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On the Entire Paranormed Triple Sequence Spaces Defined by Binomial Poisson Matrix

N. SUBRAMANIAN[1], A. ESI[2,*], TVG. SHRIPRAKASH[1]

[1]School of Arts Sciences and Humanities, Department of Mathematics, SASTRA, Deemed to be University, Thanjavur-613 401, India
[2]Department of Basic Engineering Sciences, Malatya Turgut Ozal University, Malatya-44040, Turkey
*Corresponding author: aesi23@hotmail.com; ayhan.esi@ozal.edu.tr

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Abstract In this paper the entire triple sequence space are the generalization of the classical Maddox's paranormed sequence space have been introduced and investigated some topological properties of entire triple sequence space of binomial Poisson matrix of $Ab^{rs}_{\Gamma^3}$ and $Ab^{rs}_{\Lambda^3}$.

Keywords: *Poisson matrix, triple sequence, paranormed space, entire space*

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1. Introduction

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}(\mathbb{C})$, where $\mathbb{N},\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [1,2], Esi et al. [3,4,5,6,7,8], Dutta et al. [9], Subramanian et al. [10], Debnath et al. [11] and many others. Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m,n,k\in\mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk}=\sum_{i,j,q=1}^{m,n,k} x_{ijq}\,(m,n,k=1,2,3,...).$$

A sequence $x=(x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k}\left|x_{mnk}\right|^{\frac{1}{m+n+k}}<\infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x=(x_{mnk})$ is called triple entire sequence if

$$\left|x_{mnk}\right|^{\frac{1}{m+n+k}}\to 0 \text{ as } m,n,k\to\infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. The space Λ^3 and Γ^3 is a metric space with the metric

$$d(x,y) = sup_{m,n,k}\left\{\left|x_{mnk}-y_{mnk}\right|^{\frac{1}{m+n+k}}: m,n,k:1,2,3,...\right\}, \quad (1.1)$$

for all $x=\{x_{mnk}\}$ and $y=\{y_{mnk}\}$ *in* Γ^3.

Consider a triple sequence $x=(x_{mnk})$. The mnk^{th} section $x^{[m,n,k]}$ of the sequence is defined by $x^{[m,n,k]}=\sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq}$ for all $m,n,k\in\mathbb{N}$,

$$\delta_{mnk}=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots\\ 0 & 0 & \ldots 0 & 0 & \ldots\\ \vdots & & & & \\ 0 & 0 & \ldots 1 & 0 & \ldots\\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

δ_{mnk} has 1 in the mnk^{th} position, and zero otherwise.

The Poisson matrix is defined by $A=T\otimes I+I\otimes T$.

Example: If $T=\begin{bmatrix} 4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 4\end{bmatrix}$ and $T=\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ then

$$A = T \otimes I + I \otimes T = \begin{bmatrix} T+2I & -I & 0 \\ -I & T+2I & -I \\ 0 & -I & T+2I \end{bmatrix}.$$

2. Properties of Poisson Matrix of Eigen Values and Eigen Vectors $A = T \otimes I + I \otimes T$

(1) We have $Ax_{jk} = \lambda x_{jk}$ for $j,k = 1,2,3,\cdots,m$;
(2) The eigen vectors are orthogonal
(3) A is symmetric;
(4) A is positive definite.

Example: If $A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}.$

Hence

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

and so on.

Now, we define the binomial poisson matrix $B^{rs} = Ab^{rs}_{uvw,mnk}$, where

$$b^{rs}_{uvw,mnk} = \begin{cases} \frac{1}{(s+r)^{m+n+k}}\binom{u}{m}\binom{v}{n}\binom{w}{k}s^{(m-u)+(n-v)+(k-w)}r^{u+v+w} \\ if\ 0 \le u \le m, 0 \le v \le n, 0 \le w \le k \\ 0 \\ \text{if } u > m, v > n, w > k \end{cases}$$

