

$$S_{n,\lambda}(2x,2y) = 2\sum_{k=0}^{n}\binom{n}{k}S_{n-k,\lambda}(x,y)C_{k,\lambda}(x,y).$$

From (3.7) and (3.22), we write

$$\begin{aligned}
&\sum_{n=0}^{\infty} FG_{n,\lambda}^{[k,c]}(x_1+x_2,y_1+y_2;u)\frac{t^n}{n!} \\
&= \frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u} \\
&\times e_\lambda^{(x_1)}(t)e_\lambda^{(x_2)}(t)\left\{\begin{array}{l} \cos_\lambda^{(x_1)}(t)\cos_\lambda^{(x_2)}(t) \\ -\sin_\lambda^{(y_1)}(t)\sin_\lambda^{(y_2)}(t) \end{array}\right\} \quad (3.29) \\
&= \sum_{m=0}^{\infty} FG_{m,\lambda}^{[k,c]}(x_1,y_1;u)\frac{t^m}{m!}\sum_{k=0}^{\infty} C_{k,\lambda}(x_2,y_2)\frac{t^k}{k!} \\
&- \sum_{m=0}^{\infty} FG_{m,\lambda}^{[k,s]}(x_1,y_1;u)\frac{t^m}{m!}\sum_{k=0}^{\infty} S_{k,\lambda}(x_2,y_2)\frac{t^k}{k!}.
\end{aligned}$$

Using (3.8) and (3.24), we write

$$\begin{aligned}
&\sum_{n=0}^{\infty} FG_{n,\lambda}^{[k,s]}(x_1+x_2,y_1+y_2;u)\frac{t^n}{n!} \\
&= \frac{(1-u)Ei_{k,\lambda}(\log_\lambda(1+t))}{e_q(t)-u} \\
&\times e_\lambda^{(x_1)}(t)e_\lambda^{(x_2)}(t)\left\{\begin{array}{l} \sin_\lambda^{(x_1)}(t)\cos_\lambda^{(x_2)}(t) \\ +\cos_\lambda^{(y_1)}(t)\sin_\lambda^{(y_2)}(t) \end{array}\right\} \quad (3.30) \\
&= \sum_{m=0}^{\infty} FG_{m,\lambda}^{[k,s]}(x_1,y_1;u)\frac{t^m}{m!}\sum_{k=0}^{\infty} C_{k,\lambda}(x_2,y_2)\frac{t^k}{k!} \\
&+ \sum_{m=0}^{\infty} FG_{m,\lambda}^{[k,s]}(x_1,y_1;u)\frac{t^m}{m!}\sum_{k=0}^{\infty} S_{k,\lambda}(x_2,y_2)\frac{t^k}{k!}.
\end{aligned}$$

Using Cauchy product (3.29) and (3.30), we have the following theorem.

Theorem 8. The following relations hold true:

$$FG_{n,\lambda}^{[k,c]}(x_1+x_2,y_1+y_2;u) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} FG_{n-k,\lambda}^{[k,c]}(x_1,y_1;u)C_{k,\lambda}(x_2,y_2) \\ -FG_{n-k,\lambda}^{[k,s]}(x_1,y_1;u)S_{k,\lambda}(x_2,y_2) \end{array}\right\} \quad (3.31)$$

and

$$FG_{n,\lambda}^{[k,s]}(x_1+x_2,y_1+y_2;u) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} FG_{n-k,\lambda}^{[k,s]}(x_1,y_1;u)C_{k,\lambda}(x_2,y_2) \\ +FG_{n-k,\lambda}^{[k,c]}(x_1,y_1;u)S_{k,\lambda}(x_2,y_2) \end{array}\right\}. \quad (3.32)$$

Putting $x_1=x_2=x$ and $y_1=y_2=y$ in (3.31) and (3.32), respectively, we have

$$FG_{n,\lambda}^{[k,c]}(2x,2y;u) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} FG_{n-k,\lambda}^{[k,c]}(x,y;u)C_{k,\lambda}(x,y) \\ -FG_{n-k,\lambda}^{[k,s]}(x,y;u)S_{k,\lambda}(x,y) \end{array}\right\}$$

and

$$FG_{n,\lambda}^{[k,s]}(2x,2y;u) = \sum_{k=0}^{n}\binom{n}{k}\left\{\begin{array}{l} FG_{n-k,\lambda}^{[k,s]}(x,y;u)C_{k,\lambda}(x,y) \\ +FG_{n-k,\lambda}^{[k,c]}(x,y;u)S_{k,\lambda}(x,y) \end{array}\right\}.$$

4. Conclusion

Kim and Kim [7] considered the polyexponential and unipoly functions. Kim et al. [3,11] defined and investigated the new type modified degenerate polyexponential function, the degenerate poly-Bernoulli polynomials and the degenerate poly-Genocchi polynomials. Motivated by these studying, we introduce the degenerate poly-Frobenius-Genocchi polynomials of the complex variables. We also give their some interesting properties and identities. As one of our future projects, we would like to continue to do researcher on degenerate versions of various special numbers and polynomials.

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