where $r,s > 0$. In this paper, we define the binomial Poisson triple equence spaces $b^{rs}_{\Gamma^3}$ and $b^{rs}_{\Lambda^3}$ as the set of all sequences whose $B^{rs}-$ transforms are in the spaces Γ^3 and Λ^3, respectively, that is

$$Ab^{rs}_{\Gamma^3} = lim_{u,v,w\to\infty}\left|\frac{1}{(s+r)^{u+v+w}}\sum_{m,n,w=0}^{u,v,w}\binom{u}{m}\binom{v}{n} \cdot \binom{w}{k}s^{(u-m)+(v-n)+(w-k)}r^{u+v+w}Ax^{1/m+n+k}_{mnk}\right| = 0$$

and

$$Ab^{rs}_{\Lambda^3} = sup_{u,v,w\in\mathbb{N}}\left|\frac{1}{(s+r)^{u+v+w}}\sum_{m,n,w=0}^{u,v,w}\binom{u}{m}\binom{v}{n} \cdot \binom{w}{k}s^{(u-m)+(v-n)+(w-k)}r^{u+v+w}Ax^{1/m+n+k}_{mnk}\right| < \infty.$$

Define the triple sequence $y = \{Ay_{uvw}(rs)\}$, which will be frequently used as the $B^{rs}-$ transform of a triple sequence $x = (x_{mnk})$, i.e.,

$$\{Ay_{uvw}(rs)\} := \frac{1}{(s+r)^{u+v+w}}\sum_{m,n,w=0}^{u,v,w}\binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)}r^{m+n+k}Ax^{1/m+n+k}_{mnk}; \quad (*)$$

for all $m,n,k \in \mathbb{N}$.

Now, we may begin with the following theorem which is essential in the text.

3. Definitions and Preliminaries

3.1. Definition. g is a paranorm on X if

(i) $g: X \to R$ with $g(x) \ge 0$,

(ii) $g(x) = 0 \Leftrightarrow x = 0$,

(iii) $g(x) = g(-x); \forall x \in X$,

(iv) $g(x+y) \le g(x) + g(y); \forall x, y \in X$.

(v) If $\lambda_{rst}, \lambda_0 \in C$ with $\lambda_{rst} \to \lambda_0\ (r,s,t \to \infty)$ and if $x_{rst}, a \in X$ with $x_{rst} \to a\ \ (r,s,t \to \infty)$ in the sense that $g(x_{rst} - a) \to 0\ \ (r,s,t \to \infty)$, then $\lambda_{rst}x_{rst} \to \lambda_0 a$ $(r,s,t \to \infty)$ in the sense that $g(\lambda_{rst}x_{rst} - \lambda_0 a) \to 0$ $(r,s,t \to \infty)$.

4. Main Results

4.1. Theorem. $Ab^{rs}_{\Gamma^3}$ is a complete metric space paranormed by g, defined by

$$g(x,y) = sup_{u,v,w\in\mathbb{N}}\left|\frac{1}{(s+r)^{u+v+w}}\sum_{m,n,w=0}^{u,v,w}\binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)}r^{u+v+w} \cdot A(x_{mnk} - y_{mnk})^{1/m+n+k}\right|.$$

Proof: Let $\left\{x^{(i)}\right\}$ be a Cauchy sequence in $Ab^{rs}_{\Gamma^3}$. Then given any $\epsilon > 0$ there exists a positive integer N depending on ϵ such that $g\left(x^{(i)}, x^{(j)}\right) < \epsilon$ for all $i, j \geq N$. Hence

$$sup_{u,v,w\in\mathbb{N}} \left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} \cdot A\left(x^{(i)}_{mnk} - y^{(j)}_{mnk}\right)^{1/m+n+k} \right| < \epsilon$$

for all $i, j \geq N$. Consequently

$$\left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x^{(i)}_{mnk}\right)^{1/m+n+k} \right|$$

is a Cauchy sequence in the metric space $\mathbb{C}$ of complex numbers. Since $\mathbb{C}$ is complete, so

$$\left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x^{(i)}_{mnk}\right)^{1/m+n+k} \right| \to 0$$

as $i \to \infty$. Hence there exists a positive integer i_0 such that

$$sup_{u,v,w\in\mathbb{N}} \left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} \cdot r^{u+v+w} A\left(x^{(i)}_{mnk} - x_{mnk}\right)^{1/m+n+k} \right| < \epsilon$$

for all $i \geq i_0$. In particular, we have

$$\left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x^{(i)}_{mnk} - x_{mnk}\right)^{1/m+n+k} \right| < \epsilon.$$

Now

$$\left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x_{mnk}\right)^{1/m+n+k} \right|$$
$$\leq \left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x_{mnk} - x^{(i_0)}_{mnk}\right)^{1/m+n+k} \right|$$
$$+ \left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x^{(i_0)}_{mnk}\right)^{1/m+n+k} \right|$$
$$< \epsilon + 0$$

for each *m*, *n*, *k*. Thus

$$\left| \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{u+v+w} A\left(x_{mnk}\right)^{1/m+n+k} \right| < \epsilon$$

each *m*, *n*, *k*. That is $x \in Ab^{rs}_{\Gamma^3}$. Therefore, $Ab^{rs}_{\Gamma^3}$ is a complete metric space.

4.2. Theorem. The entire triple sequence space $Ab^{rs}_{\Gamma^3}$ is linearly isomorphic to Γ^3.

Proof: Now to prove that the existence of a linear bijection between thespaces $Ab^{rs}_{\Gamma^3}$ and Γ^3 with the notation of $(*)$, define the transformation T from $Ab^{rs}_{\Gamma^3}$ and Γ^3 by $x \mapsto y = Tx$. The linearity of T is trivial. Furthermore, it is obvious that

$$x = x_{mnk} = \begin{pmatrix} \theta & \theta & ...\theta & \theta \\ \theta & \theta & ...\theta & \theta \\ \vdots \\ \theta & \theta & ...\theta & \theta \\ \theta & \theta & ...\theta & \theta \end{pmatrix}$$

whenever

$$Tx = Tx_{mnk} = \begin{pmatrix} \theta & \theta & ...\theta & \theta \\ \theta & \theta & ...\theta & \theta \\ \vdots \\ \theta & \theta & ...\theta & \theta \\ \theta & \theta & ...\theta & \theta \end{pmatrix}$$

and hence T is injective. Let $y \in \Gamma^3$ and define the sequence

$$\left(x_{mnk}\right)^{1/m+n+k} = \frac{1}{r^{m+n+k}} \sum_{a,b,c=0}^{m,n,k} \binom{m}{a}\binom{n}{b}\binom{k}{c} \cdot (-s)^{(m-a)+(n-b)+(k-c)} (s+r)^{a+b+c} A\left(y_{abc}\right)^{1/a+b+c},$$

for each $m, n, k \in \mathbb{N}$. Then, we have

$$\left(B^{rs}x\right)_{uvw} = \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} \cdot s^{(u-m)+(v-n)+(w-k)} r^{m+n+k} A\left(y_{mnk}\right)^{1/m+n+k}$$

$$= \frac{1}{(s+r)^{u+v+w}} \sum_{m,n,w=0}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k} s^{(u-m)+(v-n)+(w-k)} \sum_{a,b,c=0}^{m,n,k} \binom{m}{a}\binom{n}{b}\binom{k}{c} (-s)^{(m-a)+(n-b)+(k-c)} (s+r)^{a+b+c} A(y_{abc})^{1/a+b+c}$$

$$= \frac{1}{(s+r)^{u+v+w}} \sum_{a,b,c=0}^{u,v,w} \left(\sum_{m,n,k=a,b,c}^{u,v,w} \binom{u}{m}\binom{v}{n}\binom{w}{k}\binom{m}{a}\binom{n}{b}\binom{k}{c} s^{(u-m)+(v-n)+(w-k)} (-s)^{(m-a)+(n-b)+(k-c)} (s+r)^{a+b+c} \right) A(y_{abc})^{1/a+b+c}$$

$$= \frac{1}{(s+r)^{u+v+w}} \sum_{a,b,c=0}^{u,v,w}$$

$$\left(\sum_{m,n,k=a,b,c}^{u,v,w} \binom{u}{a}\binom{v}{b}\binom{w}{c}\binom{u-a}{m-a}\binom{v-b}{n-b}\binom{w-c}{k-c} (-s)^{(m-a)+(n-b)+(k-c)} s^{(u-a)+(v-b)+(w-c)} (s+r)^{a+b+c} A(y_{abc})^{1/a+b+c} \right)$$

$$= \frac{1}{(s+r)^{u+v+w}} \sum_{a,b,c=0}^{u,v,w} \binom{u}{a}\binom{v}{b}\binom{w}{c} s^{(u-a)+(v-b)+(w-c)} (s+r)^{a+b+c}$$

$$\cdot \left(\sum_{m,n,k=a,b,c}^{u,v,w} \binom{u-a}{m-a}\binom{v-b}{n-b}\binom{w-c}{k-c} (-s)^{(m-a)+(n-b)+(k-c)} \right) A(y_{abc})^{1/a+b+c}$$

$$= \frac{1}{(s+r)^{u+v+w}} \sum_{a,b,c=0}^{u,v,w} \binom{u}{a}\binom{v}{b}\binom{w}{c} s^{(u-a)+(v-b)+(w-c)} (s+r)^{a+b+c} \delta_{uvwmnk} A(y_{abc})^{1/a+b+c}$$

$$= \frac{1}{(s+r)^{u+v+w}} \binom{u}{u}\binom{v}{v}\binom{w}{w} s^{(u-u)+(v-v)+(w-w)} (s+r)^{u+v+w} \times 1 \times A(y_{uvw})^{1/u+v+w}$$

$$= A(y_{uvw})^{1/u+v+w}.$$

Thus, we have that $x \in Ab^{rs}_{\Gamma^3}$ and consequently T is surjective. Hence, T is a linear bijection. Hence the spaces $Ab^{rs}_{\Gamma^3}$ and Γ^3 are linearly isomorphic.

5. The Basis for the Space $Ab^{rs}_{\Gamma^3}$

Let (λ, g) be a paranormed space. Recall that a entire triple sequence (β_{mnk}) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique entire triple sequence (α_{mnk}) of scalars such that

$$g\left(x - \sum_{m=0}^{u} \sum_{n=0}^{v} \sum_{k=0}^{w} \alpha_{mnk} \beta_{mnk} \right) \to 0 \text{ as } u, v, w \to \infty.$$

The series $\sum\sum\sum \alpha_{mnk}\beta_{mnk}$ which has the sum x is then called the expansion of x with respect to (β_{uvw}), and written as $x = \sum\sum\sum \alpha_{mnk}\beta_{mnk}$. Since it is known that the poisson matrix domain λ_A of a triple sequence space λ has a basis if and only if λ has a basis we have the following, because of the isomorphism T is onto, defined in the proof of the Theorem 4.2, the inverse image of the basis of the space Γ^3 is a basis of the new space $Ab^{rs}_{\Lambda^3}$. Therefore, we have the following:

5.1. Theorem. Let $\lambda_{mnk} = (B^{rs}x)_{mnk}$ for all $m,n,k \in \mathbb{N}$. Define the sequence $b^{mnk} = \left\{ b^{(mnk)} \right\}_{m,n,k \in \mathbb{N}}$ of the elements of the space $b^{rs}_{\Gamma^3}$ by

$$b^{rs}_{uvw,mnk} = \begin{cases} \frac{1}{r^{u+v+w}} \binom{u}{m}\binom{v}{n}\binom{w}{k} (-s)^{(u-m)+(v-n)+(w-k)} (s+r)^{m+n+k} \\ if \ u > m, v > n, w > k \\ 0 \\ 0 \le u \le m, 0 \le v \le n, 0 \le w \le k \end{cases}$$

for every fixed $m,n,k \in \mathbb{N}$.

The sequence $\left\{ b^{(mnk)} \right\}_{m,n,k \in \mathbb{N}}$ is a basis for the space $Ab^{rs}_{\Gamma^3}$, and any $x \in Ab^{rs}_{\Gamma^3}$ has a unique representation of the form

$$x = \sum_m \sum_n \sum_k \lambda_{mnk} b^{(mnk)}.$$

6. The α, β and γ-duals of the Space $Ab^{rs}_{\Gamma^3}$

In this section, we state and prove the theorems determining the α, β and $\gamma-$ duals of the space $Ab^{rs}_{\Gamma^3}$ of non-absolute type.

We shall firstly give the definition of α, β and $\gamma-$ duals of triple sequence spaces and after quoting the lemmas which are needed in proving the theorems given in this section. The set $S(\lambda, \mu)$ defined by

$$S(\lambda,\mu)=\left\{\begin{array}{l} z=(z_{mnk})\in w: xz=(x_{mnk}z_{mnk})\in\mu \\ for\ all\ x=(x_{mnk})\in\lambda \end{array}\right\} (6.1)$$

is called the multiplier space of the triple sequence space λ and μ. One can easily observe for a triple sequence space ν with $\lambda\supset\nu\supset\mu$ that the inclusions $S(\lambda,\mu)\subset S(\nu,\mu)$ and $S(\lambda,\mu)\subset S(\lambda,\nu)$ hold.

The α,β and $\gamma-$ duals of a triple sequence space are also referred as Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively

For the give the α,β and $\gamma-$duals of the space $b^{rs}_{\Gamma^3}$ of non-absolute type, we need the following Lemma.

6.1. Lemma. Let A be a binomial Poisson matrix. Then the following statments hold

(1) $A\in\left(\Gamma^3:\ell^3\right)\Leftrightarrow$

$$sup_{K\in\Im}\sum_{u,v,w}\left|\sum_{(m,n,k)\in K}AM^{-1}\right|<\infty,\exists M\in\mathbb{N}. \quad (6.2)$$

(2) $A\in\left(C^3:\ell^3\right)\Leftrightarrow$ (6.2) holds and

$$\sum_{u,v,w}\left|\sum_{m,n,k}A\right|<\infty. \quad (6.3)$$

(3) $A\in\left(\Gamma^3:C^3\right)\Leftrightarrow$

$$sup_{u,v,w\in\mathbb{N}}\sum_{m,n,k}|A|M^{-1}<\infty,\exists M\in\mathbb{N}, \quad (6.4)$$

$$\begin{array}{l}\exists\alpha_{mnk}\in\mathbb{R}\ni: lim_{u,v,w\to\infty}|A-\alpha_{mnk}|=0,\\ for\ all\ m,n,k\in\mathbb{N},\end{array} \quad (6.5)$$

$$\begin{array}{l}\exists\alpha_{mnk}\in\mathbb{R}: sup_{u,v,w\in\mathbb{N}}N^{-1}\sum_{m,n,k}|A-\alpha_{mnk}|M^{-1}<\infty;\\ \exists M\in\mathbb{N}\ and\ \forall N\in\mathbb{N}.\end{array} \quad (6.6)$$

(4) $A\in\left(C^3:C^3\right)\Leftrightarrow$ (6.4), (6.5) (6.6) hold

$$\exists\alpha_{mnk}\in\mathbb{R}: lim_{u,v,w\to\infty}\left|\sum_{u,v,w}A-\alpha_{mnk}\right|=0. \quad (6.7)$$

(5) $A\in\left(\Gamma^3:\Lambda^3\right)\Leftrightarrow$

$$sup_{m,n,k\in\mathbb{N}}\left(\sum_{m,n,k}|A|M^{-1}\right)<\infty,\exists M\in\mathbb{N}. \quad (6.8)$$

(6) $A\in\left(\ell^3:\ell^3\right)\Leftrightarrow$

$$sup_{N\in\Im}sup_{m,n,k\in\mathbb{N}}\left|\sum_{(u,v,w)\in N}A\right|<\infty. \quad (6.9)$$

$$A\in\left(\ell^3:\Lambda^3\right)\Leftrightarrow sup|A|<\infty. \quad (6.10)$$

(7) $A\in\left(\ell^3:\Lambda^3\right)\Leftrightarrow$

$$sup|A|<\infty. \quad (6.11)$$

$$sup_{m,n,k\in\mathbb{N}}\sum_{m,n,k}\left|AM^{-1}\right|<\infty. \quad (6.12)$$

(8) $A\in\left(\ell^3:C^3\right)\Leftrightarrow$ {6.11} and (6.12) hold, and

$$lim_{u,v,w\to\infty}A=\beta_{mnk},\forall m,n,k\in\mathbb{N}. \quad (6.13)$$

6.2. Theorem. Let $K\in\Im$ and

$$K^{*}=\left\{(m,n,k)\in\mathbb{N}^3: m\le u, n\le v, k\le w\right\}\bigcap K$$

for $K\in\Im$. Define the sets T_1^{rs},T_2^{rs},T_3 and T_4 as follows:

$$T_1^{rs}=\bigcup_{M>1}\left\{\sup K\in\Im\sum\nolimits_{u,v,w}\left|\sum_{(m,n,k)\in K^{+}}b^{rs}_{uvw,mnk}M^{-1}\right|<\infty\right\},$$

$$T_2^{rs}=\left\{\sum_{u,v,w}\left|\sum_{m,n,k=0}^{u,v,w}b^{rs}_{uvw,mnk}M^{-1}\right| exists\ \ for\ each\ u,v,w\in\mathbb{N}\right\}$$

$$T_3=\bigcup_{M>1}\left\{sup_{N\in\Im}\sum_{m,n,k}\left|\sum_{u,v,w\in N}b^{rs}_{uvw,mnk}M^{-1}\right|<\infty\right\},$$

$$T_4=\left\{sup_{N\in\Im}sup_{m,n,k\in N}\left|\sum_{u,v,w\in N}b^{rs}_{uvw,mnk}\right|<\infty\right\},$$

where the poisson matrix $b^{rs}_{uvw,mnk}$

$$b^{rs}_{uvw,mnk}=\left\{\begin{array}{l}\frac{1}{r^{u+v+w}}\sum_{m,n,k=0}^{u,v,w}\binom{u}{m}\binom{v}{n}\binom{w}{k}\\ \cdot(-s)^{(u-m)+(v-n)+(w-k)}(s+r)^{m+n+k}A\\ if\ \ m\le u,n\le v,k\le w\\ 0\ \ if\ \ \ u<m,v<n,w<k\end{array}\right. (6.14)$$

Then $\left[Ab^{rs}_{\Gamma^3}\right]^{\alpha}=T_1$.

Proof: We choose the sequence $A\in w$. We can easily derive that with the $*$ that

$$\begin{aligned}Ax&=\frac{1}{r^{u+v+w}}\sum_{m,n,k=0}^{u,v,w}\binom{u}{m}\binom{v}{n}\binom{w}{k}\\ &\cdot(-s)^{(u-m)+(v-n)+(w-k)}(s+r)^{m+n+k}Ay_{mnk} \quad (6.15)\\ &=\left(b^{rs}y\right).\end{aligned}$$

for all $m,n,k,u,v,w\in\mathbb{N}$, where $b^{rs}=b^{rs}_{uvw,mnk}$ defined by (6.14). It follows from (6.15) that $(Ax)\in\ell^3$ whenever

$x \in b^{rs}_{\Gamma^3}$ if and only if $by \in \ell^3$ whenever $y \in \Gamma^3$. This means that $A = (A) \in \left[Ab^{rs}_{\Gamma^3}\right]^{\alpha}$ if and only if $b \in \left(\Gamma^3 : \ell^3\right)$. Then we observe that $\left[b^{rs}_{\Gamma^3}\right]^{\alpha} = T^{rs}_1$.

Competing Interests

The authors declare that there is not any con.ict of interests regarding the publication of this manuscript.

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